

# Applications of Statistical-Learning Control in Systems and Control

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## Extended abstract

It has been recently shown that many control problems are too difficult to admit analytic solutions [2, 3]. These controls problems include fixed-order controller design [11], robust multiobjective control [1], hybrid systems control [10], time-delay systems [9], and others [5]. Even though such questions may be too difficult to answer analytically, or may not be answerable exactly given a reasonable amount of computational resources, researchers have shown that we can “approximately” answer these questions “most of the time”, and have “high confidence” in the correctness of the answers. This is the gist of the probabilistic methods, which many authors have recently proposed in control analysis and design. These methods build on the standard Monte Carlo approach (with justifications based on Chernoff Bounds, Hoeffding Inequality, and other elementary probabilistic tools [14]) with ideas advanced during the 1960s and on the theory of empirical processes and statistical learning developed in the 1970s and 1980s. In control theory, some of the original (Monte Carlo) ideas have already been used in [12, 6] to solve robust analysis problems while Vidyasagar [15] and the authors [8] used learning theory to solve robust design problems. The basic idea of using learning methods in such problems is to convert a highly complex (potentially undecidable) design problem into a probabilistic optimization framework. This framework is further reduced to a sequence of testable analysis problems whose outcome is a probabilistic solution to the original design problem. Such design problems may include the congestion control of high-speed communications network, the design of power control algorithms in spread-spectrum communications and others.

Unfortunately, and as acknowledged by the various authors, probabilistic and learning methods, while more efficient than gridding techniques (which suffer from the curse of dimensionality), still require a large number of samples in order to guarantee accurate designs. This has precluded their use in a practical design setting. As an example, Vidyasagar in [15] calculates that more than 2 million samples are needed in order to probabilistically guarantee a certain performance level in a robust control design problem. On the other hand, it was conjectured and verified experimentally that much smaller bounds on the number of samples (tens of thousands instead of millions) may be sufficient to guarantee a certain level of performance [15]. In fact, Vidyasagar in [15] uses 200 samples instead of the millions implied by his bounds, while acknowledging that, in this way, the theoretical guarantees of accuracy and confidence no longer hold. In [8] the authors, invoking different versions of *bootstrap sequential learning* algorithms, found that a smaller bound on the number of samples can still guarantee the desired level of accuracy and confidence.

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To fix ideas about randomized algorithms in controller design, let us consider an uncertain plant  $G(s, X)$ , where  $X$  indicates the unknown plant parameters, and that we want to find some controller parameters  $Y$  of controller  $C(s, Y)$  such that a performance index is minimized. What is usually done in the worst-case approach is to minimize a function in the form

$$\sup_X [g(X, Y)],$$

whereas in the probabilistic approach we attempt to minimize the expected value

$$\mathbb{E}_X [g(X, Y)],$$

where  $g(X, Y)$  is a performance measure. One limitation of this approach is that in practice, we do not have the necessary information to calculate  $\mathbb{E}_X [g(X, Y)]$  since all we have are sample plants and compensators. Moreover, how do we find minimize  $\mathbb{E}_X [g(X, Y)]$  when all we have are the values of  $g(X, Y)$  at sample points? In [14, 13], the empirical mean of  $g(X, Y)$  is used instead of  $\mathbb{E}_X [g(X, Y)]$  for a given  $Y$

$$\frac{1}{n} \sum_{j=1}^n g(X_j, Y), \quad (1)$$

which then leaves us with two questions:

1. Will  $\frac{1}{n} \sum_{j=1}^n g(X_j, Y)$  be a good approximation of  $\mathbb{E}_X [g(X, Y)]$  uniformly in  $Y$  as  $n$  increases?
2. How to minimize the empirical mean  $\frac{1}{n} \sum_{j=1}^n g(X_j, Y)$  with respect to  $Y$ ?

It turns out that the first question (concerning the accuracy of empirical approximation) has been studied thoroughly in the theory of empirical processes and statistical learning. Minimization of a function defined by equation (1) is a case of empirical risk minimization. In the context of control problems, Vidyasagar [15, 13] used for this purpose a simple Monte Carlo minimization technique based on simulation of random points  $Y_i$  (independently of  $X_j$ ) and minimizing the empirical risk over a finite set of such points. We also used this approach in [8], but we emphasized there that such optimization technique becomes imprecise in high dimensional problems. This motivates the development of more efficient algorithms of empirical risk minimization that are at the same time self-bounding (i.e. they provide sharp enough data-driven bounds on the accuracy of empirical approximation).

In the following example we use the algorithm proposed in [8] to design a robust controller. The control problem we consider was originally presented in [4]. What we will do is to solve the robust control problem using our randomized algorithm and then to interpret the result using classical analysis tools.

### Example 1.

Let us consider the feedback system in Figure 1. The plant  $G(s)$  is a simplified model of a flexible beam. The input is the voltage to a power amplifier while the output is the tip deflection of the beam. The transfer function  $G(s)$  is given by

$$G(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)}. \quad (2)$$

In [4] the author designed a controller  $C(s)$  considering some time domain specifications and an amplitude constraint on the plant input. In order to use the  $\mathcal{H}_\infty$  approach, some appropriate weighting functions  $W_1(s)$  and  $W_3(s)$  were chosen and the problem was reformulated as the minimization of the mixed-sensitivity norm (see Figure 2)

$$\left\| \begin{pmatrix} W_1 S \\ W_3 C S \end{pmatrix} \right\|_\infty \quad (3)$$

where  $S$  is the sensitivity function  $S = (1 + GC)^{-1}$ . The suboptimal controller found in [4] achieves an  $\mathcal{H}_\infty$  norm of the mixed sensitivity (3) of about 0.938 and has order eight, the same order of the augmented plant (plant plus weighting function).

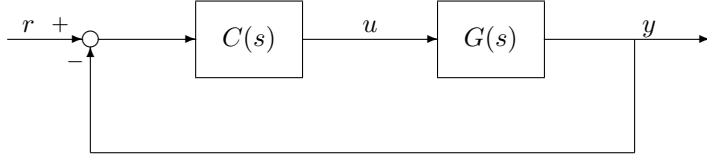


Figure 1: The closed-loop system for Example 1

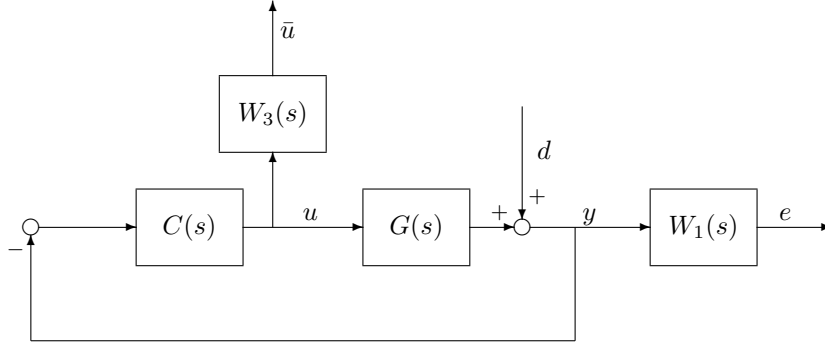


Figure 2: The closed-loop system with the weighting functions

The same example was reconsidered in [6], where the authors tried to design a fixed-structure, third order controller with a randomized algorithm. After fixing some ranges for the controller zeros, poles and gain, 10,000 controllers were randomly generated and the controller minimizing the norm (3) was chosen. This controller resulted to achieve an  $\mathcal{H}_\infty$  norm of the mixed sensitivity (3) of about 1.02, not far from the optimal one.

Now let us re-consider the beam transfer function (2) where we introduce uncertainties in the gain, the damping factor, and the natural frequency of the two complex poles which model the first flexible mode

$$\hat{G}(s) = K \frac{-6.4750s^2 + 4.0302s + 175.7700}{5s(s + 6.66 \cdot 10^{-4})(s^2 + 2\zeta\omega_n s + \omega_n^2)} \quad (4)$$

where the original plant  $G(s)$  is recovered letting  $K = 1$ ,  $\zeta = 0.0675$ ,  $\omega_n = 5.28$ .

Rewriting (4) as

$$\hat{G}(s) = K \frac{-6.4750s^2 + 4.0302s + 175.7700}{5s(s + 6.66 \cdot 10^{-4})(s^2 + As + B)} \quad (5)$$

we consider the following intervals for the coefficients  $K$ ,  $A$  and  $B$

$$K \in (0.5, 1.5), \quad A \in (0, 1.4259), \quad B \in (19.53, 36.27), \quad (6)$$

which result in the following ranges for the original parameters  $\zeta$  and  $\omega_n$

$$\zeta \in (0, 0.1184), \quad \omega_n \in (4.4176, 6.0201).$$

What we intend to find is a third-order controller which minimizes the mixed sensitivity norm (3) over all the uncertain plants (4), guaranteeing at the same time some nominal performance

$$\min \left\| \begin{pmatrix} W_1(1 + \hat{G}C)^{-1} \\ W_3C(1 + \hat{G}C)^{-1} \end{pmatrix} \right\|_\infty \quad \text{subject to} \quad \left\| \begin{pmatrix} W_1(1 + GC)^{-1} \\ W_3C(1 + GC)^{-1} \end{pmatrix} \right\|_\infty < 2 \quad (7)$$

A possible *classical* way to approach the problem could be to use the  $\mu$ -synthesis to design a controller and then reduce its order hoping that its optimal properties would be retained. Hereafter we will try a

randomized approach based on our algorithm. As in [6], we look for a stable, minimum-phase, strictly proper, third order controller. Its coefficients are randomly generated using the algorithm proposed in [6].

Next, we reformulate our original problem. We assume that the plant uncertain parameters have uniform distribution in the intervals (6). We denote by  $X \in \mathcal{X} \subset \mathbb{R}^3$  the vector of the random coefficients of the plant  $X = [K \ A \ B]$ , and by  $Y \in \mathcal{Y} \subset \mathbb{R}^6$  the the vector of the random coefficients of the controller. Let us introduce a cost function

$$\Psi(Y) = \max\{\psi_1(Y), \psi_2(Y)\} \quad (8)$$

where

$$\psi_1(Y) = \begin{cases} 0 & \text{if } \left\| \begin{pmatrix} W_1(1 + G(s)C(s, Y))^{-1} \\ W_3C(1 + G(s)C(s, Y))^{-1} \end{pmatrix} \right\|_\infty < 2 \\ 1 & \text{otherwise} \end{cases} \quad (9)$$

and

$$\psi_2(Y) = E(\zeta(X, Y))$$

with

$$\zeta(X, Y) = \begin{cases} 1 & \text{if } (1 + \hat{G}(s, X)C(s, Y))^{-1} \text{ is unstable} \\ \left\| \begin{pmatrix} W_1(1 + \hat{G}(s, X)C(s, Y))^{-1} \\ W_3C(1 + \hat{G}(s, X)C(s, Y))^{-1} \end{pmatrix} \right\|_\infty & \text{otherwise} \end{cases}$$

Now we are ready to apply the randomized algorithm proven in [8] to find a probably approximate near minimum of  $\Psi(Y)$  with confidence  $1 - \delta$ , level  $\alpha$  and accuracy  $\varepsilon$  [13].

**Algorithm 1.** Step 1. Fix the values for  $\delta$ ,  $\alpha$  and  $\varepsilon$ .

Step 2. Choose  $n$  controllers with random uniformly distributed coefficients  $Y_1, \dots, Y_n \in \mathcal{Y}$  where (we indicate by  $\lfloor \cdot \rfloor$  the floor operator)

$$n = \left\lfloor \frac{\log(2/\delta)}{\log[1/(1-\alpha)]} \right\rfloor$$

Evaluate for these controllers the function  $\psi_1(Y)$  (9) and discard those controllers for which  $\psi_1(Y) = 1$ . Let  $\hat{n}$  be the number of the remaining controllers.

Step 3. Choose  $m$  plants  $\hat{G}(s, X)$  generating random parameters  $X_1, \dots, X_m \in \mathcal{X}$  with uniform distribution, where

$$m = 2^k \left\{ \left\lfloor \frac{100}{\varepsilon^2} \log \left( \frac{8}{\delta} \right) \right\rfloor + 1 \right\}$$

Step 4. Evaluate the stopping variable

$$\gamma = \max_{1 \leq j \leq \hat{n}} \left| \frac{1}{m} \sum_{i=1}^m r_i \zeta(X_i, Y_j) \right|$$

where  $r_i$  are *Rademacher* random variables, i.e. independent identically distributed random variables taking values  $+1$  and  $-1$  with probability  $1/2$  each. If  $\gamma \leq \varepsilon/5$ , go to Step 5. If  $\gamma > \varepsilon/5$ , add  $m$  more independent plants to the plant samples, set  $m := 2m$  and repeat Step 4.

Step 5. Choose the controller which minimizes the function

$$\frac{1}{m} \sum_{i=1}^m \zeta(X_i, \cdot)$$

This is the statistical optimal controller in the sense defined above.

◇

In our simulation, we chose  $\varepsilon = 0.1$ ,  $\delta = 0.05$  and  $\alpha = 0.001$ . Therefore  $n$  evaluated to 3,688 and  $m$  initially evaluated to 50,753. Out of the  $n$  controllers, only  $\hat{n} = 15$  gave  $\psi_1 = 0$ ; the others were discarded. One iteration was sufficient.

Now let us go back to our original intent. We wanted to design a *good* third order controller attempting to minimize the norm in (7). So how *good* is our controller? To answer this question, we have analyzed the *nominal performance*, *robust stability* and *robust performance* of our closed-loop system. The tools we used have become classical control tools; for a detailed discussion see e.g. [16] and the references therein.

To analyze the nominal performance, it is sufficient to evaluate the mixed sensitivity norm (3). This norm evaluates to about 1.08, which is not far from the values achieved by the  $\mathcal{H}_\infty$  controller proposed in [4] and by the random controller proposed in [6].

The analysis of both robust stability and robust performance can be carried out using the  $\mu$ -analysis. After rearranging our plant using Linear Fractional Transformations (LFTs), and defining appropriate uncertain blocks, we evaluated a real structured singular value (for the robust stability case) and a structured singular value (for the robust performance case) and found out that the values we obtained for our controller are not far from those attained by the optimal controller proposed in [4]. Further details will be provided in the final version of the paper.  $\triangle$

The minimization part of Algorithm 1 used in Example 1 still relies on rather simplistic and inaccurate Monte Carlo type algorithm of minimization of empirical risk. Combining of measuring the accuracy of empirical approximation and determining the sample size needed to achieve the desired accuracy with more sophisticated and efficient optimization algorithms require the development of iterative structural risk minimization method as presented in [7]. The implement of these methods in the control framework is currently under investigation. The final version of the paper will include some results showing that the method of iterative structural risk minimization proposed in [7] allows us to get a solution of control problems (with probabilistic guarantees) that is substantially better than the one given by the Monte Carlo optimization.

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