Controller Synthesis for a Class of Interval Plants

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ACCEPTED FOR AUTOMATICA 1994

October 20, 1995

ABSTRACT

This paper presents a new approach to the synthesis of stabilizing controllers for a class of one-parameter interval plants. The approach is based on the concept of *analytic-real-positive* (ARP) functions.

1 Introduction

The basic problem considered in this technical communique is the problem of synthesizing a controller which robustly stabilizes one-parameter single-input-single-output interval plants of the form

$$G(s,\lambda) = \frac{\lambda n_1(s) + (1-\lambda)n_2(s)}{\lambda d_1(s) + (1-\lambda)d_2(s)} \tag{1}$$

where $0 \leq \lambda \leq 1$. It is possible to apply the one-parameter theory based on conformal mappings developed in (Khargonekar and Tannenbaum, 1985) to solve this problem. Indeed this is basically what is done in (Olbrot and Nikodem, 1992) where λ appears either in the numerator or in the denominator but not in both. However here we will solve the problem directly through the concept of an *analytic-real*

-positive (ARP) function. It should be noted that there is no loss in generality in assuming a one-parameter plant of the convex form given in (1), since by a suitable mapping the parameter in question can always be put in this form if the parameter varies in a single connected interval. The basic approach used here is to stabilize the two extreme points of the interval plant, i.e. $\lambda = 0$ and $\lambda = 1$, and then use the properties of ARP functions to guarantee that the closed-loop system remains stable for all values of λ between the two extreme values. The synthesis problem is ultimately reduced to a problem of interpolation with positive-real functions.

Since the submittal of our paper, we became aware that Theorem (3.2.2) in (Ghosh, 1988) duplicates the matrix condition in our equation (11). Also, it was brought to our attention that a paper which has been submitted for publication (Muramatsu et al., 1993) independently developed similar results. Finally, a more detailed paper of Olbrot and Nikodem (Olbrot and Nikodem, 1994) addresses a special case of the problem considered here, where the parameter λ appears either in the numerator or in the denominator.

2 Mathematical Preliminaries

Recall that a *unit in* H^∞ is defined as an H^∞ functions whose inverse is also an H^∞ function. See, for example (Vidyasagar, 1985). We will limit our discussion to real rational functions. Thus a real rational function $U(s)$ is a unit in H^∞, hereafter simply referred to as a *unit*, if and only if the numerator and denominator polynomials are real Hurwitz polynomials (polynomials with no zeros in $Re\, s \geq 0$) of the same degree. In the sequel we will make extensive use of the following definitions, where we always assume that $T(s)$ is not identically zero.

Definition 1 *A real rational function $Z(s)$ is an exactly-proper/strictly-positive-real (EP/SPR) function, see (Dorato et al., 1989)), if*

1. *The numerator and denominator polynomials of $Z(s)$ are of the same degree.*
2. *$Z(s)$ is analytic for $Re\, s \geq 0$.*
3. *$Re\, Z(j\omega) > 0$ for all real ω.*

Note that condition (3) is equivalent to $-90^o < arg\, Z(j\omega) < 90^o$ for all real ω.

Definition 2 *A real rational function $T(s)$ is an analytic-real-positive (ARP) function if*

1. *The numerator and denominator polynomials of $T(s)$ are coprime and of the same degree.*
2. *$T(s)$ is analytic for $Re\, s \geq 0$.*
3. *$T(jw) \neq 0$ and $-180^o < arg\, T(j\omega) < 180^o$, for all real ω.*

∎

We use the term *real-positive*, as distinct from *positive-real*, because an ARP function is positive when *real*, but its real part can be negative. An ARP function should not be confused with a positive-real function, however the two are related by the following lemma, which will be used in the next section to synthesize robustly stabilizing controllers. The next lemma is based upon the results in (Dorato et al., 1989), and the notation $\beta_i^{1/2}$ denotes the principal square root of the complex number β_i.

Lemma 1 *There exits an ARP function $T(s)$ which interpolates to $T(\alpha_i) = \beta_i$, where $Re\,\alpha_i > 0$, if and only if there exists an EP/SPR function $Z(s)$ which interpolates to $Z(\alpha_i) = \beta_i^{1/2}$.*

Proof: If an EP/SPR function $Z(s)$ exists which interpolates to $\beta_i^{1/2}$, then $T(s) = [Z(s)]^2$ is an ARP function which interpolates to β_i. Note that the argument of an EP/SPR is in the interval $(-90^o, 90^o)$, so that squaring an EP/SPR function results in a function whose argument lies in the interval $(-180^o, 180^o)$. Conversely if there exists an ARP function which interpolates to β_i, then $Z(s) = [T(s)]^{1/2}$ is a strictly-positive-real function which interpolates to $\beta_i^{1/2}$. Furthermore if there exists a strictly-positive-real function which interpolates given points in the right-half s-plane, there always exists a real-rational positive-real function which interpolates to these same points. See, for example (Youla and Saito, 1967). ∎

Finally we present a lemma which we use in the next section, but may be of independent interest in the study of stability of one-parameter polynomials.

Lemma 2 *The polynomial*

$$f(s, \lambda) = \lambda f_1(s) + (1 - \lambda) f_2(s)$$

where the two polynomials $f_1(s)$ and $f_2(s)$ are coprime, of the same degree, is Hurwitz for all λ in the interval $[0, 1]$ if and only if the function $T(s) = f_2(s)/f_1(s)$ is ARP.

Proof: The end point $\lambda = 0$ is treated separately. It is obvious that at that point, $f(s, \lambda)$ is Hurwitz. Next, we consider the open interval $0 < \lambda \leq 1$. From a Nyquist argument, if $T(s)$ is ARP, then the numerator polynomial of

$$1 + \frac{1 - \lambda}{\lambda} T(s)$$

is Hurwitz for all λ in the interval $(0, 1]$. Thus the polynomial $f(s, \lambda)$ is Hurwitz.

Conversely if $f(s, \lambda)$ is Hurwitz for all λ in the interval $[0, 1]$, it is Hurwitz in $(0, 1]$ and $f_1(s)$ must be Hurwitz. Let $\lambda \neq 0$, then from

$$f(s, \lambda) = \lambda f_1(s)[1 + \frac{1 - \lambda}{\lambda} \frac{f_2(s)}{f_1(s)}]$$

it follows that $[1 + \frac{1-\lambda}{\lambda} T(s)]$ cannot have any zeros in $Re\, s \geq 0$. This can only happen for all λ in the interval $(0, 1]$ if $T(s)$ is ARP. ■

3 Controller Synthesis

We consider now the problem of synthesizing a controller which stabilizes the interval plant given by equation (1). We make the following assumptions concerning the plant. *The polynomials $n_1(s)$ and $d_1(s)$, and the polynomials $n_2(s)$ and $d_2(s)$ are relatively prime, the degree of $d_1(s)$ is equal to the degree of $d_2(s)$, and the rational functions $n_1(s)/d_1(s)$ and $n_2(s)/d_2(s)$ are proper.* Consider the two extreme plants

$$\begin{aligned} G(s,1) &= G_1(s) = \frac{n_1(s)}{d_1(s)} = \frac{N_1(s)}{D_1(s)} \\ G(s,0) &= G_2(s) = \frac{n_2(s)}{d_2(s)} = \frac{N_2(s)}{D_2(s)} \end{aligned} \tag{2}$$

where $N_i(s) = n_i(s)/d(s)$ and $D_i(s) = d_i(s)/d(s)$, $i = 1, 2$ and where $d(s)$ is an arbitrary Hurwitz polynomial, with degree equal to the degree of $d_i(s)$. It is known, see (Vidyasagar, 1985), that a controller which simultaneously stabilizes the two extreme plants given in (2) is given by

$$C(s) = \frac{N_c(s)}{D_c(s)} = \frac{X_1(s) + D_1(s)R(s)}{Y_1(s) - N_1(s)R(s)} \tag{3}$$

where

$$R(s) = \frac{U(s) - D(s)}{N(s)} \tag{4}$$

where $U(s)$ is a unit which interpolates to $D(s)$ at the zeros of $N(s)$ in the right-half s-plane, with $N(s)$ and $D(s)$ given by

$$\begin{aligned} N(s) &= D_1(s)N_2(s) - N_1(s)D_2(s) \\ D(s) &= X_1(s)N_2(s) + Y_1(s)D_2(s) \end{aligned} \tag{5}$$

and with $X_1(s)$ and $Y_1(s)$ solutions of the Bezout identity

$$X_1(s)N_1(s) + Y_1(s)D_1(s) = 1. \tag{6}$$

Note that the interval plant, given in (1), may also be written

$$G(s,\lambda) = N_g(s,\lambda)/D_g(s,\lambda) = \frac{\lambda N_1(s) + (1-\lambda)N_2(s)}{\lambda D_1(s) + (1-\lambda)D_2(s)} \tag{7}$$

It is also known (Vidyasagar, 1985), that $C(s) = N_c(s)/D_c(s)$ will internally stabilize the plant $G(s,\lambda) = N_g(s,\lambda)/D_g(s,\lambda)$ if and only if

$$N_c(s)N_g(s,\lambda) + D_c(s)D_g(s,\lambda) \tag{8}$$

is a unit for all λ in the interval $[0,1]$. If the values of $N_c(s), D_c(s), N_g(s,\lambda)$ and $D_g(s,\lambda)$ given in (3) and (7) are substituted back into expression (8), one obtains the expression

$$\lambda + (1-\lambda)U(s) \tag{9}$$

where $U(s)$ is the interpolating unit in (4). But expression (9) is a unit for admissible λ, if and only if the numerator polynomial of (9) is Hurwitz for all admissible λ. If we set $U(s) = n_u(s)/d_u(s)$, where $n_u(s)$ and $d_u(s)$ are Hurwitz polynomials of the same degree, then the numerator polynomial of (9) becomes

$$\lambda d_u(s) + (1-\lambda)n_u(s) \tag{10}$$

From Lemma 2 it then follows that $U(s)$ must be an ARP function. Let α_i be the unstable zeros of $N(s)$, and let $\beta_i = D(\alpha_i)$, where $N(s)$ and $D(s)$ are given in (5). For simplicity we assume that the unstable roots of $N(s)$ are distinct and have strictly positive real parts. The controller synthesis problem then reduces to finding an ARP function which interpolates to β_i at the points α_i. But from Lemma 1, this reduces to finding an EP/SPR function which interpolates to $\beta_i^{1/2}$ at the points α_i. From (Youla and Saito, 1967) it is known that such a positive real function exists if and only if the Hermitian matrix with entries

$$\left(\frac{\beta_i^{1/2} + \bar{\beta_j}^{1/2}}{\alpha_i + \bar{\alpha}_j}\right) \tag{11}$$

is *positive definite*. This result was first obtained by Ghosh in (Ghosh, 1988) using a different proof. The synthesis procedure may then be summarized as follows.

1. Compute the unstable roots α_i of $N(s)$ and $\beta_i = D(\alpha_i)$, and check the positive definiteness of the matrix specified in (11).
2. Use the interpolation theory to compute an EP/SPR function, see (Dorato et al., 1989), which interpolates the points $\beta_i^{1/2}$. Denote this function $Z(s)$.
3. let $U(s) = [Z(s)]^2$, and use $U(s)$ to compute $R(s)$ from equation (4), and finally compute the controller $C(s)$ from equation (3).

Comments:

1. If the Hermitian matrix with entries

$$\left(\frac{\beta_i + \bar{\beta}_j}{\alpha_i + \bar{\alpha}_j}\right) \tag{12}$$

is positive definite, one can take $U(s) = Z(s)$. This reduces the order of the stabilizing controller.

2. Since the solution to the EP/SPR interpolation problem is not unique, in particular all solutions can be parameterized in terms of an arbitrary EP/SPR function, a parameterization of all robustly stabilizing controllers is available to satisfy other design specifications.

3. The case of non-distinct unstable roots of $N(s)$ may be treated as described in (Youla and Saito, 1967). The basic difference is in the formation of the Hermitian matrix (11).

4. As mentioned in the introduction, the results of Olbrot and and Nikodem (Olbrot and Nikodem, 1994) may be recovered by letting $n_1(s) = n_2(s)$ or $d_1(s) = d_2(s)$.

4 Conclusions

In this paper, we have presented a characterization of the ratio of the extreme polynomials in terms of an APR transfer function which provided us with a controller synthesis for a family of uncertain systems. The method does not place any requirements on the compensator's order which is still parameterized by a Unit. This parameterization will provide us with a design freedom that may be used to satisfy other performance objectives.

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