Finite-Time Control of Uncertain Linear Systems Using Statistical Learning Methods

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Abstract

In this paper we show how some difficult linear algebra problems can be "approximately" solved using statistical learning methods. We illustrate our results by considering the state and output feedback, finite-time robust stabilization problems for linear systems subject to time-varying norm-bounded uncertainties and to unknown disturbances. In the state feedback case, we have obtained in an earlier paper, a sufficient condition for finite-time stabilization in the presence of time-varying disturbances; such condition requires the solution of a Linear Matrix Inequality (LMI) feasibility problem, which is by now a standard application of linear algebraic methods. In the output feedback case, however, we end up with a Bilinear Matrix Inequality (BMI) problem which we attack by resorting to a statistical approach.

Keywords: Finite-Time Stability, LMIs, Disturbance Rejection, Statistical Learning Control.

1 Introduction

The interplay between linear algebra and linear control theory has been long and fruitful [4]. Until very recently, it was actually felt that most linear control problems can be solved using linear algebraic concepts, an opinion further reinforced with the introduction of linear matrix inequality (LMI) methods into control engineering [12]. However, it is now known that some apparently basic linear control questions do not admit simple solutions or any at all [5, 6, 13]. Such are the examples of fixed-order controller design, multiobjective robust control designs, and others. While such problems remain in a linear algebraic framework, more advanced and specialized techniques are needed in order to "partially" solve them (see for example [14, 15]); all of these methodologies however have resulted in specialized results which are not practically useful.

This paper advances a different notion of solving these linear control problems. As proposed by Vidyasagar [21], Koltchinskii et al. [13], and various other authors [3, 7, 16], solving some fixed-order control design

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problems for linear systems may greatly benefits from the usage of stochastic algorithms. The notion of a solution however is somehow modified from the traditional control theoretic question of presenting necessary and sufficient conditions, and an algorithm which is guaranteed to converge to the exact solution. Instead, an approximate solution is sought and all guarantees are probabilistic. This paper proceeds to apply these techniques to the problem of finite-time stabilization for uncertain linear systems; in the case of a state feedback controller, the problem is converted to an LMI problem whose solution is easily obtained, while the case of output feedback turns out to be equivalent to a bilinear matrix inequality (BMI) problem whose solution is obtained using statistical learning methods.

In order to give a general framework to our discussion, we should note that many of the engineering design problems we face are instances of decision theory [11]. Results from decision theory thus play an important role in various engineering problems, including control analysis and design. Such decision problems depend on a set of decision variables reflecting engineering choices (such as the choice of controller gains), and a set of constraints reflecting engineering specifications (such as the desired closed-loop behavior). These decision problems can be translated to optimization problems of the form:

$$\min_{Y\in\mathcal{Y}} \; f_0(Y) \text{ subject to } f_i(Y) \leq 0, \; i = 1, \cdots, m \tag{1}$$

where $f_0, \cdots, f_m$ are given scalar-valued functions of the decision vector $Y \in \mathcal{Y}$ and the set $\mathcal{Y}$ may be infinite-dimensional. In the specific case of control problems, y may denote the vector of controller gains and our setup can accommodate a multi-objective control design problem. In a more advanced setting, we may have uncertain parameters in the open-loop system leading to the robust version of the decision problem

$$\min_{Y\in\mathcal{Y}} \max_{X\in\mathcal{X}} \; f_0(X, Y) \text{ subject to } f_i(X, Y) \leq 0, \; i = 1, \cdots, m \tag{2}$$

Unfortunately, these optimization problems are in general very difficult to solve. There are a few cases (exemplified by the LMI framework in control) where the optimization problem is completely solvable as illustrated in this paper. The large majority of the optimization problems however (and by reduction, decision problems) remain hard to solve unless we resort to statistical learning methods and stochastic algorithms as will also be illustrated later in this paper.

This paper is organized as follows: Section 2 presents some linear algebra and control problems which may be solved using LMIs and some which may not. The section also contains an overview of the idea of empirical risk minimization. Section 3 presents the finite-time control problem in its various versions and solutions. Section 4 concentrates on the output feedback case of the finite-time problem which can not be solved using LMIs and instead has to be attacked using statistical methods. Our conclusions are presented in Section 5.

2 Some Linear Algebra and Control Problems

As mentioned earlier, the interplay between linear algebra and control and systems theory is deep and far reaching. Bernstein has listed in [4] various open problems lying at the boundary of linear algebra and control including the problems of robust stability, output stabilizability and pole assignment, and nonstandard matrix Riccati equations. There are however other problems which have been completely solved using LMIs.

Let us then first consider the basic LMI idea of formulating a design problem as an optimization problem with linear objective and a linear matrix inequality as follows:

$$\min_x c^T x \quad \text{subject to} \quad F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \geq 0 \tag{3}$$

where $F_i = F_i^T \in \mathrm{I\!R}^{\mathrm{N}\times\mathrm{N}}; i = 1, \cdots, m$. The notation $F(x) \geq 0$ means that $F(x)$ is symmetric and positive semidefinite, and the above program is called a semidefinite program (SDP).

On the other hand, Bilinear Matrix Inequalities (BMI) problems have the form

$$\text{Find } x \quad \text{subject to} \quad F_0 + \sum_{i=1}^{m} x_i F_i + \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j F_{ij} \geq 0 \tag{4}$$

where $F_i = F_i^T \in \mathrm{I\!R}^{\mathrm{N}\times\mathrm{N}}$, $F_{ij} = F_{ij}^T \in \mathrm{I\!R}^{\mathrm{N}\times\mathrm{N}}; i, j = 1, \cdots, m$.

While LMI problems admit efficient numerical solutions, BMI problems are notoriously hard to solve [17]. Unfortunately, many practical linear control problems (fixed-structure control design, multiobjective design) turn out to be equivalent to BMI problems. In the following, we present some linear algebra problems which are reducible to LMIs or BMIs, before presenting the finite-time control problem.

2.1 Structured Linear Algebra

The following discussion is taken from [11]. A structured linear equation is given by

$$A(\delta)x = b(\delta) \tag{5}$$

where $A(\delta) \in \mathrm{I\!R}^{\mathrm{m}\times\mathrm{n}}$ and $b(\delta) \in \mathrm{I\!R}^{\mathrm{m}}$ are affine functions of an unknown but bounded parameter vector $\delta \in \mathrm{I\!R}^{\mathrm{l}}$, $||\delta||_\infty \leq 1$, and x is an unknown variable.

Equation (5) can give rise to a number of different problems, as shown below. Some of these problems can be converted into LMIs problems, while some other cannot.

2.1.1 Robust Least Square

This is the problem of finding a vector x that minimizes

$$r_s(A, b, x) = \max_{||\delta||_\infty \leq 1} ||A(\delta)x - b(\delta)||_2 \tag{6}$$

This problem turns up in many identification and control situations and may be recast as a robust SDP [11]. Unfortunately, robust SDP remains NP-hard and thus requiring stochastic methods. This will be made clearer next.

2.1.2 Structured total least-squares

The structured total least-squares (STLS) problem arises in various inverse problem cases [8, 11] and is stated as follows:

STLS Problem: The structured total least squares (STLS) problem is

$$\min_{\delta \in \mathrm{I\!R}^{\mathrm{l}}} \{||\delta||_2 \,:\, \exists x \in \mathrm{I\!R}^{\mathrm{n}},\, A(\delta)x = b(\delta)\} \tag{7}$$

This problem can be solved approximately by solving the following robust optimization problem:
Robust Optimization Problem: A scalar ρ is said to be an upper bound on the objective in (7) if there exists $x \in \mathbb{R}^n$ such that

$$A(\delta)x = b(\delta) \text{ for some } \delta, \quad ||\delta||_2 \leq \rho\,. \tag{8}$$

Assume that we can compute an ellipsoid of center x_0 and shape matrix $P > 0$ denoted by $\mathcal{E}_p = \{x \,:\, (x - x_0)^T P(x - x_0) \leq 1\}$; then the quantity

$$\rho^* = \min\{\rho \,:\, \exists\,(\delta, x),\ \|\delta\|_2 \leq \rho,\ x \in \mathcal{E}_p,\ A(\delta)x = b(\delta)\} \tag{9}$$

is an upper bound on the STLS problem, whenever $\rho^* > 0$. This robust optimization problem turns out to be a Generalized EigenValue problem (GEVP) with variables ρ, x_0, and P, which is a specific case of an LMI problem.

The STLS Problem shows that, some problems may only be upper-bounded, in order for LMI techniques to apply. This may be motivated as follows: while the exact problem may be too difficult or impossible to solve, an easier, albeit more restrictive problem's solution will provide sufficient conditions for the solution of the original problem. On the other hand, this approach may introduce a large degree of conservatism. A different track is to "approximately" solve the original problem as detailed next.

2.2 Background on Empirical Risk Minimization

Let us start our brief discussion with a general *function learning problem.* Let $(S, \mathcal{A})$ and $(T, \mathcal{B})$ be measurable spaces and let (X, Y) be a random couple with values in $S \times T$, X being an observable instance and Y being an unobservable "label" to be predicted based on the observation of X. The goal of learning is to find a measurable function $g : S \mapsto T$ that "approximates well" the relationship between X and Y. To be more precise, consider a loss function $\ell : T \times T \mapsto \mathbb{R}$ and assume that we are looking for a function g that makes the risk $R(g) := \mathbb{E}\ell(g(X), Y)$ reasonably small (close to minimum) where $\mathbb{E}(\cdot)$ denotes the expected value.

Since the joint distribution of (X, Y) is usually unknown, a standard approach is to try to minimize *the empirical risk* $R_n(g) := n^{-1}\sum_{i=1}^{n} \ell(g(X_i), Y_i)$, where $(X_i, Y_i),\ i = 1, \ldots, n$ is a given sample of n independent "training examples" (independent identically distributed (i.i.d.) copies of (X, Y)). This approach is called *the method of empirical risk minimization* and its various versions are used in a number of particular learning algorithms. The crucial mathematical question related to this method is to find, for a given approximate solution $\hat{g}$ of the empirical risk minimization problem, sharp probabilistic bounds on the risk $R(\hat{g})$. The first comprehensive theory addressing this problem was developed by Vapnik and Chervonenkis during the 1970s and 1980s. This theory had a substantial impact on the development of the general theory of empirical processes that started with Dudley's work in 1978 (see [10]) and has resulted in a growing number of significant applications to many problems in statistics (see [18]).

As discussed so far, the learning problem is always reducible to the minimization of empirical risk. In a more abstract and simpler setting (suppressing the labels), we consider an i.i.d. sample $(X_1, \ldots, X_n)$ in a measurable space $(S, \mathcal{A})$ with common unknown distribution P. Given a class $\mathcal{F}$ of measurable functions on $(S, \mathcal{A})$, the goal is to find a function $f \in \mathcal{F}$ with a small value of Pf. Since P is unknown, it is replaced by the empirical distribution P_n of the sample $(X_1, \ldots, X_n)$ and the problem of minimization of Pf on $\mathcal{F}$

is replaced by its empirical version: to minimize $P_n f$ on $\mathcal{F}$. In the setting of the function learning problem described above, the class $\mathcal{F}$ then becomes $\{\ell(g(\cdot),\cdot) : g \in \mathcal{G}\}$.

The structured linear algebra problems may be "approximately" solved using the empirical risk minimization framework. In fact, the robust least square problem in (6), and more generally the robust SDP problem may be recast in the same framework. In order to see this, consider the robust least square problem and let $g(x(\delta)) = A(\delta)x$, $y(\delta) = b(\delta)$, then let $l(g(x,\delta), y(\delta)) = ||A(\delta)x - b(\delta)||_2$. In the empirical risk minimization framework, we substitute the average risk for the maximum risk, leading to the substitution of $R_n(g_x) = n^{-1}\sum_{i=1}^{n} l(g(x(\delta_i)), y(\delta_i))$ for r_s in (6). Note that the problem attacked is actually an inverse problem whereby a solution x is sought to minimize the empirical risk R_n.

The general control problem may also be easily stated as a risk minimization problem. Namely, given a system $G(X)$ where $X \in S$ denotes the parameters of the system, and a controller structure $C(Y)$ where $Y \in \mathcal{Y}$ denotes the design parameters of the feedback controller, let the desired closed-loop objectives be specified by functions $g_i(X,Y)$ for $i = 1,\cdots,m$. Denote $g_{i,Y}(X) := g_i(X,Y)$. If now X denotes randomly selected (uncertain) parameters of the system with probability distribution P, the design problem can be formulated as the problem of minimizing $\mathbb{E}f_Y(\cdot) = Pf_Y$ with respect to $Y \in \mathcal{Y}$, where

$$f_Y(x) := \ell(g_{1,Y}(\cdot),\ldots,g_{m,Y}(\cdot))$$

is a properly chosen cost function.

In the following, we apply these ideas to the specific problem of finite-time robust control of uncertain, time-varying linear systems with disturbances. We start our development by discussing those cases which can be solved LMIs and proceed to those which can not.

3 Finite-Time Control Problem

Consider the following linear system subject to time-varying uncertainties and to exogenous disturbances

$$\dot{x}(t) = [A + \Delta A(t)]\,x(t) + [B + \Delta B(t)]\,u(t) + Gw(t) \quad (10a)$$

$$y(t) = [C + \Delta C(t)]\,x(t) + [D + \Delta D(t)]\,u(t) \quad (10b)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, and $G \in \mathbb{R}^{n\times q}$.

We assume the following.

A1) The uncertain part in (10) is in the so-called *structured, one block form*

$$\begin{bmatrix} \Delta A(t) & \Delta B(t) \\ \Delta C(t) & \Delta D(t) \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \Delta(t) \begin{bmatrix} E_1 & E_2 \end{bmatrix}$$

where $F_1 \in \mathbb{R}^{n\times r}$, $F_2 \in \mathbb{R}^{p\times r}$, $E_1 \in \mathbb{R}^{s\times n}$ and $E_2 \in \mathbb{R}^{s\times m}$ and the unknown, real matrix-valued function Δ belongs to the class

$$\mathcal{D} := \{\Delta : [0,+\infty) \mapsto \mathbb{R}^{r\times s} |\ \Delta \text{ is Lebesgue measurable, } \Delta(t)^T\Delta(t) \le I\}.$$

A2) The exogenous disturbance w belongs to the class

$$\mathcal{W} := \{w(t) |\ \dot{w}(t) = A_w w(t)\,, \quad w^T(0)w(0) \le d\}, \quad (11)$$

where $A_w \in \mathbb{R}^{q\times q}$ and $d > 0$.

Concerning system (10), we consider the following static output feedback controller

$$u = Ky \tag{12}$$

where $K \in \mathbb{R}^{m \times p}$.

One aim of this paper is to find sufficient conditions which guarantee that the closed loop system given by the interconnection of (10) with (12) is *bounded over a finite-time interval.* The general idea of *finite-time stability* concerns the boundedness of the state of a system over a finite time interval for given initial conditions; this concept can be formalized through the following definition, which is an extension of the one given in [9].

Definition 1 (Finite-Time Stability). The time-varying linear system

$$\dot{x}(t) = A(t)x(t) \qquad t \in [0, T]$$

is said to be Finite Time Stable (FTS) with respect to (c_1, c_2, T, R), with $c_2 > c_1$ and $R > 0$ if

$$x^T(0)Rx(0) \leq c_1 \Rightarrow x^T(t)Rx(t) < c_2 \qquad \forall t \in [0, T]\,.$$

◇

Remark 1.
It is worth noting that Lyapunov Asymptotic Stability and FT Stability are *independent* concepts: a system which is FT stable may not be Lyapunov asymptotically stable, whereas a Lyapunov asymptotically stable system may not be FT stable. ◇

The idea of state *boundedness*, on the other hand, is more general, and concerns the behavior of the state in the presence both of given initial conditions and of external disturbances.

Definition 2 (Finite-Time Boundedness). Let $\mathcal{W}$ be the class of disturbance signals (11). The time-varying linear system

$$\dot{x}(t) = A(t)x(t) + G(t)w(t) \qquad t \in [0, T]$$

subject to an exogenous disturbance $w \in \mathcal{W}$, is said to be Finite-Time Bounded (FTB) with respect to (c_1, c_2, d, T, R), with $c_2 > c_1$, $d > 0$, $R > 0$ if

$$x^T(0)Rx(0) \leq c_1 \Rightarrow x^T(t)Rx(t) < c_2 \qquad \forall t \in [0, T], \forall w \in \mathcal{W}$$

◇

Remark 2 (Finite-Time Boundedness and Finite-Time Stability).
Given our Definition 2 of Finite-Time Boundedness, Finite-Time Stability can be recovered as a particular case by letting $d = 0$. ◇

On the basis of the above considerations the aim of this paper is the solution of the following *finite-time-boundedness* problems.

OP1. Given system (10) and (c_1, c_2, d, T, R), find a static output feedback controller in the form (12) such that the closed-loop system given by the interconnection of (10) with (12) is FTB with respect to (c_1, c_2, d, T, R) for all $\Delta \in \mathcal{D}$. $\diamond$

OP2. Given system (10) and the quadruple (c_1, c_2, d, R), find a static output feedback controller in the form (12) which maximizes the positive number T and renders the closed-loop system given by the connection of (10) with (12) FTB with respect to (c_1, c_2, d, T, R) for all $\Delta \in \mathcal{D}$. $\diamond$

OP3. Given system (10) and the quadruple (c_1, d, T, R), find a static output feedback controller in the form (12) which minimizes the positive number c_2 (with $c_2 > c_1$) and renders the closed-loop system given by the connection of (10) with (12) FTB with respect to (c_1, c_2, d, T, R) for all $\Delta \in \mathcal{D}$. $\diamond$

We will mainly focus our attention on Problem OP1, because the solution of Problem OP2 (OP3) can be obtained via a trivial binary search algorithm over T (c_2) based on the solution of OP1.

Concerning the disturbance free system

$$\dot{x}(t) = [A + \Delta A(t)]\, x(t) + [B + \Delta B(t)]\, u(t) \tag{13a}$$

$$y(t) = [C + \Delta C(t)]\, x(t) + [D + \Delta D(t)]\, u(t) \tag{13b}$$

we shall also consider problems similar to the problems stated above (OP1,OP2, OP3), regarding the FT *Stabilization* of system (13) with respect to (c_1, c_2, T, R). These problems will be denoted in the same way as the FT Boundedness problems, adding a prefix FTS to distinguish them, i.e. FTS-OP1, FTS-OP2, FTS-OP3.

The state-feedback case was solved using LMIs in [2], so we limit our development in the remaining of this paper to problem OP1.

4 Output Feedback via Statistical Learning

In [1] we showed that even in the simplified case when $y = Cx$, the sufficient condition for FT Boundedness leads to BMIs, which can solved only to get *local* optimal solutions. In this section, we recast our problem in a different framework. We propose to use a randomized algorithm which is described in detail and proven in [13], in order to design an output feedback controller. The designed controller does not solve our original problem in *all* cases (i.e. for all the uncertainties Δ) but in *most of the cases*. This assertion will be made rigorous in the sequel.

We shall consider the system in the form (10) with the only restriction that the uncertain term Δ is *constant*. Therefore the uncertain term Δ belongs to the set

$$\boldsymbol{\Delta} := \{\Delta \in \mathbb{R}^{\mathrm{r}\times\mathrm{s}} : \Delta^{\mathrm{T}}\Delta \le 1\}$$

These uncertainties are chosen to have uniform distribution. For the generation of these samples we used the algorithm described in [7].

We shall denote by $X \in \mathcal{X} \subseteq \boldsymbol{\Delta} \times [0, c_1] \times [0, d]$ the generic uncertain parameters, and by $Y \in \mathcal{Y} \subseteq \mathbb{R}^{\mathrm{m}}$ the vector of controller coefficients. The samples X_i and Y_j are chosen to have uniform distributions.

Let us reconsider the system of Example **??**. Now we are looking for an output feedback controller; therefore our system is in the form (10) with

$$A = \begin{pmatrix} 0 & 20 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 1 \end{pmatrix} \qquad D = 0.2$$

$$F_1 = \begin{pmatrix} -1 & 0 \\ -1 & -2 \end{pmatrix} \qquad F_2 = \begin{pmatrix} 20 & 20 \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad E_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 3 \\ \frac{3}{2} \end{pmatrix} \qquad A_w = -1$$

Moreover let $c_1 = 1$, $c_2 = 10$, $d = 1$, $T = 0.3$, and $R = I$. For this system we want to solve our original problem OP1 in a probabilistic sense. The controller K in this case is scalar; it has been chosen to have uniform distribution in the interval $[-100, -0.01]$, because a negative K is needed to stabilize. In order to use the randomized algorithm methodology, this problem has been reformulated in the following way (see also [13], [21]). Let us define a cost function

$$\Psi(Y) = \max\{\psi_1(Y), \psi_2(Y)\} \tag{14}$$

where

$$\psi_1(Y) = \begin{cases} 0 & \text{if the closed-loop system with the nominal plant is FTB} \\ 1 & \text{otherwise} \end{cases}$$

and

$$\psi_2(Y) = E\left(\zeta(X, Y)\right), \tag{15}$$

where E indicates the *expected value* with respect to X, and

$$\zeta(X, Y) = \begin{cases} 1 & \text{if the closed-loop system with the randomly generated plant is not FTB} \\ 0 & \text{otherwise} \end{cases}$$

Our aim is to minimize the cost function (14) over $\mathcal{Y}$. The optimal controller is then characterized by the vector of parameters Y^* for which

$$\Psi^* := \Psi(Y^*) = \inf_{Y \in \mathcal{Y}} \Psi(Y) \tag{16}$$

Finding the vector Y^* which minimizes (16) would imply the evaluation of the expected value in (15) and then the minimization of (14) over the set $\mathcal{Y}$. What we shall find is a *suboptimal* solution, a probably approximate near minimum of $\Psi(Y)$ with confidence $1 - \delta$, level α and accuracy ϵ (see [19]).

Definition 3. Suppose $\Psi : \mathcal{Y} \to \mathbb{R}$, that P is a given probability measure on $\mathcal{Y}$, and that $\alpha \in (0, 1)$, $\delta \in (0, 1)$ and $\epsilon > 0$ are given. A number Ψ_0 is a probably approximate near minimum of $\Psi(Y)$ with confidence $1 - \delta$, level α and accuracy ϵ, if

$$\text{Prob}\left\{ \inf_{Y \in \mathcal{Y}} \Psi(Y) - \epsilon \leq \Psi_0 \leq \inf_{Y \in \mathcal{Y} \setminus \mathcal{S}} \Psi(Y) + \epsilon \right\} \geq 1 - \delta \tag{17}$$

with some measurable set $\mathcal{S} \subseteq \mathcal{Y}$ such that $P(S) \leq \alpha$. In (17), $\mathcal{Y} \setminus \mathcal{S}$ indicates the complement of the set $\mathcal{S}$ in $\mathcal{Y}$. $\diamond$

An interpretation of the definition is that we are not searching for the minimum over all of the set $\mathcal{Y}$ but only over its subset $\mathcal{Y} \setminus \mathcal{S}$, where $\mathcal{S}$ has a small measure (at most α). Unless the actual infimum Ψ^* is attained in the exceptional set $\mathcal{S}$, Ψ_0 is within ϵ from the actual infimum with confidence $1-\delta$. Although using Monte Carlo type minimization, it is unlikely to obtain a better estimate of Ψ^* than Ψ_0 (since the chances of getting into the set $\mathcal{S}$ are small), nothing can be said in practice about the size of the difference $\Psi_0 - \Psi^*$.

Based on the randomized algorithms discussed in [13], a probably approximate near minimum of $\Psi(Y)$ with confidence $1-\delta$, level α and accuracy ϵ, can be found with the following Procedure, which was derived in [13].

Procedure

1. Let $k = 0$

2. Choose n controllers with random uniformly distributed coefficients $Y_1, \ldots, Y_n \in \mathcal{Y}$, where (we indicate by $\lfloor \cdot \rfloor$ the floor operator)

$$n = \left\lfloor \frac{\log(2/\delta)}{\log[1/(1-\alpha)]} \right\rfloor$$

   Evaluate for these controllers the function ψ_1 (15) and discard those controllers for which $\psi_1 = 1$. Let $\hat{n}$ be the number of the remaining controllers.

3. Choose m plants and initial conditions on $x(0)$ and $w(0)$, generating random uncertainties $X_1, \ldots, X_m \in \mathcal{X}$ with uniform distribution, where

$$m = 2^k \left\{ \left\lfloor \frac{100}{\epsilon^2} \log\left(\frac{8}{\delta}\right) \right\rfloor + 1 \right\}$$

4. Evaluate the stopping variable

$$\gamma = \max_{1 \le j \le \hat{n}} \left| \frac{1}{m} \sum_{i=1}^{m} r_i \zeta(X_i, Y_j) \right|$$

   where r_i are *Rademacher* random variables, i.e. independent identically distributed random variables taking values $+1$ and -1 with probability $1/2$ each. If $\gamma > \epsilon/5$, let $k = k+1$ and go back to step 3

5. Choose the controller which minimizes the function

$$\frac{1}{m} \sum_{i=1}^{m} \zeta(X_i, \cdot)$$

   This is the *suboptimal* controller in the sense defined above.

Remark 3.
The proposed algorithm consists of two distinct parts: the estimate of the expected value in (15), which is given with an accuracy ϵ and a confidence $1-\delta/2$, and the minimization procedure which is carried out with a confidence $1-\delta/2$ and introduces the level α. As it can be seen from the Procedure, the number m of samples in $\mathcal{X}$ which are needed to achieve the estimate of the expected value (15), known as the *sample complexity*, is not known *a priori* but is itself a random variable. The upper bounds for this random sample complexity however, are of the same order of those that can be found in [20].

In our case, the procedure needed just one iteration to converge, i.e. $k = 1$. Therefore, for $\delta = 0.05$, $\alpha = 0.005$ and $\epsilon = 0.1$, n evaluated to 736 controllers and m evaluated to $50,753$ plants. The *suboptimal* controller is $K = -8.7025$, and the corresponding value of the cost function is $\Psi_0 = 0.146$.

5 Conclusions

We have shown in this paper that various control and linear algebra problem may be "approximately" solved using statistical learning methods. Our argument is that while LMIs and other standard methods can solve many control problems, there remain a host of important problems whose solutions have defied the standard techniques. Statistical learning methods and stochastic algorithms provide one possible outlet to such an impasse and are becoming much more practical design tools. We illustrated this by considering the state and output feedback finite-time robust control problem: while in the state feedback case we end up with an LMI feasibility problem, in the output feedback case we need to resort to a statistical approach in order to get a solution. We have also applied our techniques to design various fixed-structure controllers for linear and nonlinear systems, and to various communications problems. We are currently investigating various optimization techniques to reduce the conservativeness in the number m of plants (more generally the number of X_i) and ways to adjust the controller structure and order adaptively.

References

[1] F. Amato, M. Ariola, C. T. Abdallah, and P. Dorato. Finite-Time Stabilization of Uncertain Linear Systems in the Presence of Disturbances. In *Proc. American Control Conference*, pages 1176–1780, San Diego, CA, 1999.

[2] F. Amato, M. Ariola, and P. Dorato. Robust Finite-Time Stabilization of Linear Systems Depending on Parameter Uncertainties. In *Proc. Conference on Dec. and Control*, pages 4100–4102, Tampa, FL, 1998.

[3] B. Barmish and C. Lagoa. The uniform distribution: A rigorous justification for its use in robustness analysis. *Mathematics of Control, Signals, and Systems*, 10:203–222, 1997.

[4] D.S. Bernstein. Some Open Problems in Matrix Theory Arising in Linear Systems and Control. *Lin. Alg. Appl.*, pages 409–432, 1992.

[5] V. Blondel and J. Tsitsiklis. NP-hardness of some linear control design problems. *SIAM Journal of Control and Optimization*, 35:2118–2127, 1997.

[6] V. Blondel and J. Tsitsiklis. A survey of computational complexity results in systems and control. *Automatica*, 36:1249–1274, 2000.

[7] G. Calafiore, F. Dabbene, and R. Tempo. Randomized Algorithms for Probabilistic Robustness with Real and Complex Structured Uncertainty. *IEEE Trans. Auto. Contr.* To appear.

[8] B. de Moor. Total least Square for Affinely Structured Matrices and the Noisy Realization Problem. *IEEE Trans. Acoust., Speech, Signal Processing*, 42:3104–3113, 1994.

[9] P. Dorato. Short time stability in linear time-varying systems. In *Proc. IRE International Convention Record Part 4*, pages 83–87, 1961.

[10] R. Dudley. Central limit theorems for empirical measures. *Annals of Probability*, 6:899–929, 1978.

[11] L. El-Ghaoui and S.-I. Niculescu. Robust decision problems in engineering: A linear matrix inequality approach. In L. El-Ghaoui and S.-I. Niculescu, editors, *Advances in Linear Matrix Inequality Methods in Control*, pages 3–37. SIAM, Philadelphia, 2000.

[12] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali. *LMI Control Toolbox*. The Mathworks Inc, Natick, MA, 1995.

[13] V. Koltchinskii, C. T. Abdallah, M. Ariola, P. Dorato, and D. Panchenko. Improved Sample Complexity Estimates for Statistical Learning Control of Uncertain Systems. *IEEE Trans. Auto. Contr.*, 45(12):2383–2388, Dec. 2000.

[14] J. Leventides and N. Karcanias. Global asymptotic linearization of the pole placement map: A closed form solution for the constant output feedback problem. *Automatica*, 31:1303–1309, 1995.

[15] J. Rosenthal and X. Wang. Output feedback pole placement with dynamic compensators. *IEEE Trans. Auto. Control*, 41:830–843, 1996.

[16] R. Tempo, E. Bai, and F. Dabbene. Probabilistic robustness analysis: Explicit bounds for the minimum number of samples. *Systems and Control Letters*, 30:237–242, 1997.

[17] O. Toker and H. Özbay. On the NP-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback. In *Proc. IEEE American Control Conf.*, pages 2056–2064, Seattle, WA, 1995.

[18] A. van der Vaart and J. Wellner. *Weak Convergence and Empirical Processes*. Springer-Verlag, Berlin, 1996.

[19] M. Vidyasagar. Statistical Learning Theory and Its Applications to Randomized Algorithms for Robust Controller Synthesis. In *ECC97*, volume Plenary Lectures and Mini-Courses, pages 162–190, Brussels, Belgium, 1997.

[20] M. Vidyasagar. *A Theory of Learning and Generalization: With Applications to Neural Networks and Control Systems*. Springer-Verlag, London, 1997.

[21] M. Vidyasagar. Statistical Learning Theory and Randomized Algorithms for Control. *IEEE Control Systems Magazine*, 18(6):69–85, Dec. 1998.