Robust Nonlinear Feedback Design via Quantifier Elimination Theory

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Abstract

In this paper symbolic-computation methods are used to design simple, fixed-structure, robust controllers for nonlinear systems. Design specifications are reduced to logically quantified polynomial inequalities. The quantifier-elimination (QE) software package QEPCAD is used to eliminate quantifiers on state and plant parameters, to obtain regions of admissible controller parameters, and to guarantee robust stability and performance.

Key Words-Nonlinear control; robust stability; robust performance; Lyapunov function.

I. Introduction

It is known that many linear robust multiobjective feedback design problems with fixed controller structures can be reduced to the study of multivariate polynomial inequalities (MPIs). See, for example, [1], [2]. A function of many variables is said to be a *multivariate polynomial* function if it is a polynomial in any given variable when all the other variables are held fixed. MPIs are then collections of multivariate polynomial inequalities.

In this brief paper we show that some robust nonlinear problems can also be reduced to MPI problems, with logic quantifiers of the type "for all", $\forall$ and "there exists", $\exists$, on various variables, and we explore the use of *quantifier elimination* (QE) theory and software to solve problems of modest complexity. In particular we will use the Hamilton-Jacobi-Bellman (HJB) inequality to design fixed-structure controllers which are robustly stable and satisfy given guaranteed-cost bounds. The key assumption we need is that the plant nonlinearities, feedback laws, and guaranteed-cost bounds are all given multivariate polynomial functions in the variables of interest. Quantifier elimination theory is then used to define a region of controller parameter space where the quantified MPIs in question are satisfied.

The robust nonlinear feedback design problem we will consider is defined in section II, followed by a discussion of quantifier elimination theory and software in section III. Some numerical examples are given section IV and our conclusions are given in section V.

II. The Robust Nonlinear Feedback Design Problem

Consider a nonlinear plant with dynamics

$$\dot{x} = f(x, u, p), \quad x \in \Omega, \ \ p \in \mathcal{P} \tag{1}$$

where x is the state of the plant within a specified set Ω, p is an uncertain parameter vector within the specified range $\mathcal{P}$ and u is a control input. The components of the vector $f(x, u, p)$ are assumed to be multivariate polynomials with respect to the entries of the vectors x, u and

p. This does restrict the class of nonlinear systems that can be considered, but more general systems can often be approximated by polynomial functions via Taylor series expansions. The control input is assumed to be a fixed-structure control law of the form

$$u = \psi(x, q)\,,$$

where $\psi(x, q)$ is polynomial in the components of the vectors x and q , and q is a vector of design parameters.

We assume a performance measure of the form

$$\tilde{V}(x(0)) = \int_0^\infty \ell(x, u)\, dt\,,$$

where $\ell(x, u)$ is a non-negative multivariable polynomial function in the components of x ad u. Let $V(x)$ be a given multivariate polynomial function which is positive for all nonzero x. It is known, see for example reference [3], that if the HJB inequality

$$F(x, p, q) = \ell(x, \psi(x, q)) + (\frac{dV}{dx})^T f(x, \psi(x, q), p) < 0\,, \tag{2}$$

is satisfied *for all* $x \neq 0$, and *for all* admissible p, then $V(x(0))$ is a guaranteed bound on the performance measure $\tilde{V}(x(0))$. Since the function $F(x, p, q)$ is a multivariate polynomial, given the assumptions on V, f, and ψ, QE methods can then be used to explore admissible values of the design vector q. Robust stability may also be explored by letting $\ell(x, u) = 0$ in (2) and using $V(x)$ as a Lyapunov function.

III. QE Computation and Software

Given the set of polynomials with integer coefficients $P_i(X, Y)$, $\quad 1 \leq i \leq s$ where X represents a k-dimensional vector of quantified real variables and Y represents a l dimensional vector of unquantified real variables, let $X^{[i]}$ be a block of k_i quantified variables, Q_i be one of the quantifiers $\exists$ (there exists) or $\forall$ (for all), and let $\Phi(Y)$ be the quantified formula

$$\Phi(Y) = \left(Q_1 X^{[1]}, ..., Q_w X^{[w]}\right)\; F\left(P_1, \ldots, P_s\right)\,, \tag{3}$$

where $F(P_1, ..., P_s)$ is a quantifier free *Boolean* formula, that is a formula containing the Boolean operators $\wedge$ (and) and $\vee$ (or), operating on *atomic predicates* of the form:
$P_i(Y, X^{[1]}, ..., X^{[w]}) \geq 0$ or $P_i(Y, X^{[1]}, ..., X^{[w]}) > 0$ or $P_i(Y, X^{[1]}, ..., X^{[w]}) = 0$. We can now

state the general quantifier elimination problem

General Quantifier Elimination Problem: Find a quantifier-free Boolean formula $\Psi(Y)$ such that $\Phi(Y)$ is true if and only if $\Psi(Y)$ is true.

In linear control problems, the unquantified variables are generally the compensator parameters, represented by the parameter vector $Y = q$, and the quantified variables are the plant parameters, represented by the plant parameter vector p, and the frequency variable ω. Uncertainty in plant parameters are characterized by quantified formulas of the type $\forall(p_i)\ \ [\underline{p_i} \leq p_i \leq \overline{p_i}]$ where $\underline{p_i}$ and $\overline{p_i}$ are rational numbers. *The quantifier-free formula* $\Psi(q)$ *then represents a characterization of the compensator design.*

An important special problem is the QE problem with no unquantified variables (free variables), i.e. $l = 0$. This problem is referred to as the *General Decision Problem.*

General Decision Problem: With no unquantified variables, i.e. $l = 0$, determine if the quantified formula given in (3) is true or false.

The general decision problem may be applied to the problem of *existence* of compensators that meet given specifications, in which case an "existence" quantifier is applied to the compensator parameter q.

Quantifier elimination (QE) theory in nonlinear systems allows one to eliminate logic quantifiers such as, $\forall$, and, $\exists$, from multivariable inequalities of the form

$$\left(Q_1 X^{[1]}, ..., Q_w X^{[w]}\right) \quad F(x, p, q) > 0, \quad x \in \Omega, \;\; p \in \mathcal{P}\,, \tag{4}$$

where F is a *multivariate polynomial function* (see (3)), and where x, p and q are vectors related to the nonlinear feedback systems, in particular x represents the state of the system being controlled (plant) which is within the set specified by Ω, p represents the uncertain plant parameters which are within the set specified by $\mathcal{P}$, and q represents the controller "design" parameters. When the quantifiers on the controller parameters are "existence" quantifiers, i.e.

$$(\exists q)\,(\forall p \in \mathcal{P})\,(\forall x \in \Omega)\ [F(x, p, q) > 0]\,,$$

the theory tells us if a solution to the robust nonlinear feedback problem exists. By eliminating only the "for all" quantifiers on the plant parameters and the system state, i.e.

$$(\forall p \in \mathcal{P})(\forall x \in \Omega)[F(x, p, q) > 0]\,,$$

a quantifier-free Boolean formula $\Psi(q)$ is obtained which defines the set of controller parameters which satisfy given specifications.

The application of QE theory to the design of linear feedback systems is discussed in [4], [2]. Algorithms for solving general QE problems were first given by [5] and [6], and are commonly called Tarski-Seidenberg decision procedures. Tarski showed that QE is solvable in a finite number of "algebraic" steps, but his algorithm and later modifications are doubly exponential in the size of the problem. Researchers in control theory have been aware of Tarski's results and their applicability to control problems since the 1970's [7], but the complexity of the computations and lack of software limited their applicability. In [8] a theoretically more efficient QE algorithm that uses a cylindrical algebraic decomposition (CAD) approach is introduced. However, this algorithm was not capable of effectively handling nontrivial problems. More recently Hong and Collins [9], [10], [11] introduced a significantly more efficient Partial CAD QE algorithm.

The Partial Cylindrical Algebraic Decomposition algorithm, has been developed, see [10], for the computer elimination of quantifiers on polynomial inequalities. This algorithm requires a *finite number* of "algebraic" operations. A key assumption for all computer elimination algorithms is that the coefficients in the polynomial functions be *integers*. This is not a serious limitation since one can generally approximate a real number by a rational number, and one can clear fractions in the polynomial functions to obtain integer coefficients. However the number of operations is still doubly exponential in the number of variables, so that only problems of modest complexity can actually be computed. See [12] for a discussion of computational complexity in the quantifier elimination problem. A software package called QEPCAD (Quantifier Elimination by Partial Cylindrical Algebraic Decomposition) has been developed for the solution of quantifier elimination problems.

An excellent introduction to quantifier elimination theory and its applications to control system design may be found in the monograph of [13].

In the examples that follow we use the software package QEPCAD to solve some simple robust nonlinear control problems. It should be noted that numerical techniques can also be used to "eliminate quantifiers". For example in [1] and [14], branch-and-bound/Bernstein (BBB) polynomial methods are used for this purpose. Indeed numerical techniques may be applicable to more complex problem than those that can be handled by QE algorithm. However numerical techniques generally require a priori bounds on design parameter range, and are also limited by problem size. A major advantage of QE algorithm is that they require no approximations or a priori parameter ranges. Recently, [15], randomized algorithms have been suggested for the solution of the class of polynomial problems considered here. Such algorithms are applicable to much more complex systems than can be treated with QE or BBB methods, but one must settle for "approximate probabilistic" results, rather than "exact deterministic" results.

IV. Examples

Example 1: This example is taken from [16]. The nonlinear system is given by

$$\begin{cases} \dot{x}_1 &= x_1x_2 + px_1^3 \\ \dot{x}_2 &= u \end{cases},$$

where $p \in [-0.25, 0.25]$. We are interested in designing a simple controller which guarantees robust local asymptotic stability of the origin. In the above reference, the nominal system ($p = 0$) is *globally* stabilized by the control law

$$u = -8x_1^3 - 8x_2 - 8x_1^2x_2\,,$$

as may be obtained from the control Lyapunov function

$$V(x) = \frac{x_1^2 + (x_2 + x_1^2)^2}{2}\,.$$

Here we use this same Lyapunov function, and a control law of the form

$$u = -8x_1^2 - qx_2 - 8x_1^2x_2\,,$$

with design parameter q, to design a locally robustly stabilizing feedback system. In this case

$$\dot{V}(x) = F(x_1, x_2, q,p)\,,$$

where

$$F(x_1, x_2, q, p) = (x_1 + 2x_1(x_2 + x_1^2))(x_1x_2 + px_1^3)(-8x_1^2 - qx_2 - 8x_1^2x_2)\,.$$

The condition $\dot{V}(x) < 0$, for all (x_1, x_2) in the region

$$\Omega_x = \{0 < |x_1| \leq 0.5, 0 < |x_2| \leq 0.5\} ,$$

and all $p \in [-0.25, 0.25]$ results in the quantifier formula

$$(\forall(x_1, x_2) \in \Omega_x)(\forall p \in [-0.25, 0.25]) \left[F(x_1, x_2, q, p) > 0\right] \; .$$

If QEPCAD is used to eliminate the quantifiers on x_1, x_2, and p, the following quantifier-free formula in q is produced

$$\begin{array}{l} \Psi(q) = [q^2 + 17q + 49 \leq 0 \wedge \\ 16q^3 + 1996q^2 + 40729q + 151424 \geq 0 \\ \vee[64q^2 + 880q + 1561 \leq 0 \wedge \\ 16q^3 + 1996q^2 + 40729q + 151424 \leq 0] \end{array} . \quad (5)$$

¿From the computation of roots of the various polynomials in (5), the following admissible range of design-parameter values is obtained

$$-11.6758 \leq q \leq -3.6782.$$

Example 2: This problem is taken from [17] where the stabilization of the following system is studied:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & u + p(x_1^3 - x_1) \end{array} \right. ,$$

where p is an uncertain parameter with $p \in [-1, 1]$. The problem is to design a state-feedback controller such that the closed-loop system is at least locally stable and the following performance index

$$\begin{array}{ll} \tilde{V}(x(0)) & = \int_0^\infty \ell(x, u) dt \\ & = \int_0^\infty (x_1^2 + x_2^2 + u^2) dt \, , \end{array}$$

is bounded (at least locally) by the function

$$V(x) = 10(x_1^2 + x_1 x_2 + x_2^2) \, .$$

We assume a simple linear control law of the form

$$u(x, q) = q x_1 - 3 x_2 \, ,$$

with design parameter q. In this case

$$\dot{V}(x) + \ell(x,u) = F(x, q_1, p)\,,$$

where

$$\begin{aligned} F(x,q,p) &= 10[(2x_1+x_2)x_2 + (2x_2+x_1)(qx_1 - 3x_2 + p(x_1 - x_1^3))] \\ &\quad +(1+q^2)x_1^2 + 10x_2^2 - 6qx_1x_2\,. \end{aligned}$$

If $\Omega_x = \{0 < |x_1| \le 3/2, 0 < |x_2| \le 3/2\}$, the following quantified formula guarantees local robust stability and performance

$$(\forall (x \in \Omega_x))\,(\forall p \in [-1,1])\,[F(x,q,p) < 0]\,.$$

The quantifier-free formula returned by QEPCAD is given by

$$\Psi(q) = [P_1(q_1) \le 0 \wedge P_2(q_1) \le 0] \vee [P_3(q_1) \le 0 \wedge P_2(q_1) \ge 0]\,, \tag{6}$$

where the polynomials P_1, P_2, P_3 have the following expressions

$$\begin{aligned} P_1(q_1) &= 356q_1^2 + 620q_1 - 615, \\ P_2(q_1) &= 2q_1^8 + 80q_1^7 + 1213q_1^6 + 9020q_1^5 + \\ &\quad 39832q_1^4 + 112400q_1^3 + 803q_1^2 - \\ &\quad 78840q_1 + 382482, \\ P_3(q_1) &= 356q_1^2 + 2020q_1 + 2385\,. \end{aligned} \tag{7}$$

By computing the roots of the equations involved in formula (7) we obtain the range of q_1 which satisfies (6) are

$$-2.4474 \le q_1 \le -1.6754$$

Note the complexity of the polynomials in the quantifier-free formula (6). It is obvious that quantifier-elimination "by hand" in this case would have been very difficult.

V. Conclusions

Many robust nonlinear feedback design problems can be reduced to quantifier-elimination problems. While software exists for the computer solution of quantifier elimination problems, the complexity of the problem is severely limited due to the inherent complexity of the basic QE problem. However with the assumption of simple, fixed structures for feedback, some modest problems can be solved as illustrated by the examples above.

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