# CONTINUOUS AND DISCRETE TIME SPR DESIGN USING FEEDBACK

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ABSTRACT

This paper presents necessary and sufficient conditions for the existence of a feedback compensator that will render a given continuous-time or discrete-time linear system SPR. When these conditions hold, the controller is explicitely found.

## 1 INTRODUCTION

The concepts of Positive-Real (PR) and Strictly-Positive-Real (SPR) functions and matrices have been very useful in network theory [1], adaptive control [2] and robust control [3]. These concepts have also been generalized to include discrete-time systems [4] and [5]. The importance of PR and SPR matrices is obvious when dealing with uncertain systems. In this situation, a nominal SPR transfer function allows for large passive uncertainties without the loss of stability [2] and [5]. The standard definition of SPR matrices [6], here termed strong SPR, is usually difficult to apply. Moreover, it was recently shown [7] that the strong SPR definition is overly restrictive for control theory applications. In this paper we will use the term SPR to denote weak SPR matrices as defined in [7] and [8] and reviewed in the next section. On the other hand, if a given transfer matrix is not SPR, the question of whether a feedback controller might make the closed-loop system SPR is of considerable interest. This problem was termed "Almost Strict Positive Real " and studied in [9]. What has been lacking, however, is a set of conditions that will answer the existence question: Given a transfer matrix $P(s)$, does a controller that will make it SPR exist?. Moreover, a construction of the controller (when it exists) is desirable. A necessary condition was found in [9] and a partial answer to the existence and construction questions was given in [10] for continuous-time systems. Sufficient existence conditions were also found in [11] for the single-input-single-output (SISO) continuous-time case and in [9] for the Multi-Input-Multi-Output case. In the present paper, we provide a simple proof of the results in [10] and [11], and extend these results to the discrete-time case and to a more general class of systems. This paper is organized as follows: In Section 2, we review the available SPR definitions for continuous and discrete-time transfer matrices. In Section 3, we define the problem and present our results on designing controllers to make a closed-loop system SPR. Our conclusions are presented in Section 4.

# 2 WHICH SPR?

In order to keep the exposition clear, we will treat the continuous-time case first, then present the discrete-time results.

## 2.1 Continuous-Time Case:

Consider the multi-input-multi-output linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x$ is an $n$ vector, $u$ is an $m$ vector, $y$ is a $p$ vector, $A, B, C$, and $D$ are of the appropriate dimensions. The corresponding transfer function matrix is

$$P(s) = C(sI - A)^{-1}B + D \tag{2}$$

We will first assume that the system has an equal number of inputs and outputs, i.e. $p = m$. Then define the relative degree $n^*$ as follows: $n^* = 0$ if $\det(D) \neq 0$, and $n^* = m$ if $\det(D) = 0$ but det $(CB) \neq 0$. A formalism for the poles and zeros of multivariable systems is given in [12] and may be used to justify the definition of $n^*$. To simplify our notation we will denote the Hermitian part of a real, rational transfer matrix $T(s)$ by $He[T(s)] = \frac{1}{2}[T(s) + T^T(s^*)]$ where $s^*$ is the complex conjugate of $s$. A number of definitions have been given for SPR functions and matrices [6] and [8]. It appears that the most useful definition for control applications is the following [7]

**Definition 1** *An mxm matrix $T(s)$ of proper real rational functions which is not identically zero is (weak) SPR if*

1. *All elements of $T(s)$ are analytic in the closed right half plane, i.e. in the region $Re(s) \geq 0$, and*

2. *The matrix $He[T(s)]$ is positive definite for $Re(s) \geq 0$.*

□

As a result of this definition, a necessary condition for a given transfer function to be SPR is that $n^* = -1,\ 0,\ 1$. The more standard definition of SPR matrices advocated in [6] is more restrictive than Definition 1. In fact, a long-held view was that strong SPR was needed to prove the Meyer-Kalman-Yakubovitch (MKY) lemma, which is, after all, the major application of SPR concepts in control systems. However, As shown in [7], the weak SPR definition is just as useful in this regard and will therefore be adopted in this paper. Note that, from minimum real-part arguments given in [1], condition 2) of Definition 1 is equivalent to $He[T(jw)] > 0$ for all $w$.

### 2.2 Discrete-Time Case:

Consider now the discrete-time multi-input-multi-output linear time-invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{3}$$

where $x(k)$ is an $n$ vector, $u(k)$ is an $m$ vector, $y(k)$ is a $p$ vector, $A, B, C$, and $D$ are of the appropriate dimensions. The corresponding transfer function matrix is then

$$P(z) = C(zI - A)^{-1}B + D \tag{4}$$

Similar to the continuous-time case, we assume that the system has an equal number of inputs and outputs i.e. $p = m$ and define the relative degree $n^*$ as follows: $n^* = 0$ if $D \neq 0$, and $n^* = m$ if $D = 0$ but $CB \neq 0$. Also, we denote the Hermitian part of $T(z)$ by $He[T(z)] = \frac{1}{2}[T(z) + T^T(z^*)]$ where $z^*$ is the complex conjugate of $z$ and $z = \rho e^{jw}$. The concept of discrete PR matrices is defined in [4]. The following definition is motivated by [4] and by the continuous-time counterpart.

**Definition 2** *An mxm matrix $T(z)$ of real rational functions is SPR if*

1. *All elements of $T(z)$ are analytic on and outside the unit circle, i.e. in the region $|z| \geq 1$, and*
2. *The matrix $He[T(e^{jw})]$ is positive definite Hermitian for all real $w$.*

□

Note first that a transfer function $T(z)$ is SPR only if the corresponding $T(s)$ with $s = (z-1)/(z+1)$ is SPR. In addition, a necessary condition for $T(z)$ to be SPR is that it relative degree $n^* = 0$.

## 3 SPR USING FEEDBACK

We will again separate our results into continuous-time and discrete-time results.

### 3.1 Continuous-Time Case:

The question addressed in this section is to find conditions on (1) or (2) so that a feedback controller will render the closed-loop system SPR. The result of Theorem 1 appeared in [10] for the case of a continuous-time plant and a static output feedback, i.e. $u = -\gamma K y + K r$. The closed-loop system is then given by

$$\begin{aligned} \dot{x} &= (A - \gamma BKC)x + BKr \\ y &= Cx \end{aligned} \tag{5}$$

or in the frequency-domain

$$Y(s) = [I + \gamma P(s)K]^{-1} P(s) K R(s) \tag{6}$$

We present a simple frequency domain proof to show the existence of $K$ and $\gamma$ that will render the closed-loop system SPR.

**Theorem 1** *Let system (1) be stabilizable and detectable and let its relative degree be $n^* = m$. Then there exists a nonsingular $K$ and a positive scalar $\gamma$ such that the closed-loop system (5) is SPR, if and only if $P(s)$ is minimum phase.*

**Proof:**
*Sufficiency:* Consider the closed-loop transfer function

$$T(s) = [I + \gamma P(s)K]^{-1} P(s) K$$

or

$$T(s) = [K^{-1}P^{-1}(s) + \gamma I]^{-1}$$

Since $P(s)$ is minimum phase with a relative degree $n^* = m$, its inverse $P^{-1}(s)$ will be given by

$$P^{-1}(s) = sL + P_1(s)$$

where $P_1(s)$ is proper and stable, and det $(L)\ != 0$. In fact, $\det(CB) \neq 0$ and $L = (CB)^{-1}$. On the other hand, since $P(s)$ is minimum phase, $P_1(s)$ cannot have any poles in Re $(s) \geq 0$. It is now obvious that $T(s)$ will be stable if and only if $W(s) = [K^{-1}P^{-1}(s) + \gamma I]$ has no zeros in $\mathrm{Re}(s) \geq 0$. Let $K$ be given by

$$K = (CB)^{-1}$$

then

$$\begin{aligned} W(s) &= sI + CBP_1(s) + \gamma I \\ He[W(jw)] &= He[CBP_1(jw)] + \gamma I \end{aligned}$$

Since $P_1(jw)$ has no poles on the $jw$ axis, $He[W(jw)]$ may be made positive-definite by a large enough positive scalar $\gamma$. This then implies that $W(s)$ is weak SPR. Since $T(s)$ is the inverse of $W(s)$, it is also weak SPR [6].

*Necessity:* Suppose now that a nonsingular $K$ and a $\gamma$ were found to make the closed-loop system $T(s)$ SPR and that $D = 0$. Then

$$W(s) = [K^{-1}P^{-1}(s) + \gamma I]$$

is also SPR. Writing $P^{-1}(s)$ as $sL + P_1(s)$, with $L = (CB)^{-1}$ we get

$$W(s) = K^{-1}(CB)^{-1} + K^{-1}P_1(s) + \gamma I$$

Since $W(s)$ is SPR, $P_1(s)$ must be stable, hence $P(s)$ must be minimum-phase.

□

This result indicates that with the given assumptions on $P(s)$, static output feedback can always be found to stabilize the closed-loop system $T(s)$. Moreover, $T(s)$ can also be made SPR to give the desired robustness against passive uncertainties. It can also be seen that a dynamic output feedback compensator will not relax the conditions of the theorem since output compensation can not move the open-loop zeros nor change the relative degree of the plant. The choice of $K = (CB)^{-1}$ in the proof of the theorem is not unique. In fact, it is sufficient to choose $K = Q(CB)^{-1}$ where $Q$ is any symmetric positive-definite matrix. Next, note that the condition $\det(CB) \neq 0$ (or that $P(s)$ has a relative degree $n^* = m$), also reveals that the system (1) has an inverse obtained by cascading one differentiator and a dynamical system [13]. Note that the inverse system given in the proof of Theorem 1 may be written in state-space as

$$\begin{aligned} \dot{x} &= [A - B(CB)^{-1}CA]x + B(CB)^{-1}\dot{y} \\ u &= -(CB)^{-1}CAx + (CB)^{-1}\dot{y} \end{aligned} \tag{7}$$

Now recall that the invertibility of the system (1) may still be inferred even though det $(CB) = 0$. In fact, a sufficient condition for the inverse to exist is that the first nonzero matrix in the sequence, $D, CB, CAB, CA^2B, ..., CA^{n-1}B$, be nonsingular [13]. It is then obvious that for a nonzero matrix $D$, the condition for $T(s)$ to be SPR is that $D$ be invertible and $P(s)$ be minimum phase, i.e. an exactly-proper, minimum-phase transfer function may be made SPR with a static output feedback if its high frequency gain is nonsingular. On the other hand, the following general result may be established.

**Theorem 2** *Suppose that (1) is both stabilizable and detectable, and det* $(CA^iB) \neq 0$ *where* $CA^iB$ *is the first nonzero matrix in the sequence*

$$D, CB, CAB, CA^2B, ..., CA^{n-1}B$$

*Then the closed-loop system from $r$ to $\frac{d^i y}{dt^i}$ given by*

$$T_i(s) \;=\; CA^i(sI - A + \gamma BKCA^i)^{-1}BK$$

*is SPR if and only if $P(s)$ is minimum phase.*

**Proof:** Given system (1), repeated here for convenience

$$\begin{aligned} \dot{x} &= Ax \;+\; Bu \\ y &= Cx \;+\; Du \end{aligned}$$

let us define an output $z_i$ by

$$z_i \;=\; \frac{d^i y}{dt^i} \;=\; y^{(i)}$$

then

$$z_{i+1} \;=\; CA^{i+1}x \;+\; CA^iBu$$

If det $(CA^iB) \neq 0$, the inverse system of $P(s)$ is given by

$$P^{-1}(s) \;=\; (CA^iB)^{-1}s^{i+1} \;+\; P_2(s)$$

where $P_2(s)$ is stable. Repeating the arguments of theorem 3.1 and using the controller

$$u \;=\; -\gamma K y^{(i)} \;+\; Kr$$

we obtain the desired result.

□

## 3.2 Discrete-Time Case:

Next, we turn our attention to the discrete-time case. Specifically, we find conditions on (3) or (4) so that a feedback controller will render the closed-loop system SPR. Consider then the static output feedback, i.e. $u(k) \;=\; -\gamma K y(k) + Kr(k)$. The closed-loop system is then given by

$$\begin{aligned} x(k+1) &= (A - \gamma BKC)x(k) \;+\; BKr(k) \\ y(k) &= Cx(k) \;+\; Dr(k) \end{aligned} \qquad (8)$$

or in the $z-$ domain

$$Y(z) \;=\; [I + \gamma P(z)K]^{-1}P(z)KR(z) \qquad (9)$$

The following theorem parallels theorem reft1 and shows the existence of $K$ and $\gamma$ that will render the closed-loop system (8) SPR if $D$ is invertible.

**Theorem 3** *Let system (3) be stabilizable and detectable and let its relative degree be $n^* = 0$. Then there exist a nonsingular $K$ and a positive scalar $\gamma$ such that the closed-loop system (8) is SPR, if and only if $P(z)$ is minimum phase.*

**Proof:**
*Sufficiency:* Consider the closed-loop transfer function

$$\begin{aligned} T(z) &= [I \ + \ \gamma P(z)K]^{-1}P(z)K \\ &= [K^{-1}P^{-1}(z) \ + \ \gamma I]^{-1} \ = \ W^{-1}(z) \end{aligned}$$

Since $P(z)$ is minimum phase with a relative degree $n^* = 0$, its inverse $P^{-1}(z)$ will be given by

$$P^{-1}(z) \ = \ P_1(z)$$

where $P_1(z)$ is proper and stable. Note that $W(z) \ = \ K^{-1}P^{-1}(z) \ + \ \gamma I$ can be made SPR by choosing $\gamma$ positive and large enough since

$$||P^{-1}(z)|| \ \le \ c \ \ \infty \ ; \ \ \text{for all } |z| \ \ge \ 1$$

Since the inverse of an SPR matrix is SPR, $T(z)$ is SPR and therefore analytic in $|z| \ \ge \ 1$.
*Necessity:* Similar to the continuous-time case.

□

Note that the discrete-time case requires that $P(z)$ be of relative degree $n^* \ = \ 0$. Thus for example, and unlike the continuous-time plant $\frac{1}{s-a}$, the discrete-time plant $\frac{1}{z-a}$ may not be made SPR using the output feedback suggested. Next, we discuss the inverse-system interpretation of Theorem 3. Note that the condition det $D \neq 0$ (or that $P(z)$ has a relative degree $n^* \ = \ 0$), also reveals that the system (3) has an inverse system given in state-space by

$$\begin{aligned} x(k+1) &= [A - BD^{-1}C]x(k) \ + \ BD^{-1}y(k) \\ u(k) &= -D^{-1}Cx(k) \ + \ D^{-1}y(k) \end{aligned} \tag{10}$$

The following general result may then be established.

**Theorem 4** *Suppose that (3) is both stabilizable and detectable, and det $(CA^iB) \neq 0$ where $CA^iB$ is the first nonzero matrix in the sequence*

$$D, CB, CAB, CA^2B, ..., CA^{n-1}B$$

*Then the closed-loop system from $r(k)$ to $y(k+i)$ given by*

$$T_i(z) \ = \ CA^i(zI - A + \gamma BKCA^i)^{-1}BK$$

*is SPR if and only if $P(z)$ is minimum phase.*

**Proof:** See the proof of Theorem 2.

□

## 4 CONCLUSIONS

In this paper we found necessary and sufficient conditions for a transfer function to be rendered SPR using output feedback. These results generalize a previously published result and establish a connection with the invertibility problem. The design is useful when a passive uncertainty enters the system such as in the Lure's problem [6].

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