CONTINUOUS AND DISCRETE TIME SPR DESIGN USING FEEDBACK

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ABSTRACT

This paper presents necessary and sufficient conditions for the existence of a feedback compensator that will render a given continuous-time or discrete-time linear system SPR. When these conditions hold, the controller is explicitly found.

1 INTRODUCTION

The concepts of Positive-Real (PR) and Strictly-Positive-Real (SPR) functions and matrices have been very useful in network theory [1], adaptive control [2] and robust control [3]. These concepts have also been generalized to include discrete-time systems [4] and [5]. The importance of PR and SPR matrices is obvious when dealing with uncertain systems. In this situation, a nominal SPR transfer function allows for large passive uncertainties without the loss of stability [2] and [5]. The standard definition of SPR matrices [6], here termed strong SPR, is usually difficult to apply. Moreover, it was recently shown [7] that the strong SPR definition is overly restrictive for control theory applications. In this paper we will use the term SPR to denote weak SPR matrices as defined in [7] and [8] and reviewed in the next section. On the other hand, if a given transfer matrix is not SPR, the question of whether a feedback controller might make the closed-loop system SPR is of considerable interest. This problem was termed "Almost Strict Positive Real" and studied in [9]. What has been lacking, however, is a set of conditions that will answer the existence question: Given a transfer matrix \( P(s) \), does a controller that will make it SPR exist?. Moreover, a construction of the controller (when it exists) is desirable. A necessary condition was found in [9] and a partial answer to the existence and construction questions was given in [10] for continuous-time systems. Sufficient existence conditions were also found in [11] for the single-input-single-output (SISO) continuous-time case and in [9] for the Multi-Input-Multi-Output
case. In the present paper, we provide a simple proof of the results in [10] and [11], and extend these results to the discrete-time case and to a more general class of systems. This paper is organized as follows: In Section 2, we review the available SPR definitions for continuous and discrete-time transfer matrices. In Section 3, we define the problem and present our results on designing controllers to make a closed-loop system SPR. Our conclusions are presented in Section 4.

2 WHICH SPR?

In order to keep the exposition clear, we will treat the continuous-time case first, then present the discrete-time results.

2.1 Continuous-Time Case:

Consider the multi-input-multi-output linear time-invariant system

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

where \( x \) is an \( n \) vector, \( u \) is an \( m \) vector, \( y \) is a \( p \) vector, \( A, B, C, \) and \( D \) are of the appropriate dimensions. The corresponding transfer function matrix is

\[
P(s) = C(sI - A)^{-1}B + D
\]

We will first assume that the system has an equal number of inputs and outputs, i.e. \( p = m \). Then define the relative degree \( n^* \) as follows: \( n^* = 0 \) if \( \det(D) \neq 0 \), and \( n^* = m \) if \( \det(D) = 0 \) but \( \det(CB) \neq 0 \). A formalism for the poles and zeros of multivariable systems is given in [12] and may be used to justify the definition of \( n^* \).

To simplify our notation we will denote the Hermitian part of a real, rational transfer matrix \( T(s) \) by \( H\!e[T(s)] = \frac{1}{2}[T(s) + T^H(s^*)] \) where \( s^* \) is the complex conjugate of \( s \). A number of definitions have been given for SPR functions and matrices [6] and [8]. It appears that the most useful definition for control applications is the following [7]

**Definition 1** An \( m \times m \) matrix \( T(s) \) of proper real rational functions which is not identically zero is (weak) SPR if

1. All elements of \( T(s) \) are analytic in the closed right half plane, i.e. in the region \( \text{Re}(s) \geq 0 \), and
2. The matrix \( H\!e[T(s)] \) is positive definite for \( \text{Re}(s) \geq 0 \).

\[\Box\]
As a result of this definition, a necessary condition for a given transfer function to be SPR is that \( n^* = -1, \ 0, \ 1 \). The more standard definition of SPR matrices advocated in [6] is more restrictive than Definition 1. In fact, a long-held view was that strong SPR was needed to prove the Meyer-Kalman-Yakubovitch (MKY) lemma, which is, after all, the major application of SPR concepts in control systems. However, As shown in [7], the weak SPR definition is just as useful in this regard and will therefore be adopted in this paper. Note that, from minimum real-part arguments given in [1], condition 2) of Definition 1 is equivalent to \( H e^*[T(jw)] > 0 \) for all \( w \).

### 2.2 Discrete-Time Case:

Consider now the discrete-time multi-input-multi-output linear time-invariant system

\[
\begin{align*}
    x(k+1) &= A x(k) + B u(k) \\
    y(k) &= C x(k) + D u(k)
\end{align*}
\]

where \( x(k) \) is an \( n \) vector, \( u(k) \) is an \( m \) vector, \( y(k) \) is a \( p \) vector, \( A, B, C, \) and \( D \) are of the appropriate dimensions. The corresponding transfer function matrix is then

\[
P(z) = C (zI - A)^{-1} B + D
\]

Similar to the continuous-time case, we assume that the system has an equal number of inputs and outputs i.e. \( p = m \) and define the relative degree \( n^* \) as follows: \( n^* = 0 \) if \( D \neq 0 \), and \( n^* = m \) if \( D = 0 \) but \( CB \neq 0 \). Also, we denote the Hermitian part of \( T(z) \) by \( H e^*[T(z)] = \frac{1}{2} [T(z) + T^*(z^*)] \) where \( z^* \) is the complex conjugate of \( z \) and \( z = \rho e^{jw} \). The concept of discrete PR matrices is defined in [4]. The following definition is motivated by [4] and by the continuous-time counterpart.

**Definition 2** An \( m \times m \) matrix \( T(z) \) of real rational functions is SPR if

1. All elements of \( T(z) \) are analytic on and outside the unit circle, i.e. in the region \( |z| \geq 1 \), and
2. The matrix \( H e^*[T(e^{jw})] \) is positive definite Hermitian for all real \( w \).

\( \Box \)

Note first that a transfer function \( T(z) \) is SPR only if the corresponding \( T(s) \) with \( s = (z-1)/(z+1) \) is SPR. In addition, a necessary condition for \( T(z) \) to be SPR is that it relative degree \( n^* = 0 \).

### 3 SPR USING FEEDBACK

We will again separate our results into continuous-time and discrete-time results.
3.1 Continuous-Time Case:

The question addressed in this section is to find conditions on (1) or (2) so that a feedback controller will render the closed-loop system SPR. The result of Theorem 1 appeared in [10] for the case of a continuous-time plant and a static output feedback, i.e. \( u = -\gamma Ky + Kr \). The closed-loop system is then given by

\[
\dot{x} = (A - \gamma BK C)x + BK r \\
y = Cx
\]

or in the frequency-domain

\[
Y(s) = [I + \gamma P(s)K]^{-1}P(s)KR(s)
\]  

We present a simple frequency domain proof to show the existence of \( K \) and \( \gamma \) that will render the closed-loop system SPR.

**Theorem 1** Let system (1) be stabilizable and detectable and let its relative degree be \( n^* = m \). Then there exists a nonsingular \( K \) and a positive scalar \( \gamma \) such that the closed-loop system (5) is SPR, if and only if \( P(s) \) is minimum phase.

**Proof:**

**Sufficiency:** Consider the closed-loop transfer function

\[
T(s) = [I + \gamma P(s)K]^{-1}P(s)K
\]

or

\[
T(s) = [K^{-1}P^{-1}(s) + \gamma I]^{-1}
\]

Since \( P(s) \) is minimum phase with a relative degree \( n^* = m \), its inverse \( P^{-1}(s) \) will be given by

\[
P^{-1}(s) = sL + P_1(s)
\]

where \( P_1(s) \) is proper and stable, and \( \det(L) \neq 0 \). In fact, \( \det(CB) \neq 0 \) and

\[
L = (CB)^{-1}
\]

On the other hand, since \( P(s) \) is minimum phase, \( P_1(s) \) cannot have any poles in \( \text{Re}(s) \geq 0 \). It is now obvious that \( T(s) \) will be stable if and only if

\[
W(s) = [K^{-1}P^{-1}(s) + \gamma I] \text{ has no zeros in } \text{Re}(s) \geq 0.
\]

Let \( K \) be given by

\[
K = (CB)^{-1}
\]

then

\[
W(s) = sI + CBP_1(s) + \gamma I
\]

\[
H[eW(j\omega)] = He[CBP_1(j\omega)] + \gamma I
\]
Since $P_1(jw)$ has no poles on the $jw$ axis, $He[W(jw)]$ may be made positive-definite by a large enough positive scalar $\gamma$. This then implies that $W(s)$ is weak SPR. Since $T(s)$ is the inverse of $W(s)$, it is also weak SPR [6].

**Necessity:** Suppose now that a nonsingular $K$ and a $\gamma$ were found to make the closed-loop system $T(s)$ SPR and that $D = 0$. Then

$$W(s) = [K^{-1}P^{-1}(s) + \gamma I]$$

is also SPR. Writing $P^{-1}(s)$ as $sL + P_1(s)$, with $L = (CB)^{-1}$ we get

$$W(s) = K^{-1}(CB)^{-1} + K^{-1}P_1(s) + \gamma I$$

Since $W(s)$ is SPR, $P_1(s)$ must be stable, hence $P(s)$ must be minimum-phase.

\[\square\]

This result indicates that with the given assumptions on $P(s)$, static output feedback can always be found to stabilize the closed-loop system $T(s)$. Moreover, $T(s)$ can also be made SPR to give the desired robustness against passive uncertainties. It can also be seen that a dynamic output feedback compensator will not relax the conditions of the theorem since output compensation can not move the open-loop zeros nor change the relative degree of the plant. The choice of $K = (CB)^{-1}$ in the proof of the theorem is not unique. In fact, it is sufficient to choose $K = Q(CB)^{-1}$ where $Q$ is any symmetric positive-definite matrix. Next, note that the condition $\det(CB) \neq 0$ (or that $P(s)$ has a relative degree $n^* = m$), also reveals that the system (1) has an inverse obtained by cascading one differentiator and a dynamical system [13]. Note that the inverse system given in the proof of Theorem 1 may be written in state-space as

$$\begin{align*}
\dot{x} &= [A - B(CB)^{-1}CA]x + B(CB)^{-1}y \\
u &= -(CB)^{-1}CAx + (CB)^{-1}y
\end{align*}$$

(7)

Now recall that the invertibility of the system (1) may still be inferred even though $\det(CB) = 0$. In fact, a sufficient condition for the inverse to exist is that the first nonzero matrix in the sequence, $D, CB, CAB, CA^2B, \ldots, CA^{n-1}B$, be nonsingular [13]. It is then obvious that for a nonzero matrix $D$, the condition for $T(s)$ to be SPR is that $D$ be invertible and $P(s)$ be minimum phase, i.e. an exactly-proper, minimum-phase transfer function may be made SPR with a static output feedback if its high frequency gain is nonsingular. On the other hand, the following general result may be established.

**Theorem 2** Suppose that (1) is both stabilizable and detectable, and $\det(CA^iB) \neq 0$ where $CA^iB$ is the first nonzero matrix in the sequence

$$D, CB, CAB, CA^2B, \ldots, CA^{n-1}B$$
Then the closed-loop system from $r$ to $\frac{dy}{dt}$ given by

$$T_1(s) = CA^i(sI - A + \gamma BKCA^i)^{-1}BK$$

is SPR if and only if $P(s)$ is minimum phase.

**Proof:** Given system (1), repeated here for convenience

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

let us define an output $z_i$ by

$$z_i = \frac{d^iy}{dt^i} = y^{(i)}$$

then

$$z_{i+1} = CA^{i+1}x + CA^Bu$$

If det $(CA^iB) \neq 0$, the inverse system of $P(s)$ is given by

$$P^{-1}(s) = (CA^iB)^{-1}s^{i+1} + P_2(s)$$

where $P_2(s)$ is stable. Repeating the arguments of theorem 3.1 and using the controller

$$u = -\gamma Ky^{(i)} + Kr$$

we obtain the desired result.

$\square$

### 3.2 Discrete-Time Case:

Next, we turn our attention to the discrete-time case. Specifically, we find conditions on (3) or (4) so that a feedback controller will render the closed-loop system SPR. Consider then the static output feedback, i.e. $u(k) = -\gamma Ky(k) + Kr(k)$. The closed-loop system is then given by

$$x(k+1) = (A - \gamma BKC)x(k) + BKr(k)$$
$$y(k) = Cx(k) + Dr(k)$$

or in the $z-$ domain

$$Y(z) = [I + \gamma P(z)K]^{-1}P(z)KR(z)$$

The following theorem parallels theorem reft1 and shows the existence of $K$ and $\gamma$ that will render the closed-loop system (8) SPR if $D$ is invertible.
**Theorem 3** Let system (3) be stabilizable and detectable and let its relative degree be $n^* = 0$. Then there exist a nonsingular $K$ and a positive scalar $\gamma$ such that the closed-loop system (8) is SPR, if and only if $P(z)$ is minimum phase.

**Proof:**

**Sufficiency:** Consider the closed-loop transfer function

$$T(z) = [I + \gamma P(z)K]^{-1}P(z)K$$

$$= [K^{-1}P^{-1}(z) + \gamma I]^{-1} = W^{-1}(z)$$

Since $P(z)$ is minimum phase with a relative degree $n^* = 0$, its inverse $P^{-1}(z)$ will be given by

$$P^{-1}(z) = P_1(z)$$

where $P_1(z)$ is proper and stable. Note that $W(z) = K^{-1}P^{-1}(z) + \gamma I$ can be made SPR by choosing $\gamma$ positive and large enough since

$$||P^{-1}(z)|| \leq c \infty; \text{ for all } |z| \geq 1$$

Since the inverse of an SPR matrix is SPR, $T(z)$ is SPR and therefore analytic in $|z| \geq 1$.  

**Necessity:** Similar to the continuous-time case.

\[ \square \]

Note that the discrete-time case requires that $P(z)$ be of relative degree $n^* = 0$. Thus for example, and unlike the continuous-time plant $\frac{1}{s-a}$, the discrete-time plant $\frac{1}{z-a}$ may not be made SPR using the output feedback suggested. Next, we discuss the inverse-system interpretation of Theorem 3. Note that the condition $D \neq 0$ (or that $P(z)$ has a relative degree $n^* = 0$), also reveals that the system (3) has an inverse system given in state-space by

$$x(k+1) = [A - BD^{-1}C]x(k) + BD^{-1}y(k)$$

$$u(k) = -D^{-1}Cx(k) + D^{-1}y(k)$$

(10)

The following general result may then be established.

**Theorem 4** Suppose that (3) is both stabilizable and detectable, and $det(CA^iB) \neq 0$ where $CA^iB$ is the first nonzero matrix in the sequence

$$D, CB, CAB, CA^2B, ..., CA^{n-1}B$$

Then the closed-loop system from $r(k)$ to $y(k+i)$ given by

$$T_i(z) = CA^i(zI - A + \gamma BKCA^i)^{-1}BK$$

is SPR if and only if $P(z)$ is minimum phase.

**Proof:** See the proof of Theorem 2.

\[ \square \]
4 CONCLUSIONS

In this paper we found necessary and sufficient conditions for a transfer function to be rendered SPR using output feedback. These results generalize a previously published result and establish a connection with the invertibility problem. The design is useful when a passive uncertainty enters the system such as in the Lure's problem [6].

References


