

PHASE-CONVEX ARCS IN ROOT SPACE AND THEIR APPLICATION TO ROBUST SPR PROBLEMS

F. Pérez *, C. Abdallah †

* ETSI Telecomunicacion, Universidad de Vigo, 36200-VIGO, SPAIN

† EECE Dept., University of New Mexico, Albuquerque, NM 87131, USA.

ABSTRACT

This paper considers the problem of identifying regions in the complex-plane, such that the phase of polynomials having roots in those regions, is bounded by that of a few extreme polynomials. Applications of the results are also described.

1 Introduction

This short paper considers the problem of identifying regions in the complex-plane, such that the phase of polynomials having roots in those regions, is bounded by that of a few extreme polynomials. More specifically, given a family of polynomials $P(z^{-1})$

$$P(z^{-1}) = \{p(z^{-1}); p(z_i^{-1}) = 0 \Rightarrow z_i \in \Omega_i\} \quad (1)$$

where Ω_i is a set about the corresponding roots of a nominal polynomial $p_0(z^{-1})$. We are interested in finding the sets Ω_i such that the phase of every member in $P(z^{-1})$ is determined by that of a few extreme members. Our development parallels that of [?] and the motivation given there and elsewhere [?], [?], is that an SPR condition is frequently invoked to prove convergence of adaptive algorithms. We differ from [?] and [?] however, in that we

do not assume any knowledge of the spectral density of the regressor. Section 2 presents a simple version of the problem, while Section 3 generalizes the results to roots in arbitrary domains. Some applications are given in Section 4, and our conclusions are given in Section 5.

2 Phase-Convex arcs

Let $\lambda \rightarrow F(\lambda)$, $\lambda \in [\lambda_0, \lambda_1]$ be a continuously differentiable parameterization of an arc in the complex plane. We write $F(\lambda) = X(\lambda) + jY(\lambda)$, where X and Y are two real functions of the real variable λ . We first consider polynomials with the following structure

$$p(z^{-1}, \lambda) = [1 - (X + jY)z^{-1}][1 - (X - jY)z^{-1}]$$

Throughout the text we will assume that $|F(\lambda)| < 1/r_0$, $\lambda \in [\lambda_0, \lambda_1]$ which will be necessary for stability of the polynomials considered. Let $\phi(\omega, \lambda) = \arg\{p(r_0^{-1}e^{-j\omega}, \lambda)\}$ we seek conditions that guarantee that $\phi(\omega, \lambda)$ is bounded by $\phi(\omega, \lambda_0)$ and $\phi(\omega, \lambda_1)$, for all $\omega \in [0, 2\pi)$. We will call an arc satisfying this property a phase-convex arc or simply say that an extreme-point property holds for the phase. Next we drop the dependence on λ wherever there is no possible confusion, and state our first result

Theorem 1 *An extreme-point result holds for $\phi(\omega, \lambda)$ if for any $\lambda \in (\lambda_0, \lambda_1)$ the functions*

$$g_1(\lambda) = (1 - \alpha^2) \frac{\partial X}{\partial \lambda} - 2\alpha \frac{\partial Y}{\partial \lambda} \quad (2)$$

$$g_{-1}(\lambda) = (1 - \beta^2) \frac{\partial X}{\partial \lambda} + 2\beta \frac{\partial Y}{\partial \lambda} \quad (3)$$

both have the same sign, where

$$\alpha = \frac{Yr_0}{(1 - Xr_0)}; \quad \beta = \frac{Yr_0}{(1 + Xr_0)}$$

Proof: See [?].

For a geometric interpretation, provided that $\partial X/\partial \lambda \neq 0$, we make $\gamma = \partial Y/\partial X$ so that the sign condition can be restated in terms of h_1 and h_{-1}

$$h_1(\lambda) = |\alpha + \gamma| - \sqrt{\gamma^2 + 1}; \quad h_{-1}(\lambda) = |\beta - \gamma| - \sqrt{\gamma^2 + 1}$$

thus giving a feasible region of points $X + jY$ for which the condition holds for a fixed γ , which can be thought as the direction of the tangent to the arc. In [?] we show that the feasible region is a rectangle. Next, we discuss the particular cases of straight line segments and circular arcs.

2.1 Straight Line Segments

In this case, we can write $X(\lambda) = x_0 + \lambda a$, $Y(\lambda) = y_0 + \lambda b$ where $\lambda \in [0, 1]$, and a and b are real numbers. Now, whenever $a \neq 0$ we can write $\gamma = b/a$ and therefore an extreme point result will hold when h_1 and h_{-1} have the same sign for (x_0, y_0) and $(x_0 + a, y_0 + b)$. For the case $a = 0$ (vertical lines) it is not possible to use the sign condition. However, it is shown in [?] that with an adequate restructuring of the problem, an extreme-point result holds also for vertical segments. The case of real roots can be treated as a special case of horizontal segments for which we have proven that an extreme point result holds as long as $F(\lambda) \subset (-1/r_0, 1/r_0)$.

2.2 Circular Arcs

Circular arcs are described by $F(\psi) = (x_0 + jy_0) + re^{j\psi}$, $\psi \in [\psi_0, \psi_1]$. Calculating g_1 and g_{-1} in (2-3) and after some straightforward algebraic manipulations, it can be shown that the set of values of $\psi \in [0, 2\pi)$ satisfying the sign condition can be calculated from the roots of two equations of the form $A \sin \psi + B \cos \psi + C = 0$, with A, B, C real numbers [?]. These solutions divide the interval $[0, 2\pi)$ into subintervals where the sign condition holds or fails and, obviously, ψ_0 and ψ_1 must belong to one subinterval of the first type in order to have an extreme-point result. As a particularly simple case, we mention $y_0 = 0$, that gives an extreme point result as long as $x_0 \in (-1/r_0, 1/r_0)$.

3 Generalization to Domains

Let Ω be an open region, simply connected and symmetric with respect to the real axis. Let $P(z^{-1})$ be the family of second order polynomials with roots in Ω and assume that

the boundary of Ω , $\partial\Omega \in \mathbf{C}$, is given by

$$\partial\Omega = \bigcup_{i=1}^N F_i(\lambda_i), \lambda_i \in [\lambda_i^0, \lambda_i^1] \subset \mathbf{R} \quad (4)$$

where $F_i(\lambda_i)$ is a continuously differentiable phase-convex arc. Let $\phi_p(z) = \arg\{p(z^{-1})\}$ and define

$$\bar{\phi}(z) = \sup_{p \in P} \phi_p(z), \quad \underline{\phi}(z) = \inf_{p \in P} \phi_p(z) \quad (5)$$

These functions are respectively termed lead and lag functions of the family $P(z^{-1})$ [?]. Also define the extreme polynomials as $p_i(z^{-1}) = [1 - F_i(\lambda_i^0)z^{-1}][1 - F_i^*(\lambda_i^0)z^{-1}]$, $i = 1, \dots, N$ and define the corresponding extreme phase functions $\phi_i(z) = \arg\{p_i(z^{-1})\}$, $i = 1, \dots, N$. Then, the following result can be stated

Theorem 2 *The lead and lag functions of $P(z^{-1})$ for a given $z = r_0 e^{j\omega}$, $\omega \in [0, 2\pi)$, are given by*

$$\bar{\phi}(r_0 e^{j\omega}) = \max_{i=1, \dots, N} \phi_i(r_0 e^{j\omega}); \quad \underline{\phi}(r_0 e^{j\omega}) = \min_{i=1, \dots, N} \phi_i(r_0 e^{j\omega})$$

Proof: See [?].

Obviously, it is possible that only a subset of the N extremes is necessary to bound the phase, and a very important special case is that in which the lead and lag functions correspond to the same extreme-phase function for every $\omega \in [0, 2\pi)$, so that only two extremes are necessary. Note that the arcs considered here (except for vertical line segments) satisfy the conditions for this result.

Theorem 2 can be also used to produce conservative (but simple) bounds for families of the form

$$Q(z^{-1}) = \{q(z^{-1}); q(z_0^{-1}) = 0 \Rightarrow z_0 \in \Gamma \text{ or } z_0 \in \Gamma^*\} \quad (6)$$

for which $\Gamma \subset \Omega$. The results given so far can be extended to families of polynomials of any degree n . Hereafter n is an even number and the roots appear only in complex conjugate pairs, although the results can be easily extended to the remaining cases. In short, in [?] it is shown that a extreme-point result also holds for families of the form

$$P(z^{-1}) = \{p(z^{-1}) = \prod_{l=1}^m p_l(z^{-1}), \quad p_l(z^{-1}) \in P_l(z^{-1})\} \quad (7)$$

where for $l = 1, \dots, m$

$$P_l(z^{-1}) = \{p_l(z^{-1}) = \prod_{i=1}^{M_l} (1 - z_i z^{-1})(1 - z_i^* z^{-1}), z_i \in \Omega_l\}$$

$$\partial\Omega_l = \bigcup_{i=1}^{N_l} F_{i,l}(\lambda_{i,l}), \quad \lambda_{i,l} \in [\lambda_{i,l}^0, \lambda_{i,l}^1] \subset \mathbf{R}$$

and where the $F_{i,l}(\lambda_{i,l})$ are continuously differentiable phase-convex arcs.

4 Application to SPR problems

The first important application has to do with SPR checking of families of polynomials. Suppose that the family $P(z^{-1})$ described in (7) is stable, then, the condition for SPRness on the complement of the circle of radius $1/r_0$ is

$$\pi/2 < \underline{\phi}(r_0 e^{j\omega}) \leq \bar{\phi}(r_0 e^{j\omega}) < \pi/2 \quad (8)$$

so if the root domains are bounded by closed curves that are piecewise phase-convex, then the family will be SPR if and only if some adequately selected extreme polynomials are SPR.

The second application deals with filter design to enforce SPRness of a family of polynomials. In this case, we want to find a stable polynomial $f(z^{-1})$ such that $\frac{p(z^{-1})}{f(z^{-1})}$ is SPR for every $p \in P$. Again, if the root domains for the family of polynomials satisfy the requirements for phase-convexity, it can be shown that the necessary and sufficient condition for the existence of a stable $f(z^{-1})$ can be stated in terms of a finite set of conditions involving the extreme polynomials [?].

Finally, if the problem is to find $f(z^{-1})$ to make $P(z^{-1})$ SPR, we can split the uncertain family into a product of even-order families, each having its roots inside a domain for which the phase can be bounded by two extremes. The optimal solution for the synthesis of $f(z^{-1})$ is then obtained from [?]

5 Conclusions

In this paper we have considered the problem of identifying regions in the complex-plane, such that the phase of

polynomials having roots in those regions, is bounded by that of a few extreme polynomials. Our results are simple to check and generalize previously available ones. Applications of the results were also described.

References