A Reduction in Conservatism for Convex Linear-Quadratic Simultaneous Performance Design

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Abstract

In this paper a fixed state feedback control law which minimizes upper bounds on linear-quadratic performance measures for m distinct plants is studied. Previous work [8] by the authors demonstrated a convex semidefinite programming solution thereby guaranteeing global optimality. The present work extends that result by proposing an algorithm which reduces the conservatism of the minimum guaranteed-cost upper bounds for each of the m performance measures.

Keywords: simultaneous stabilization, simultaneous control, semidefinite programming, state feedback control, linear matrix inequalities (LMIs)

1 Introduction

The problem considered here is the design of a fixed state feedback control law $u(t) = -Kx(t)$ which minimizes an upper bound on the performance measures

$$\mathbb{E}\left\{\int_0^\infty \left[x_j^T(t)Q_jx_j(t) + u_j^T(t)R_ju_j(t)\right] dt\right\} \tag{1}$$

for $Q_j > 0$ and $R_j > 0$, each associated with one of the plants described by state space equations

$$\dot{x}_j(t) = A_jx_j(t) + B_ju_j(t) \tag{2}$$

for all $j \in I_m \triangleq \{1, \ldots, m\}$. In (1) the expectation operator $\mathbb{E}\{\cdot\}$ is taken over random initial conditions satisfying $\mathbb{E}\{x(0)\} = 0$ and $\mathbb{E}\{x(0)x^T(0)\} = I$. We refer to this as a *simultaneous performance design problem.*

The paper begins with the convex reformulation of the Chang and Peng [3] guaranteed-cost control method discussed in Luke, et al. [8]. The problem can be solved, albeit conservatively, using widely available semidefinite programming software (El Ghaoui, et al. [5]; Nesterov and Nemirovskii [9]; and Vandenberghe and Boyd [11]). Further, the property of convexity guarantees that any solution found will be *globally* optimal. However the guaranteed-cost upper bounds on performance measures (1) were kept invariant over all systems $j \in I_m$ in order to achieve that convexity. These invariant bounds necessarily introduce conservatism to the solution and the goal of the present work is to reduce conservatism while retaining problem convexity.

The problem appears to be approachable in at least two ways. The first involves a search for a parameterization of the simultaneously stabilizing gain matrix K such that nonconvexity does not arise. This effort would be similar to that suggested by Kar [7] in a somewhat different context. But it may limit the reduction in conservatism because the search would be constrained to the set of all variable gain matrices K satisfying the pattern imposed by the parameterization. The second method involves a decomposition of the larger nonconvex problem into a set of two convex, but coupled, systems. We deal with the latter option and follow a direction suggested by the work of Grigoriadis and Skelton [6]. This course of action also tends to limit the reduction in conservatism and this limiting action is analyzed herein. Finally an example frequently appearing in the literature is used to demonstrate a significant reduction in the guaranteed-cost upper bounds, compared to what was previously achieved.

2 Convex Guaranteed-Cost Control

Using the methods of guaranteed-cost control developed by Chang and Peng [3], it is possible to show (Dorato, et al. [4]) that an upper bound on the performance measures (1) involving the transient state response $x_j(t)$ and the control effort $u_j(t)$ are optimized for each of the systems by assigning to each an integral quadratic performance function (1). The expectation operator taken over all initial conditions $x(0)$ results in

$$\mathbb{E}\left\{\int_0^\infty \left[x_j^T(t)Q_jx_j(t) + u_j^T(t)R_ju_j(t)\right]dt\right\} \leqslant \mathrm{tr}\left\{P\right\} \ ,\forall j \in I_m \tag{3}$$

where the single matrix $P = P^T > 0$ satisfies each of the m matrix Lyapunov inequalities

$$\left(A_j - B_jK\right)^T P + P\left(A_j - B_jK\right) + Q_j + K^TR_jK < 0 \tag{4}$$

for all $j \in I_m$. The existence of such a $P = P^T > 0$ is also sufficient to guarantee that the state feedback control law $u_j(t) = -Kx_j(t)$ simultaneously stabilizes the m distinct plants (2). See Boyd, et al. [2, pp. 100–101]. It turns out that this guaranteed-cost control problem can be reduced to a convex programming problem, the advantage being that any converged solution is guaranteed to be *globally* optimal. Consider the change of variables used by Bernussou, et al. [1]: for some real matrix $Y = Y^T > 0$, let $P = Y^{-1}$ and $K = XY^{-1}$. An equivalent convex programming problem can be formulated as the following Optimization Problem 2.1 (see [8]).

Optimization Problem 2.1 (Conservative Gain)

$$\min_{X,Y,Z} tr\{Z\} \tag{5}$$

for real matrix variables $Y = Y^T > 0$ and $Z = Z^T > 0$, subject to $m+1$ separate linear matrix inequality constraints

$$\begin{bmatrix} -YA_j^T - A_jY + B_jX + X^TB_j^T & Y & X^T \\ Y & Q_j^{-1} & 0 \\ X & 0 & R_j^{-1} \end{bmatrix} > 0 \ , \qquad \begin{bmatrix} Z & I \\ I & Y \end{bmatrix} > 0 \tag{6}$$

for all $j \in I_m$. The optimal gain is calculated as $K^* = X^*(Y^*)^{-1}$.

An example found in Paskota, et al. [10]; Luke, et al. [8], and others is used to demonstrate Optimization Problem 2.1 and to produce a point for comparison with results of reduced conservatism to follow in Section 3. A static state feedback gain K is sought which simultaneously stabilizes four different operating points of an airplane trajectory in the vertical plane. The operating points are specified with a set of $m = 4$ state differential equations (2) assuming a scalar input u. The state coefficient matrices are

$$A_1 = \begin{bmatrix} -0.9896 & 17.41 & 96.15 \\ 0.2648 & -0.8512 & -11.89 \\ 0 & 0 & -30 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.6607 & 18.11 & 84.34 \\ 0.08201 & -0.6587 & -10.81 \\ 0 & 0 & -30 \end{bmatrix}, \tag{7}$$

$$A_3 = \begin{bmatrix} -1.702 & 50.72 & 263.5 \\ 0.2201 & -1.418 & -31.99 \\ 0 & 0 & -30 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} -0.5162 & 26.96 & 178.9 \\ -0.6896 & -1.225 & -30.38 \\ 0 & 0 & -30 \end{bmatrix}, \tag{8}$$

and the control coefficient vectors b_j are

$$b_1 = \begin{bmatrix} -97.78 \\ 0 \\ 30 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -272.2 \\ 0 \\ 30 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -85.09 \\ 0 \\ 30 \end{bmatrix}, \quad b_4 = \begin{bmatrix} -175.6 \\ 0 \\ 30 \end{bmatrix}.$$

The state coefficient matrices Q_j and control coefficient matrices R_j used in objective functions (1) are each set to an identity matrix of appropriate dimensions. Using numerical interior point programming methods discussed by Nesterov and Nemirovskii [9], LMITOOL by El Ghaoui, et al. [5], and the semidefinite programming software package SP by Vandenberghe and Boyd [11], the results turn out to be

$$X^* = \begin{bmatrix} -0.2593 \\ 0.0061 \\ 0.0560 \end{bmatrix}^T, \; Y^* = \begin{bmatrix} 3.3514 & -0.3781 & -0.1683 \\ -0.3781 & 0.0569 & 0.0208 \\ -0.1683 & 0.0208 & 0.0387 \end{bmatrix}, \; Z^* = \begin{bmatrix} 1.2380 & 7.7939 & 1.2021 \\ 7.7939 & 70.9534 & -4.1972 \\ 1.2021 & -4.1972 & 33.3622 \end{bmatrix},$$

indicating an optimal performance matrix bound of

$$\mathrm{tr}\{P^*\} = \mathrm{tr}\left\{\left(Y^{-1}\right)^*\right\} = 105.3 \,. \tag{9}$$

The optimal gain vector $K^* = X^*(Y^*)^{-1}$ is

$$K^* = \begin{bmatrix} -0.2063 & -1.8247 & 1.5305 \end{bmatrix} . \tag{10}$$

3 Distinct Performance Bounds

Now the use of *distinct* scalar functions $\mathrm{tr}\{P_j\}$ is addressed to bound each of the integral quadratic performance indices (3). Consider the simultaneously stabilizing gain matrix K satisfying $K = X_1Y_1^{-1} = \cdots = X_mY_m^{-1}$ and rewrite Optimization Problem 2.1 as the following Optimization Problem 3.1.

Optimization Problem 3.1 (Nonlinear Problem)

$$\min_{K,\{(Y_j,Z_j)\}_{j\in I_m}} \sum_{j=1}^{m} tr\{Z_j\} \tag{11}$$

for real matrix variables K and $\{(Y_j, Z_j) : Y_j = Y_j^T > 0, Z_j = Z_j^T > 0\}_{j\in I_m}$, subject to $2m$ separate nonlinear matrix inequality constraints

$$\begin{bmatrix} -Y_jA_j^T - A_jY_j + B_jKY_j + Y_jK^TB_j^T & Y_j & Y_jK^T \\ Y_j & Q_j^{-1} & 0 \\ KY_j & 0 & R_j^{-1} \end{bmatrix} > 0 \,, \qquad \begin{bmatrix} Z_j & I \\ I & Y_j \end{bmatrix} > 0 \tag{12}$$

for all $j \in I_m$.

Using distinct matrix bounds on the performance functions has the advantage of reducing conservatism in the solution but has the disadvantage of the loss of matrix constraint convexity. Such convexity is required for the application of the interior point algorithms. However the reduction in conservatism can be recovered to some extent with a decomposition of the nonconvex Optimization Problem 3.1 into a coupled set of convex problems. We consider the following Algorithm 3.2 to be a numerical analog of the alternating projections method of Grigoriadis and Skelton [6].

Algorithm 3.2

Step 1 *Guess initial values of the $3m+1$ matrix optimization variables K and $\{(X_j, Y_j, Z_j) : Y_j = Y_j^T > 0, Z_j = Z_j^T > 0\}_{j\in I_m}$ and a scalar variable t. Calculate $f_{old} = tr\{Z_1\} + \cdots + tr\{Z_m\} + t$ for use in a convergence tolerance test in Step 4.*

Step 2 *Minimize the magnitude (that is, the maximum singular value) of the simultaneously stabilizing gain matrix K for fixed values of matrices $\{(X_j, Y_j, Z_j)\}_{j\in I_m}$. Vandenberghe and Boyd [12] show that this minimization can be posed in terms of linear matrix inequalities as*

$$\min_{K,t} t$$

subject to the linear matrix equality constraints

$$X_j = KY_j, \;\; \forall j \in I_m \tag{13}$$

for given matrix pairs $\{(X_j, Y_j)\}_{j\in I_m}$*; and subject to the linear matrix inequality constraint*

$$\begin{bmatrix} tI & K \\ K^T & tI \end{bmatrix} > 0 \ .$$

The converged values of variables K *and* t *are then saved for use as* <u>*fixed*</u> *quantities in Step 3.*

Step 3 *Minimize the guaranteed-cost bounds for each of the* m *systems by*

$$\min_{\{(Y_j, Z_j)\}_{j\in I_m}} \sum_{j=1}^{m} tr\{Z_j\}$$

subject to the $2m$ *linear matrix inequality constraints*

$$\begin{bmatrix} -Y_j A_j^T - A_j Y_j + B_j K Y_j + Y_j K^T B_j^T & Y_j & Y_j K^T \\ Y_j & Q_j^{-1} & 0 \\ K Y_j & 0 & R_j^{-1} \end{bmatrix} > 0 \ , \qquad \begin{bmatrix} Z_j & I \\ I & Y_j \end{bmatrix} > 0$$

for given K *from Step 2 for all* $j \in I_m$*. Save the converged values of optimization variables* $\{Y_j\}_{j\in I_m}$ *for use as* <u>*fixed*</u> *quantities in a return to Step 2. Calculate matrices* $X_j = KY_j$ *for all* $j \in I_m$*, also for use as* <u>*fixed*</u> *quantities in Step 2.*

Step 4 *Check for convergence: let* $f_{new} = tr\{Z_1\} + \cdots + tr\{Z_m\} + t$*. If* $\|f_{new} - f_{old}\| < \varepsilon$ *(some user-defined tolerance), then stop. Otherwise let* $f_{old} := f_{new}$ *and go to Step 2.*

An immediate and obvious wrinkle is that constraints (13) specify an overdetermined system. For example, in the flight trajectory problem described in Section 2, constraints (13) involve four matrix equations (twelve scalar equations) in three scalar unknowns (three components of the gain K array). However it is well known[1] that linear matrix inequalities (13) admit at least one solution K if $\text{rank}(Y^T) = \text{rank}([Y^T \mid X^T])$ where $X := [X_1, \ldots, X_m]$ and $Y := [Y_1, \ldots, Y_m]$. Analysis of the example problem using MATLAB indicates that the ranks are in fact equal. Further, we find that solution K is *unique* because Y^T has full rank.

It is already known that there is at least one solution, namely the "conservative" $K^* = X^*(Y^*)^{-1}$ from Optimization Problem 2.1, assuming $X := X_1 = \cdots = X_m$; $Y := Y_1 = \cdots = Y_m$; and $Z := Z_1 = \cdots = Z_m$. Step 2 can thus be replaced entirely, obviating the need for iteration. Also since K is constant in Step 3, the sufficient condition matrix Lyapunov inequality (4) becomes, after replacing matrix variable P with distinct matrix variables $\{P_j\}_{j\in I_m}$,

$$(A_j - B_j K)^T P_j + P_j (A_j - B_j K) + Q_j + K^T R_j K < 0 \ .$$

This is already a linear matrix inequality and Algorithm 3.2 is therefore simplified to the following Algorithm 3.3.

Algorithm 3.3

Step 1 *Solve the "conservative" Optimization Problem 2.1 and calculate the resulting gain matrix* K *as*

$$K = XY^{-1} \ .$$

Save the converged value of K *and use it as a* <u>*fixed*</u> *quantity in Step 2. Use the resulting conservatively optimized values of* X*,* Y*, and* Z *as initial guesses of the matrix variables* $\{(X_j, Y_j, Z_j) : Y_j = Y_j^T > 0, Z_j = Z_j^T > 0\}_{j\in I_m}$*, respectively, in Step 2.*

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[1] A solution x to an overdetermined system $Ax = b$ exists if $\text{rank}(A) = \text{rank}([A \mid y])$. This is due to the fact that the equality of ranks implies that b is in the column space of A. The solution is unique if A is of full rank because that implies that the null space of A is of dimension zero. If A were *not* of full rank then there would exist a nonzero vector $z \neq x$ contained by the null space of A. Vector z would be a solution of $Az = b$ because $b = A(x + z) = Ax + Az$. Since z is in the null space of A, $Az = 0$, leaving $Ax = b$.

Step 2 *Minimize the guaranteed-cost bounds for each of the m systems by*

$$\min_{\{(P_j\}_{j\in I_m}} \sum_{j=1}^{m} tr\{P_j\}$$

subject to the m linear matrix inequality constraints

$$-A_j^T P_j - P_j A_j + K^T B_j^T P_j + P_j B_j K - Q_j - K^T R_j K > 0$$

for all $j \in I_m$.

The example used above for the "conservative" Optimization Problem 2.1 is revisited for use with the noniterative Algorithm 3.3 of reduced conservatism so that results can be compared. The optimized matrix variables turn out to be

$$P_1^* = \begin{bmatrix} 0.0726 & -0.3125 & 0.1774 \\ -0.3125 & 1.8192 & -0.9305 \\ 0.1774 & -0.9305 & 0.5175 \end{bmatrix}, \quad P_2^* = \begin{bmatrix} 0.0479 & -0.5360 & 0.3625 \\ -0.5360 & 7.2761 & -4.6376 \\ 0.3625 & -4.6376 & 3.0744 \end{bmatrix},$$

$$P_3^* = \begin{bmatrix} 0.2593 & 1.6214 & 0.6648 \\ 1.6214 & 12.9840 & 3.3934 \\ 0.6648 & 3.3934 & 2.0399 \end{bmatrix}, \quad P_4^* = \begin{bmatrix} 0.1472 & 0.1014 & 0.8008 \\ 0.1014 & 0.5332 & 0.5158 \\ 0.8008 & 0.5158 & 4.5380 \end{bmatrix}.$$

The maximal guaranteed-cost bound over all $m = 4$ systems is calculated as

$$\max_{j\in I_4} \text{tr}\left\{P_j^*\right\} = 15.3 ,$$

a reduction of an order of magnitude (at least for this example) over the conservative bound (9). Note that the ability to simultaneously control all m systems with the single gain matrix K is retained.

4 Summary

This paper demonstrates a decomposition of the convex linear-quadratic simultaneous performance design method that can produce a significant reduction in solution conservatism. Previous work by the authors is reviewed, numerical analysis and simplification of the decomposition is provided, and examples are given for the purpose of comparison.

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