Guaranteed-Cost Control of the Nonlinear Benchmark Problem Using Model-Based Fuzzy Systems

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Abstract

In this paper we design a state-feedback controller for the nonlinear benchmark problem. Our approach relies on the use of Takagi-Sugeno fuzzy models to approximate the nonlinear system. Once the fuzzy model is obtained, we develop a guaranteed-cost framework to design the controller using Linear Matrix Inequality methods and recently obtained relaxed stability conditions. We show that our proposed controller will not only stabilize the system, but also has satisfactory disturbance attenuation properties.

1 Introduction

The nonlinear benchmark problem was introduced in [18] as a simplified model of a dual-spin spacecraft to study the resonance capture phenomenon [12]. It has since been clarified and proposed as a benchmark to study nonlinear control techniques [3]. After simplification, the system is described by the non-dimensional equations

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + d(x)w \\ f(x) &= \begin{bmatrix} x_2 \\ \frac{\epsilon x_4^2 \sin(x_3) - x_1}{1-\epsilon^2\cos^2(x_3)} \\ x_4 \\ \frac{\epsilon\cos(x_3)(x_1 - \epsilon x_4^2 \sin(x_3))}{1-\epsilon^2\cos^2(x_3)} \end{bmatrix}; \quad g(x) = \begin{bmatrix} 0 \\ \frac{-\epsilon\cos(x_3)}{1-\epsilon^2\cos^2(x_3)} \\ 0 \\ \frac{1}{1-\epsilon^2\cos^2(x_3)} \end{bmatrix}; \quad d(x) = \begin{bmatrix} 0 \\ \frac{1}{1-\epsilon^2\cos^2(x_3)} \\ 0 \\ \frac{-\epsilon\cos(x_3)}{1-\epsilon^2\cos^2(x_3)} \end{bmatrix} \end{aligned} \quad (1)$$

where ϵ is a nonnegative constant less than one. This problem has since been studied and reported on in [1, 7, 8, 9, 10, 15]. In this paper, we introduce a new control approach, based on the linearization of the system equations around two different operating points. The nonlinear system is then approximated by a convex combination of these two linear models. This way of modeling has recently become quite popular. The key point of this modeling approach is that once linear models are obtained, linear control methodology can be used to design controllers for each linear model. The overall controller for the original nonlinear system is obtained by aggregating the local models. Stability conditions for these systems were first given in [14]. These conditions required the existence of a common Lyapunov matrix which would simultaneously satisfy a set of Lyapunov Matrix Inequalities. It was later shown in [17] that these stability conditions can be relaxed and that they can be transformed into Linear Matrix Inequalities which are efficiently solvable using interior-point convex optimization methods [2, 11]. Recently, less conservative stability conditions were derived for these systems [13].

In [5], we developed a guaranteed-cost approach for design of stabilizing Takagi-Sugeno (T-S from now on) controllers which would also minimize an upper bound on a quadratic performance measure. In this paper, we combine our results from [5] and those of [13] to develop a T-S controller for the nonlinear benchmark problem. Our simulation results indicate that the controller will satisfy the required design specifications and will also attenuate the effect of disturbance.

Most of the papers which studied the benchmark problem (with the exception of [15]) did not deal with the disturbance rejection issues directly. Some, such as [9] showed that the effect of disturbance is attenuated by the design. Others, such as the passivity designs of [8] are robust with respect to L_2 disturbances entering at the right place into the system. As discussed in [9], and further evidenced by the many approaches to the problem, the regulation problem in the absence of disturbance may be efficiently solved. The present paper attempts to design controllers which will not only stabilize the system, but will do so in the face of sinusoidal disturbances and within the limits set forth in the benchmark problem.

This paper is organized as follows: Section 2 gives a brief review of T-S fuzzy systems and their stability conditions. Section 3 presents a guaranteed-cost approach for the design of T-S fuzzy controllers. In section 4, we apply our design to the nonlinear benchmark problem and present numerical simulations. Our conclusions are given in section 5.

2 T-S Fuzzy Systems and Closed-loop Stability Conditions

A dynamic T-S fuzzy model is described by a set of fuzzy "IF... THEN" rules with fuzzy sets in the antecedents and dynamic LTI systems in the consequents. A generic T-S plant rule can be written as follows:

i^{th} Plant Rule: IF $x_1(t)$ is $\tilde{M}_{i1}$ and $\ldots, x_n(t)$ is $\tilde{M}_{in}$ THEN $\dot{x} = A_i x + B_i u$

where $x \in \mathrm{I\!R}^{n\times 1}$ is the state vector, $i = \{1, \cdots, r\}$, r is the number of rules, $\tilde{M}_{ij}$ are input fuzzy sets, $A_i \in \mathrm{I\!R}^{n\times n}, \mathrm{B_i} \in \mathrm{I\!R}^{n\times m}$, and $u \in \mathrm{I\!R}^{m\times 1}$.
Using singleton fuzzifier, max-product inference and center average defuzzifier, we can write the aggregated fuzzy model as

$$\dot{x} = \frac{\sum_{i=1}^{r} w_i(x)(A_i x + B_i u)}{\sum_{i=1}^{r} w_i(x)} \tag{2}$$

where w_i is defined as

$$w_i(x) = \prod_{j=1}^{n} \mu_{ij}(x_j) \tag{3}$$

where μ_{ij} is the membership function of jth fuzzy set in the ith rule. Defining

$$\alpha_i(x) = \frac{w_i(x)}{\sum_{i=1}^{r} w_i(x)} \tag{4}$$

we can write (2) as

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(x)(A_i x + B_i u) \tag{5}$$

where $\alpha_i > 0$ and $\sum_{i=1}^{r} \alpha_i = 1$
Using the same method for generating T-S fuzzy rules for the controller, we have

i^{th}Controller Rule: IF $x_1(t)$ is M_{i1} and $\ldots x_n(t)$ is M_{in} THEN $u = -K_i x$

The overall controller is given by

$$u = -\sum_{j=1}^{r} \alpha_j(x) K_j x \tag{6}$$

Replacing (6) in (5), we obtain the following equation for the closed loop system:

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(x)\alpha_j(x)(A_i - B_i K_j)x \tag{7}$$

We the have the following theorem for closed-loop stability

Theorem 1 *[17]:The closed-loop fuzzy system (7) is globally asymptotically stable if there exist a common, positive-definite matrix P which satisfies the following Lyapunov inequalities:*

$$\begin{array}{rcll} (A_i - B_i K_i)^T P + P(A_i - B_i K_i) & < & 0 & i = 1, \cdots, r \\ G_{ij}^T P + P G_{ij} & < & 0 & j < i \leq r \\ P & > & 0 & \end{array} \tag{8}$$

where G_{ij} is defined as

$$G_{ij} = A_i - B_i K_j + A_j - B_j K_i \tag{9}$$

Although the conditions given in the above theorem guarantee stability, they can be quite conservative. The reason is that these onditions are independent of the shape of membership functions, and are the same whether the α_i's are membership functions or uncertain parameters. Recently these stability conditions have been relaxed in [13]. The relaxed stability conditions are given in the following theorem.

Theorem 2 *[13]:The closed-loop fuzzy system (7) is globally asymptotically stable, if there exist a common, positive-definite matrix P, and a positive-semidefinite matrix U which satisfy the following Lyapunov inequalities:*

$$\begin{array}{rcll} (A_i - B_i K_i)^T P + P(A_i - B_i K_i) + (s-1)U & < & 0 & i = 1, \cdots, r \\ G_{ij}^T P + P G_{ij} - U & < & 0 & j < i \leq r \\ U & \geq & 0 & \\ P & > & 0 & \end{array} \tag{10}$$

where s is the maximum number of rules fired at every instance of time, and G_{ij} is defined as in (9).

These stability conditions are less conservative when the membership functions overlap. For example, in the most common case of 50% overlap between the membership functions, i.e., two rules being fired for all times, s is equal to 2.

Pre-multiplying and post-multiplying both sides of the inequalities in (10) by P^{-1} and using the following change of variables

$$\begin{aligned} Y &= P^{-1} \\ X_i &= K_i Y \\ Z &= YUY \end{aligned} \tag{11}$$

we obtain the following LMIs [13]:

$$\begin{aligned} YA_i^T + A_iY - B_iX_i - X_i^TB_i^T + (s-1)Z &< 0 \quad i = 1, \cdots, r \\ Y(A_i + A_j)^T + (A_i + A_j)Y - M_{ij} - M_{ij}^T - Z &< 0 \quad j < i \leq r \\ Y &> 0 \\ Z &\geq 0 \end{aligned} \tag{12}$$

where M_{ij} is defined as:

$$M_{ij} = B_iX_j + B_jX_i \tag{13}$$

The feasibility of the above LMIs guarantees stability, but in most practical problems, stability is just a primary goal and performance is also usually required. In the next section, we develop a guaranteed-cost framework for the design of T-S fuzzy systems [5].

3 Guaranteed-Cost Performance Design Using LMIs

It is a well known result that the problem of minimizing an upper bound on the linear quadratic performance measure

$$J = \int_0^\infty (x(t)^TQx(t) + u(t)^TRu(t))dt \tag{14}$$

subject to the LTI system

$$\dot{x} = Ax + Bu \qquad u = -Kx \tag{15}$$

can be transformed into the following optimization problem, subject to a set of Matrix Inequalities [2]:

minimize: $x_0{}^TPx_0$

Subject to:

$$\begin{aligned} (A - BK)^TP + P(A - BK) + Q + K^TRK &< 0 \\ P &> 0 \end{aligned} \tag{16}$$

The above optimization problem can then be transformed into a convex optimization problem using the first two change of variables in (11). To avoid the dependency of the minimum cost on initial conditions, we assume that initioal conditions are randomized with zero mean and a covariance equal to the identity, therefore we minimize the expected value of the cost function J with respect to all possible initial conditions x_0 with [4]:

$$\begin{aligned} \mathbb{E}\{\mathrm{x_0 x_0^T}\} &= I \\ \mathbb{E}\{\mathrm{x_0}\} &= 0 \end{aligned} \tag{17}$$

Therefore, our optimization problem will be transformed to a trace minimization problem subject to the matrix inequalities in (16). This result can be extended to nonlinear systems approximated by T-S fuzzy models as follows [5].

Theorem 3 *Consider the closed-loop fuzzy system (7). We have the following bound on the performance objective J*

$$J = \mathbb{E}_{x_0} \int_0^\infty (x^T Q x + u^T R u) dt < tr(P) \tag{18}$$

where P is the solution of the following inequalities

$$\begin{aligned} (A_i - B_i K_i)^T P + P(A_i - B_i K_i) + Q + (s-1)U + \sum_{i=1}^{r} K_i^T R K_i < 0 \quad i = 1, \cdots, r \\ G_{ij}^T P + P G_{ij} + Q - U + \sum_{i=1}^{r} K_i^T R K_i < 0 \quad j < i \leq r \end{aligned} \tag{19}$$

u is defined in equation (6), and M_{ij} is the same as in (13).

The proof can be easily obtained by combining the proof given in [5] with the relaxed stability conditions in [13]. The key point in the proof is to note that

$$(\sum_{i=1}^{r} \alpha_i K_i)^T R (\sum_{i=1}^{r} \alpha_i K_i) < \sum_{i=1}^{r} K_i^T R K_i \tag{20}$$

■

Using the change of variables in (11) and utilizing the LMI lemma [2, 4], the inequalities in (19) can be transformed into the following LMIs:

$$\begin{aligned} \begin{bmatrix} YA_i^T + A_i Y - B_i X_i - X_i^T B_i^T + (s-1)Z & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ Q^{1/2} Y & -I_{n\times n} & 0 & \cdots & 0 \\ R^{1/2} X_1 & 0 & -I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2} X_r & 0 & 0 & \cdots & -I_{m\times m} \end{bmatrix} &< 0 \\ \begin{bmatrix} Y(A_i + A_j)^T + (A_i + A_j)Y - M_{ij} - M_{ij}^T - Z & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ Q^{1/2} Y & -I_{n\times n} & 0 & \cdots & 0 \\ R^{1/2} X_1 & 0 & -I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2} X_r & 0 & 0 & \cdots & -I_{m\times m} \end{bmatrix} &< 0 \\ Y &> 0 \\ Z &\geq 0 \\ i = 1, \cdots, r \quad j < i \leq r & \end{aligned} \tag{21}$$

To obtain the least possible upper-bound using a quadratic Lyapunov function, we have the following optimization problem

Min $tr(Y^{-1})$
Subject To: LMIs in(21)
This is a convex optimization problem which can be solved in polynomial time [11] using any of the available LMI toolboxes. To make it possible to use *Matlab*'s LMI Toolbox, we introduce an artificial variable Z, which is an upper bound on Y^{-1}, and minimize $tr(Z)$ instead, i.e, we recast the problem in the following form

Min $tr(Z)$
Subject To LMIs in(21), and

$$\begin{bmatrix} Z & I_{n\times n} \\ I_{n\times n} & Y \end{bmatrix} > 0 \tag{22}$$

$$\tag{23}$$

If the above LMIs are feasible, we can calculate the controller gains as

$$K_i = X_i Y^{-1}$$

The global controller can then be obtained as in (6).

4 Control of the Benchmark Problem

In this section, we apply the results obtained so far to the nonlinear benchmark problem (1). We linearize the equations of the benchmark system around two points, 0° and 80°. The linearized model around zero is obtained by finding the Jacobian of the system, while for the second point, $\cos x$ is approximated with $\beta = \cos 80°$. Simulations are performed for $\epsilon = 0.5$. We obtain the following T-S fuzzy model for the system
Plant Rule (1): If x_3 is *close to zero* **Then** $\dot{x} = A_1 x + B_1 u$
Plant Rule (2): If x_3 is *close to* $\pm\pi/2$ **Then** $\dot{x} = A_2 x + B_2 u$
where *close to zero* and *close to* $\pm\pi/2$ are the input fuzzy sets defined by the membership functions

$$\mu_1 = 1 - \frac{2}{\pi}|x_3| \quad \mu_2 = \frac{2}{\pi}|x_3|$$

respectively, (see Figure 1), and A_1, A_2, B_1, B_2 are given as follows

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{1-\epsilon^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon}{1-\epsilon^2} & 0 & 0 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -\frac{\epsilon}{1-\epsilon^2} \\ 0 \\ \frac{1}{1-\epsilon^2} \end{bmatrix} \\
A_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{1-\epsilon^2\beta^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon\beta}{1-\epsilon^2\beta^2} & 0 & 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -\frac{\epsilon\beta}{1-\epsilon^2\beta^2} \\ 0 \\ \frac{1}{1-\epsilon^2\beta^2} \end{bmatrix} \\
\beta &= \cos(80°)
\end{aligned} \tag{24}$$

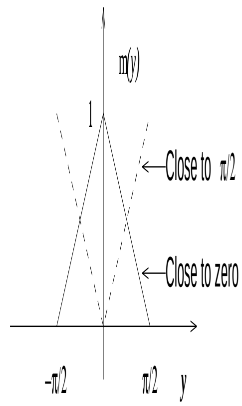

Figure 1: Membership functions for the two fuzzy sets *close to zero* and *close to* $\pm\pi/2$.

Simulation results indicate that our control method can stabilize the system for initial conditions wit x_3 up to 80° even in the presence of disturbances. The system is simulated in the presence of the sinusoidal disturbance of $\sin 20t$. The results are depicted in Figure 2, and clearly show that the disturbance is attenuated in magnitude. The control torque in Figure 3 ois within the se limits in the statement of the benchmark problem.

5 Conclusions

In this paper, we presented a new approach to control the nonlinear benchmark problem using a T-S modeling methodology and LMIs. We used a recently developed guaranteed-cost approach with relaxed stability conditions and then reduced these conditions to LMIs. The results turned out to be quite satisfactory and in the ranges set forth for the benchmark system. Note that teh approach is quite general and may be applied to other nonlinear systems by using more rules and different linearization points. Further research can be done in this area by trying to design dynamic T-S output feedback controllers using an asymptotic T-S observer [6].

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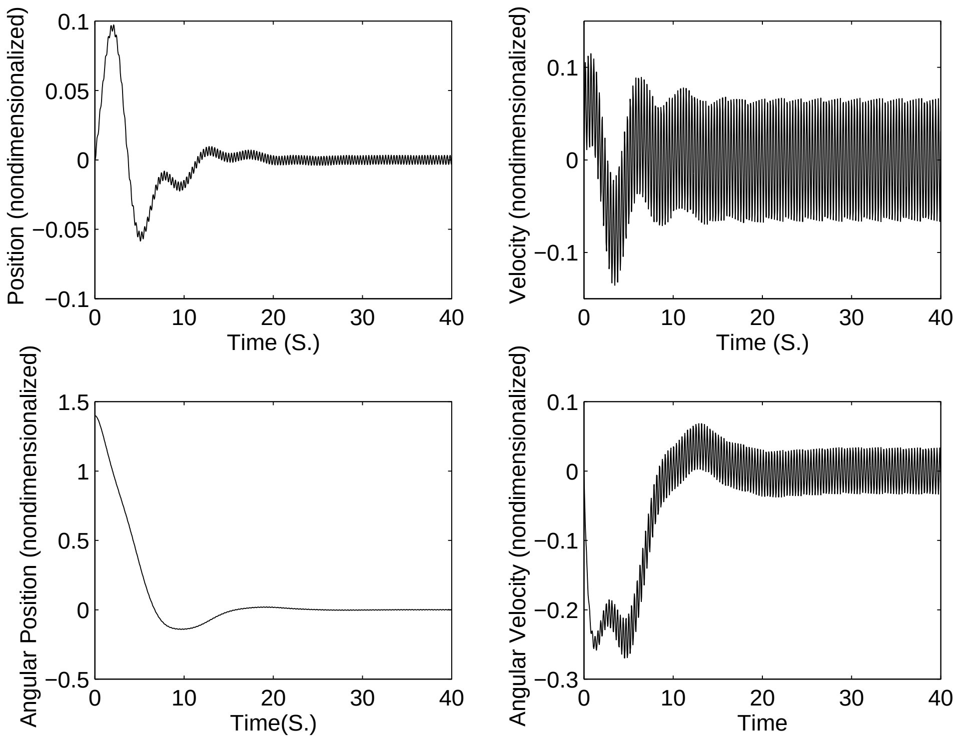

Figure 2: Initial Condition Response of the benchmark system with sinusoidal disturbance.

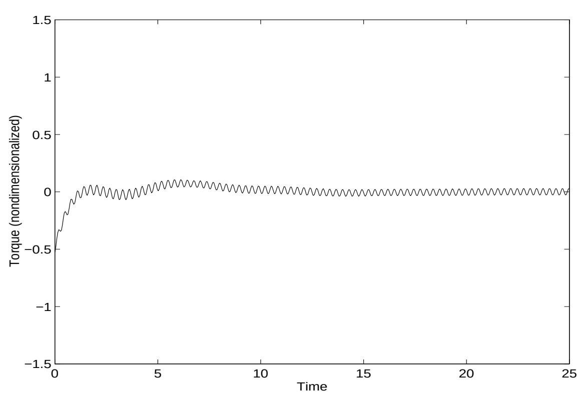

Figure 3: Control action in the presence of disturbance.

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