Internet-like Protocols for the Control and Coordination of Multiple Agents with Time Delay

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Abstract

In this paper we show how Internet-like protocols may be used to coordinate and control the usage of a resource by n agents. Lyapunov second method is used to provide sufficient stability conditions of the dynamics of the n agents in the presence of time delay.

Keywords: Coordination, multiple agents, time delay, utility function.

I. INTRODUCTION

The coordination of multiple systems presents unique challenges and remains a worthy goal in situations where humans need to project efforts and control over large distances. Particularly challenging problems arise when multiple robots are used in space applications to construct large space structures or to manipulate material. For example, various teleautonomous applications have been proposed (see [15], [17]) to assist in the construction of space systems meant to provide beamed power for commercial applications [16]. While the physics and economics of such systems are being worked out, our particular research focuses on the distributed, coordinated, and robust control of multiple agents representing robots engaged in an assembly task in space.

The idea of internet-like protocols in control can be applied to such systems in which a group of n users (clients) share a common resource (server)of finite capacity C. While Information systems such as the Internet are only concerned with transferring the information (with high fidelity), networked control applications are more involved due to the effects that dated information has on the control of dynamical systems [9], [10], [11], [12], [13]. To quote Traub [14], information is "incomplete, priced, and corrupted". However, and for control purposes, it is also "timed". To make the general idea of Internet-like control more precise, we consider a network of n users of a resource C. If for user i, the state variable x_i represents its usage of the resource, then, it is desired that the system reaches the equilibrium point at which

$$\sum_{i=1}^{n} x_i = C \tag{1}$$

It is also necessary to define a feedback signal, from the resource to the users, which communicates the availability or shortage of the resource. In [1], [3] a priced scheme has been used for this feedback. A low price is an indication of resource availability, and a resource shortage is represented by an increase in price.

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A simple dynamical equation for the resource price is the integral of the difference between the resource usage and the size of the resource, scaled by a positive gain γ, as follows

$$\dot{p}(t) = \gamma \left[\sum_{i=1}^{n} x_i(t) - C \right] \tag{2}$$

Different dynamical interactions between the resource usage $x_i(t)$, and the resource price feedback may be used. The common thread in these models is the inverse relation, such as the additive increase multiplicative decrease used in TCP Reno [3], e.g.

$$\dot{x}(t) = \frac{1 - p(t)}{\tau^2} - \frac{1}{2} p(t) x^2(t) \tag{3}$$

Where $p(t)$ is the price for the resource, and τ is the propagation delay between the user and the resource. Equation (3) yields the following equilibrium point, with the natural assumption that $x \geq 0$ and $p \geq 0$,

$$\frac{2 - 2p}{p\tau^2} = x^2$$

Another interesting inverse relation between the resource price p, and the resource usage x, is $x = \frac{a}{p}$, where a is a constant positive parameter of the user. As we will see below, this parameter a dictates the speed of convergence of the user. Figure 1 illustrates the relation between x and p, with $a = 1$.

For a given resource price p, there corresponds a unique resource usage, or equilibrium point x. Such

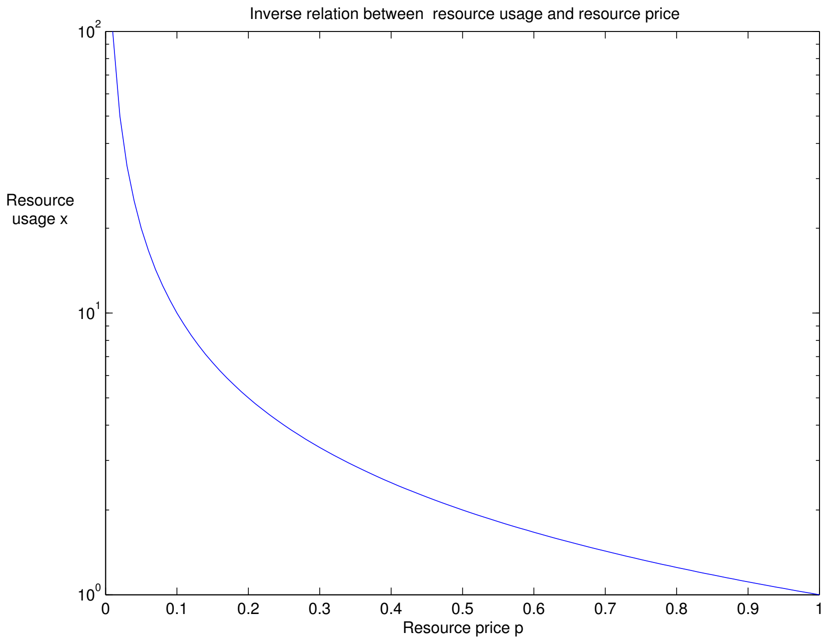

Fig. 1. Plot of resource usage versus resource price, showing the inverse relation

equilibrium point, or inverse relation, can be obtained from six different differential equations, although the dynamic behavior of these equations may be very different

$$\begin{array}{ll} \dot{x}(t) = \dfrac{a}{x(t)} - p(t) & \dot{x}(t) = -\frac{a}{x(t)} + p(t) \\ \dot{x}(t) = x(t) - \dfrac{a}{p(t)} & \dot{x}(t) = -x(t) + \frac{a}{p(t)} \\ \dot{x}(t) = x(t)p(t) - a & \dot{x}(t) = -x(t)p(t) + a \end{array}$$

In this paper we focus our attention on the last differential equation, given that we are interested in the additive increase multiplicative decrease behavior:

$$\dot{x}(t) = -x(t)p(t) + a. \tag{4}$$

This model has been used ([1], [3]) to associate a utility function $U(x)$ to each user, which in turns adjusts its resource usage x to optimize its utility function. These utility functions must satisfy the following conditions:

- If the i^{th} user attempts to maximize $U_i(x_i)$, then $U_i(x_i)$ must be strictly concave, such that there exists a unique maximizer x_i^*.
- If the i^{th} user attempts to minimize $U_i(x_i)$, then $U_i(x_i)$ must be strictly convex, such that there exists a unique minimizer x_i^*.

The condition for a unique optimizer may be obtained by forcing the gradient $\frac{d}{dx}U(x)$ to be the update equation for the resource usage, i.e. by making the resource update a gradient system [6]. One interesting application of the utility function is its use as a Lyapunov function for stability analysis [2].
Following this line of thought and integrating (4) over the resource usage, results in the following utility function

$$\int_0^x \dot{x}\, dx = U(x(t), p(t)) = ax(t) - \frac{1}{2}x^2(t)p(t) \tag{5}$$

A plot of the utility function as a function of the resource usage and resource pricing is shown in Figure 2.

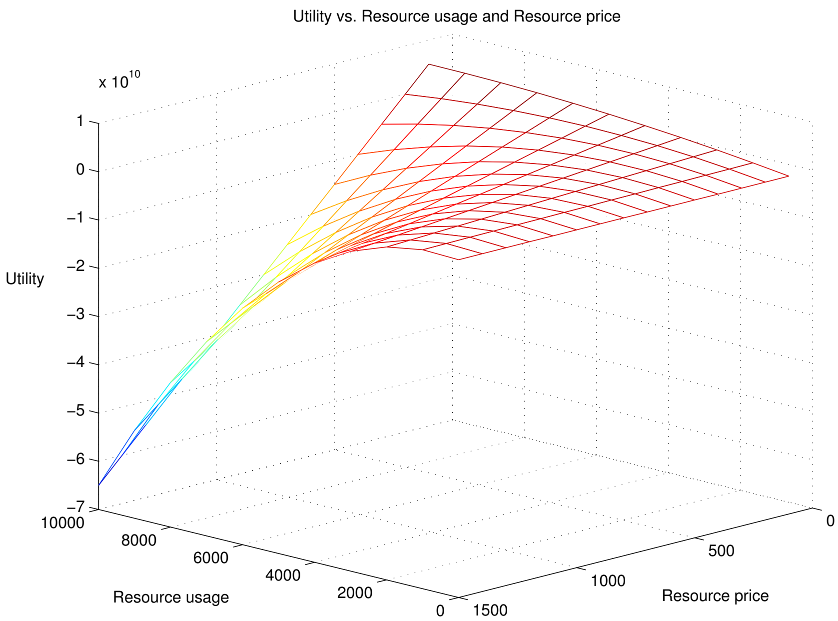

Fig. 2. Utility function as a function of the resource usage and resource price

After this brief overview, we present in section II the case of n users and illustrate the behavior of the coupled n-users system. Section III presents the stability analysis for a 1-user system in the presence of time delays, while section IV presents the same study in the case of n-users system, and we conclude our paper in section V.

II. System with n users

In this section we extend the system (4) into a multiuser case. The system with n users and one resource of size C is represented by the following differential equations,

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t)p(t) + a_1 \\ \dot{x}_2(t) &= -x_2(t)p(t) + a_2 \\ &\vdots \\ \dot{x}_n(t) &= -x_n(t)p(t) + a_n \\ \dot{p}(t) &= \gamma[x_1(t) + x_2(t) + \cdots + x_n(t) - C] \end{aligned} \tag{6}$$

where $p(t) \geq 0, x_i(t) \geq 0, \forall t \in \mathbb{R}^+$. The equilibrium point $(x_1^*, x_2^*, \ldots, x_n^*, p^*)$ for the system (6) is given by

$$x_i^* = \frac{a_i}{p^*}, \quad p^* = \frac{\sum_{i=1}^n a_i}{C}. \tag{7}$$

In order to study the stability of the equilibrium point using a Lyapunov function, we first translate the equilibrium point of (6) to the origin with the following change of variables

$$\begin{aligned} &y_i(t) = x_i(t) - x_i^*, \quad \text{for } 1 \leq i \leq n \\ &y_m(t) = p(t) - p^*, \quad m = n + 1 \end{aligned} \tag{8}$$

Given that $\dot{y}_i(t) = \dot{x}_i(t)$, then

$$\begin{aligned} \dot{y}_i(t) &= -\Big(y_i(t) + \frac{a_i C}{S}\Big)\Big(y_m(t) + \frac{S}{C}\Big) + a_i \\ \dot{y}_m(t) &= \gamma\Big[\sum_{i=1}^n \Big[y_i(t) + \frac{a_i C}{S}\Big] - C\Big] \end{aligned}$$

where $S \triangleq \sum_{i=1}^n a_i$. Simplifying,

$$\begin{aligned} \dot{y}_i(t) &= -\frac{S}{C}y_i(t) - y_i(t)y_m(t) - \frac{a_i C}{S}y_m(t) \\ \dot{y}_m(t) &= \gamma\Big[\sum_{i=1}^n y_i(t)\Big] \end{aligned} \tag{9}$$

The translated nonlinear system (9) can be arranged in order to have the general bilinear representation as in [4].

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{y}_n(t) \\ \dot{y}_m(t) \end{bmatrix} = \begin{bmatrix} -\frac{S}{C} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{S}{C} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{S}{C} & 0 \\ \gamma & \gamma & \cdots & \gamma & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \\ y_m(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \\ y_m(t) \end{bmatrix} y_m(t) + \frac{C}{S} \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \\ 0 \end{bmatrix} y_m(t) \tag{10}$$

or more compactly:

$$\begin{bmatrix} \dot{\bar{y}} \\ \dot{y}_m \end{bmatrix} = \left[\begin{array}{c|c} A & 0_{(n-1)\times 1} \\ \hline \gamma_{1\times(n-1)} & 0 \end{array}\right] \begin{bmatrix} \bar{y} \\ y_m \end{bmatrix} + \left[\begin{array}{c|c} -I_{(n-1)\times(n-1)} & 0_{(n-1)\times 1} \\ \hline 0_{1\times(n-1)} & 0 \end{array}\right] \begin{bmatrix} \bar{y} \\ y_m \end{bmatrix} y_m + \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] y_m \tag{11}$$

Theorem 1: The system (10) is globally asymptotically stable.
Proof: To analyze the stability of system (10), we use the quadratic Lyapunov function

$$V(y) = \frac{1}{2} y^T P y$$

where $P > 0$ is an $m \times m$ diagonal matrix. Then $V(y) > 0$ for $y \neq 0$, and $V(0) = 0$. Taking the time derivative of $V(y)$ we obtain

$$\dot{V}(y) = \frac{1}{2}\left[\dot{y}^T P y + y^T P \dot{y}\right] = \sum_{i=1}^{m} y_i P_i \dot{y}_i$$

Expanding the terms

$$\dot{V}(y) = \sum_{i=1}^{n} \left(-\frac{S}{C} P_i y_i^2(t) - P_i \frac{a_i C}{S} y_i(t) y_m(t) - P_i y_i^2(t) y_m(t) + \gamma P_m y_i(t) y_m(t) \right)$$

We can cancel out the cross product terms by choosing $P_i = \frac{S\gamma}{a_i C}$, for $1 \leq i \leq n$, and $P_m = 1$, thus simplifying

$$\dot{V}(y) = -\sum_{i=1}^{n} y_i^2(t) \left(\frac{S\gamma}{a_i C} y_m(t) + \frac{S^2 \gamma}{a_i C^2} \right)$$

In order to ensure that $\dot{V}(y) < 0$ we need the term inside the parenthesis to be positive, leading to

$$\frac{S\gamma}{a_i C} y_m(t) + \frac{S^2\gamma}{a_i C^2} = \frac{S\gamma}{a_i C}\left(y_m(t) + \frac{S}{C} \right) > 0$$

By definition, a_i, γ, $S = \sum_{i=1}^{n} a_i$, and C are positive, and since

$$y_m(t) + \frac{S}{C} = p(t) > 0 \quad \text{for all } t$$

then, the system (6) is asymptotically stable for all $x_i(t) > 0$ and $p(t) > 0$.

□

A. Linearization Approach

We can get some insight into the effects of the parameters a_i and γ on the speed of convergence by linearizing the system in (10) about the equilibrium point, which leads to:

$$A = \frac{d}{dy} f(\mathbf{0}) = \begin{pmatrix} -\frac{S}{C} & 0 & \cdots & 0 & -\frac{a_1 C}{S} \\ 0 & -\frac{S}{C} & \cdots & 0 & -\frac{a_2 C}{S} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{S}{C} & -\frac{a_n C}{S} \\ \gamma & \gamma & \cdots & \gamma & 0 \end{pmatrix}$$

where

$$f(y) = \begin{pmatrix} \dot{y}_1(t) \\ \vdots \\ \dot{y}_n(t) \\ \dot{y}_m(t) \end{pmatrix}$$

Solving for the eigenvalues of A, we obtain

$$\begin{aligned} \lambda_k &= -\frac{1}{C} \sum_{i=1}^{n} a_i \qquad 1 \leq k \leq n-1 \\ \lambda_n &= -\frac{1}{2C} \sum_{i=1}^{n} a_i + \frac{1}{2C} \sqrt{\sum_{i=1}^{n} a_i^2 + 2 \sum_{i=1}^{n-1} a_i \sum_{j=i+1}^{n} a_j - 4C^3 \gamma} \\ \lambda_{n+1} &= -\frac{1}{2C} \sum_{i=1}^{n} a_i - \frac{1}{2C} \sqrt{\sum_{i=1}^{n} a_i^2 + 2 \sum_{i=1}^{n-1} a_i \sum_{j=i+1}^{n} a_j - 4C^3 \gamma} \end{aligned} \tag{12}$$

The first $n-1$ eigenvalues in (12) are always negative, and larger a_i's result in large negative real parts of the eigenvalues. Note in addition that the gain γ defines the root loci for the last 2 eigenvalues. The parameter γ can then be selected to move all the eigenvalues into the left half plane.

Example 1: In order to see the response of this system, we ran a Simulink program for the case of 2 users. The parameters used are $a_1 = 1 * 10^5$, $a_2 = 2 * 10^5$, $C = 1 * 10^4$, and $\gamma = 0.018$. The left plot of Figure 3 shows the resource usage, the right plot of Figure 3 shows the resource price. The accumulated usage converges smoothly to the size of the resource C. This smooth convergence was attained for the specified γ, using the linearization derived in Section II-A, to avoid an overshoot. Also needed is an initial price larger than the equilibrium price to avoid an initial overshoot.

We can see that the user with the larger a_i receives proportionally more resource. The allocated resource x_i, is given by

$$x_i = \frac{a_i}{\sum_{i=1}^{n} a_i} C.$$

Note 1: In order to increase the speed of convergence to the equilibrium point, the user parameters a_i have to be increased by the same factor.

III. Analysis of a one-user system with time delay

Starting with the non-delayed system (see equation (9)), the differential equations for a single user and price feedback are

$$\begin{aligned} \dot{y}_1(t) &= -\frac{a}{C} y_1(t) - y_1(t) y_2(t) - C y_2(t) \\ \dot{y}_2(t) &= \gamma\, y_1(t) \end{aligned} \tag{13}$$

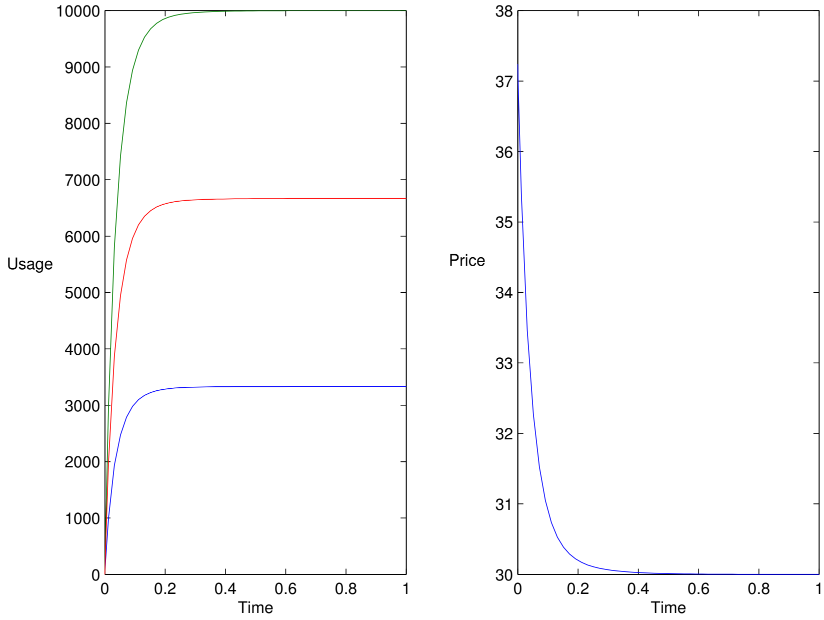

Fig. 3. Resource usage and resource price versus time, the 2 users case. In the left plot, user 1 is denoted by the blue line, user 2 by the red line, and the green line denotes the accumulated usage.

In what follows, we consider the case when the forward propagation time delay (τ_f), from the users to the resource, is equal to the backward propagation time delay (τ_b), from the resource to the users. This is the symmetric propagation delay case. Let $\tau = \tau_f = \tau_b$, then the delayed system is

$$\dot{y}_1(t) = -\frac{a}{C} y_1(t) - y_1(t) y_2(t-\tau) - C y_2(t-\tau) \quad (14)$$

$$\dot{y}_2(t) = \gamma \, y_1(t-\tau) \quad (15)$$

Grouping the terms multiplied by $y_2(t)$, and adding and subtracting $y_2(t+\tau)$ as in [5]

$$\begin{aligned} \dot{y}_1(t) &= -\frac{a}{C} y_1(t) - (y_1(t) + C) \cdot y_2(t+\tau) \\ &\quad + (y_1(t) + C) \cdot [y_2(t+\tau) - y_2(t-\tau)] \end{aligned} \quad (16)$$

To remove the positive time shift, we use the delayed differential equation

$$\begin{aligned} \dot{y}_1(t-\tau) &= -\frac{a}{C} y_1(t-\tau) - (y_1(t-\tau) + C) \cdot y_2(t) \\ &\quad + (y_1(t-\tau) + C) \cdot [y_2(t) - y_2(t-2\tau)] \end{aligned} \quad (17)$$

In order to eliminate $y_2(t)$ from the last term of (17), we integrate the differential equation for the price (15) from -2τ to 0.

$$\int_{-2\tau}^{0} \dot{y}_2(t+s) ds = \gamma \int_{-2\tau}^{0} y_1(t+s-\tau) ds \quad (18)$$

Applying the fundamental theorem of calculus,

$$y_2(t) - y_2(t-2\tau) = \gamma \int_{-2\tau}^{0} y_1(t+s-\tau) ds \quad (19)$$

and substituting (19) in (17), yields

$$\begin{aligned}\dot{y}_1(t-\tau) &= -\frac{a}{C}y_1(t-\tau) - (y_1(t-\tau)+C)\cdot y_2(t) \\ &+(y_1(t-\tau)+C)\cdot\gamma\int_{-2\tau}^{0} y_1(t+s-\tau)ds \qquad (20)\end{aligned}$$

Theorem 2: The system (20) is globally asymptotically stable if $\tau < \frac{p(t-2\tau)}{2\gamma C}$ for all t.
Proof: We propose the following Lyapunov function,

$$V(y) = y_1^2(t-\tau) + \frac{C}{\gamma}y_2^2(t) + \gamma C\int_{-2\tau}^{0}\int_{t+s}^{t} y_1^2(u-\tau)duds \qquad (21)$$

Taking time derivative to the Lyapunov function

$$\begin{aligned}\dot{V}(y) &= 2y_1(t-\tau)\left[-\frac{a}{C}y_1(t-\tau) - \Big(y_1(t-\tau)+C\Big)y_2(t-2\tau)\right] \\ &+2Cy_2(t)y_1(t-\tau) + \gamma C\int_{-2\tau}^{0}\left[y_1^2(t-\tau) - y_1^2(t+s-\tau)\right]ds \qquad (22)\end{aligned}$$

Expanding the terms

$$\begin{aligned}\dot{V}(y) &= -2\frac{a}{C}y_1^2(t-\tau) - 2y_1^2(t-\tau)y_2(t-2\tau) \\ &-2Cy_1(t-\tau)y_2(t-2\tau) + 2Cy_2(t)y_1(t-\tau) \\ &+\gamma C\int_{-2\tau}^{0}\left[y_1^2(t-\tau) - y_1^2(t+s-\tau)\right]ds \qquad (23)\end{aligned}$$

Using (18) and (19) with the third and fourth terms of (23), we get

$$\begin{aligned}\dot{V}(y) &= -2\frac{a}{C}y_1^2(t-\tau) - 2y_1^2(t-\tau)y_2(t-2\tau) \\ &+2\gamma Cy_1(t-\tau)\int_{-2\tau}^{0} y_1(t+s-\tau)ds \\ &+\gamma C\int_{-2\tau}^{0}\left[y_1^2(t-\tau) - y_1^2(t+s-\tau)\right]ds \qquad (24)\end{aligned}$$

Grouping the first two terms and the last two, yields

$$\begin{aligned}\dot{V}(y) &= -2y_1^2(t-\tau)\left[y_2(t-2\tau) + \frac{a}{C}\right] \\ &+\gamma C\int_{-2\tau}^{0}\Big[y_1^2(t-\tau) - y_1^2(t+s-\tau) \\ &+2y_1(t-\tau)y_1(t+s-\tau)\Big]ds \qquad (25)\end{aligned}$$

Using the following inequality

$$y_1^2(t-\tau) + y_1^2(t+s-\tau) \geq 2y_1(t-\tau)y_1(t+s-\tau)$$

We can place an upper bound on $\dot{V}(y)$, as follows

$$\dot{V}(y) \leq -2y_1^2(t-\tau)\left[y_2(t-2\tau) + \frac{a}{C}\right] + 2\gamma C\int_{-2\tau}^{0}\left[y_1^2(t-\tau)\right]ds \qquad (26)$$

The integrand ($y_1^2(t-\tau)$) is independent of s, so we can move it outside and complete the integration, thus

$$\dot{V}(y) \leq -2y_1^2(t-\tau)\left[y_2(t-2\tau)+\frac{a}{C}\right]+2\gamma C y_1^2(t-\tau)2\tau \tag{27}$$

Simplifying

$$\dot{V}(y) \leq -y_1^2(t-\tau)\left[y_2(t-2\tau)+\frac{a}{C}-2\gamma C\tau\right] \tag{28}$$

From the definition of the translated state variables $y_i(t)$ in (8) and the equilibrium point in (7), we can see that the first two terms inside the square brackets are the original price

$$p(t-2\tau) = y_2(t-2\tau)+\frac{a}{C}$$

Then, if we impose the following condition in (28)

$$p(t-2\tau)-2\gamma C\tau > 0$$

The original time-delay system in (10) is asymptotically stable.

□

This leads to the upper bound on the propagation delay τ,

$$\tau < \frac{p(t-2\tau)}{2\gamma C} \tag{29}$$

Note 2: Note that the upper bound on the delay τ decreases for larger values of the resource size C, but this can be compensated for by changing the gain γ. In fact, note that the delay upper bound is inversely proportional to the product γC.

IV. Analysis of a system with time delay and Multiple users

In this section, we analyze the multiuser case with different propagation delays. Thus, starting from (9) and assuming symmetric propagation delays for each user, we have

$$\begin{aligned} \dot{y}_i(t) &= -\frac{S}{C}y_i(t) - y_i(t)y_m(t-\tau_i) - \frac{a_iC}{S}y_m(t-\tau_i) \\ \dot{y}_m(t) &= \gamma\left[\sum_{i=1}^{n} y_i(t-\tau_i)\right] \end{aligned} \tag{30}$$

Adding and subtracting the positive time shift of $y_m(t)$, as we did in the single user case,

$$\begin{aligned} \dot{y}_i(t) &= -\frac{S}{C}y_i(t) - \left[y_i(t)+\frac{a_iC}{S}\right]y_m(t+\tau_i) \\ &\quad + \left[y_i(t)+\frac{a_iC}{S}\right]\left[y_m(t+\tau_i)-y_m(t-\tau_i)\right] \end{aligned} \tag{31}$$

leading to the following delayed differential equation:

$$\begin{aligned} \dot{y}_i(t-\tau_i) &= -\frac{S}{C}y_i(t-\tau_i) - \left[y_i(t-\tau_i)+\frac{a_iC}{S}\right]y_m(t) + \cdots \\ &\quad + \left[y_i(t-\tau_i)+\frac{a_iC}{S}\right]\left[y_m(t)-y_m(t-2\tau_i)\right] \end{aligned} \tag{32}$$

Using

$$y_m(t) - y_m(t-2\tau_i) = \gamma \int_{-2\tau_i}^{0} \sum_{j=1}^{n} y_j(t+s-\tau_j)ds \tag{33}$$

We obtain

$$\begin{aligned} \dot{y}_i(t-\tau_i) &= -\frac{S}{C}y_i(t-\tau_i) - \left[y_i(t-\tau_i) + \frac{a_iC}{S}\right] y_m(t) \\ &\quad + \left[y_i(t-\tau_i) + \frac{a_iC}{S}\right] \gamma \int_{-2\tau_i}^{0} \sum_{j=1}^{n} y_j(t+s-\tau_j)ds \end{aligned} \tag{34}$$

Theorem 3: The system (34) is globally asymptotically stable if:

$$\tau_i < \frac{S^2}{2n\gamma C^2 a_i}; \ \ \forall i = 1, \cdots, n \tag{35}$$

Proof: We purpose the following Lyapunov function

$$\begin{aligned} V(y) &= \sum_{i=1}^{n} \frac{1}{a_i} y_i^2(t-\tau_i) + \frac{C}{S\gamma} y_m^2(t) \\ &\quad + \frac{n\gamma C}{S} \sum_{i=1}^{n} \int_{-2\tau_i}^{0} \int_{t+s}^{t} y_i^2(u-\tau_i)duds \end{aligned} \tag{36}$$

Taking the time derivative of $V(y)$, leads to

$$\begin{aligned} \dot{V}(y) &= \sum_{i=1}^{n} \frac{2}{a_i} y_i(t-\tau_i)\Big[-\frac{S}{C}y_i(t-\tau_i) - \Big[y_i(t-\tau_i) + \frac{a_iC}{S}\Big] y_m(t-2\tau_i)\Big] \\ &\quad + \frac{2C}{S\gamma} y_m(t)\gamma\Big[y_1(t-\tau_1) + y_2(t-\tau_2) + \cdots + y_n(t-\tau_n)\Big] \\ &\quad + \frac{n\gamma C}{S}\Big[\int_{-2\tau_1}^{0} \big[y_1^2(t-\tau_1) - y_1^2(t+s-\tau_1)\big]ds + \cdots \\ &\quad + \int_{-2\tau_n}^{0} \big[y_n^2(t-\tau_n) - y_n^2(t+s-\tau_n)\big]ds\Big] \end{aligned} \tag{37}$$

Expanding terms

$$\begin{aligned} \dot{V}(y) &= -\frac{2S}{a_1C} y_1^2(t-\tau_1) - \frac{2}{a_1} y_1^2(t-\tau_1) y_m(t-2\tau_1) - \frac{2C}{S} y_1(t-\tau_1) y_m(t-2\tau_1) \\ &\quad \vdots \\ &\quad -\frac{2S}{a_nC} y_n^2(t-\tau_n) - \frac{2}{a_n} y_n^2(t-\tau_n) y_m(t-2\tau_n) - \frac{2C}{S} y_n(t-\tau_n) y_m(t-2\tau_n) \\ &\quad + \frac{2C}{S} y_1(t-\tau_1) y_m(t) + \cdots + \frac{2C}{S} y_n(t-\tau_n) y_m(t) \\ &\quad + \frac{n\gamma C}{S} \int_{-2\tau_1}^{0} \big[y_1^2(t-\tau_1) - y_1^2(t+s-\tau_1)\big]ds + \cdots \\ &\quad + \frac{n\gamma C}{S} \int_{-2\tau_n}^{0} \big[y_n^2(t-\tau_n) - y_n^2(t+s-\tau_n)\big]ds \end{aligned} \tag{38}$$

Grouping the last terms of the first n rows with the $(n+1)$th row using (33), and also grouping the first two terms together for the n first rows.

$$\begin{aligned}\dot{V}(y) &= -\sum_{i=1}^{n} y_i^2(t-\tau_i)\left[\frac{2}{a_i}\left[y_m(t-2\tau_i)+\frac{S}{C}\right]\right] \\ &+\frac{2\gamma C}{S}\sum_{i=1}^{n}\int_{-2\tau_i}^{0} y_i(t-\tau_i)\sum_{j=1}^{n} y_j(t+s-\tau_j)ds \\ &+\frac{n\gamma C}{S}\sum_{i=1}^{n}\int_{-2\tau_i}^{0}\left[y_i^2(t-\tau_i)-y_i^2(t+s-\tau_i)\right]ds \end{aligned} \tag{39}$$

Using the following inequality for the second term

$$y_i^2(t-\tau_i)+y_j^2(t+s-\tau_j)\geq 2y_i(t-\tau_i)y_j(t+s-\tau_j)$$

After applying this to the $n\times n$ cross products, we can set an upper bound on $\dot{V}(y)$ as follows

$$\begin{aligned}\dot{V}(y) &\leq -\sum_{i=1}^{n} y_i^2(t-\tau_i)\left[\frac{2}{a_i}\left[y_m(t-2\tau_i)+\frac{S}{C}\right]\right] \\ &+\frac{n\gamma C}{S}\sum_{i=1}^{n}\int_{-2\tau_i}^{0}\left[y_i^2(t-\tau_i)+y_i^2(t+s-\tau_i)\right]ds \\ &+\frac{n\gamma C}{S}\sum_{i=1}^{n}\int_{-2\tau_i}^{0}\left[y_i^2(t-\tau_i)-y_i^2(t+s-\tau_i)\right]ds \end{aligned} \tag{40}$$

We can reduce $\dot{V}(y)$ to

$$\begin{aligned}\dot{V}(y) &\leq -\sum_{i=1}^{n} y_i^2(t-\tau_i)\left[\frac{2}{a_i}\left[y_m(t-2\tau_i)+\frac{S}{C}\right]\right] \\ &+\frac{2n\gamma C}{S}\sum_{i=1}^{n} y_i^2(t-\tau_i)\int_{-2\tau_i}^{0} ds \end{aligned} \tag{41}$$

Evaluating the integral

$$\begin{aligned}\dot{V}(y) &\leq -\sum_{i=1}^{n} y_i^2(t-\tau_i)\left[\frac{2}{a_i}\left[y_m(t-2\tau_i)+\frac{S}{C}\right]\right] \\ &+\frac{2n\gamma C}{S}\sum_{i=1}^{n} y_i^2(t-\tau_i)(2\tau_i) \end{aligned} \tag{42}$$

And finally collecting terms

$$\dot{V}(y)\leq -\sum_{i=1}^{n} y_i^2(t-\tau_i)\left[\frac{1}{a_i}\left(y_m(t-2\tau_i)+\frac{S}{C}\right)-\frac{2n\gamma C\tau_i}{S}\right] \tag{43}$$

We thus require the term inside the square brackets to be positive. In fact, it is sufficient for stability to assume:

$$\sum_{i=1}^{n}\tau_i<\frac{S}{2n\gamma C}\sum_{i=1}^{n}\frac{y_m(t-2\tau_i)+\frac{S}{C}}{a_i} \tag{44}$$

In order to simplify this condition further, note from the definition of the translation variables that

$$y_m(t-2\tau_i)+\frac{S}{C}=p(t-2\tau_i) \tag{45}$$

Thus, we have the following inequality

$$\frac{p(t-2\tau_i)}{a_i}>\frac{2n\gamma C\tau_i}{S} \tag{46}$$

In order to guarantee asymptotic stability, we can thus set the following upper bound for each delay

$$\tau_i<\frac{Sp(t-2\tau_i)}{2n\gamma Ca_i} \tag{47}$$

We can see that this condition is conservative, given that we are forcing each one of the n terms inside the square brackets to be positive, when all that is needed is for (43) to be negative definite. Moreover, note that (43) is an upper bound to (39).

From the plot of the price in Figure 3, we can see that if the initial value for the price is greater than the equilibrium price, and with the appropriate selection of the gain γ, the price $p(t)$ will be, for all $t\geq 0$, greater or equal to the equilibrium price. Then, if we substitute $p(t-2\tau_i)$ by $\frac{S}{C}$ in (47), we can evaluate the upper bounds, although these bounds will be even more conservative. The expression for these upper bounds is

$$\tau_i<\frac{S^2}{2n\gamma C^2a_i} \tag{48}$$

□

As we can see in (48) the upper bound in the time delay for each user decreases for large values of the resource capacity, this condition is similar to what the authors in [3] found for their models of TCP Reno and RED. Also with a large number of users the upper bound is decreased. But decreasing the value of γ accordingly will diminish the effect of these previous parameters. Is important to note that the parameter a_i will give each user a different upper bound; users favored with a large resource allocation, given a large a_i, will have a lower upper bound than the others users.

Example 2: Using the values of the example 1, $a_1=100,000$, $a_2=200,000$, $\gamma=0.018$, $C=10,000$. Then $n=2$, and $S=\sum_{i=1}^{2}a_i=300,000$. The upper bounds are

$$\begin{aligned}\tau_1&<\frac{(300,000)^2}{2\cdot 2\cdot 0.018\cdot(10,000)^2\cdot(100,000)}=125\ msec\\ \tau_2&<\frac{(300,000)^2}{2\cdot 2\cdot 0.018\cdot(10,000)^2\cdot(200,000)}=62.5\ msec\end{aligned} \tag{49}$$

Using the same Simulink model of example 1, with the addition of the following delays, $\tau_1=125\ msec$, and $\tau_2=62.5\ msec$, results in the responses shown in Figure 4. As we can see in Figure 4, the overshoot is almost about 200%, but the accumulated usage converges to the resource size after 3 seconds.

As we mentioned before, (48) gives conservative upper bounds, but how conservative are such bounds is yet to be quantified.

V. CONCLUSIONS

The use of a bilinear equation as the inverse relation between resource usage and resource price let us obtain a Lyapunov function that in clean way gave us a proof for the stability of the multiuser system without delay. For the multiuser system with symmetric propagation time delay, we could obtain a Lyapunov

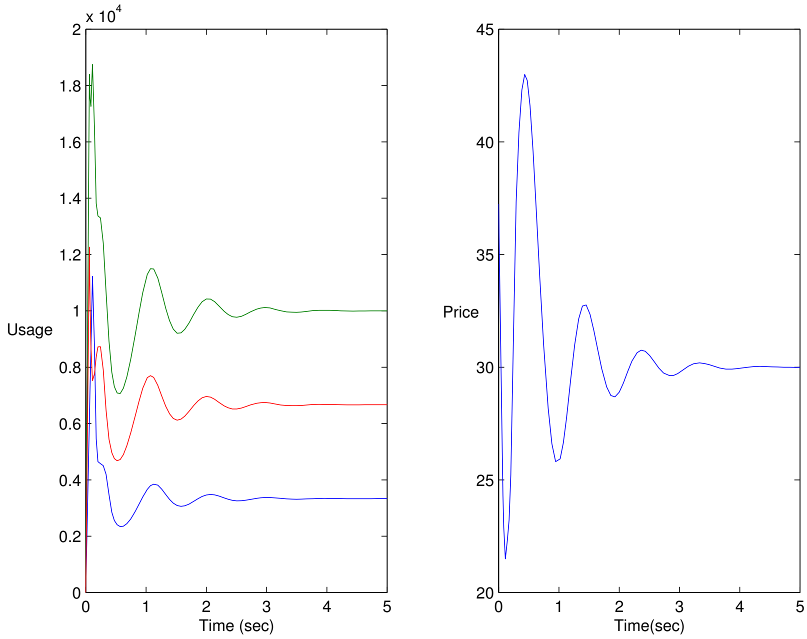

Fig. 4. Resource usage and resource price versus time, the 2 users case with the nominal delays in (49).

function which gave us upper bounds for the delay of each user, however, these upper bounds are too conservative. Moreover, can particular users increase their delays beyond the conservative bound if other users have much shorter delays? We are currently using sampling techniques [7], [8] to answer these questions and to obtain more practical bounds on the time delays. The use of sampling techniques allows us to obtain more practical bounds for the delays, and other metrics of the system responses. It will also allow us analyze the effects of varying the forward and backward delays, on the convergence and overshoot of the accumulated resource usage.

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