Advances in Undergraduate Control Education: The Analytical Design Approach

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Abstract

Introductory undergraduate control courses in the USA are generally limited to trial-and-error design techniques, based largely on the Nyquist stability criterion and root-locus analysis. The corresponding theory is well over fifty years old. Very little is presented on analytic design, where one has an existence theorem, and a computable algorithm to find a solution when one exists. One reason for the lack of *analytic design* in introductory courses is the level of mathematics required to understand much of this theory. Here we summarize some of the existing analytic design techniques, and their mathematical pre-requisites, and then we propose the interpolation approach for analytic design, as one requiring the least amount of mathematics.

1. Introduction

In this paper, we focus on one aspect of undergraduate control education which has been ignored in most undergraduate textbooks. One of the standard tools used for the design of feedback control systems is the *Nyquist stability criterion*. While this tool provides considerable insight into the analysis of feedback systems, it suffers from one very important limitation, that is it can be used only in an *trial-and-error* way in the design of a compensator, essentially because of the complicated way magnitude and phase are related for rational transfer functions. For example, given the Nyquist diagram of an unstable plant, it is difficult to answer even the simple question, does a stable compensator *exist* that will stabilize the closed-loop system? The compensator design problem is further complicated in the multivariable case where the criterion involves the computation of a determinant at each frequency point.

In contrast to *trial-and-error* techniques, *analytical design* techniques, always include the following two elements:

- *Conditions for the existence of a solution.*
- *An algorithm which is guaranteed to find the solution, when it exists.*

While analytical techniques appear to be very appropriate for design, they do have some limitations. One important limitation is that the compensator is generally more complex than that obtained by trial-and-error methods. Another is that most analytical techniques deal only with limited performance measures. Thus it is to the advantage of the designer to be familiar with both design techniques. It should be noted that *analytical design* techniques are also often referred to as *synthesis* techniques.

We will present one particular analytical approach to the design of compensators for feedback systems, the so-called *interpolation approach*. This approach has been successfully presented to our undergraduates students at the EECE Department of the University of New Mexico for the last five years. In the interpolation approach various feedback design problems are converted into problems of finding special rational functions which interpolate to given values at given points in the complex s-domain. We assume here that the plant (system being controlled) can be characterized by a rational transfer function in the Laplace-transform variable s (or in the Z-transform variable z, in the discrete-time case). This approach can be introduced with a minimal amount of mathematics, e.g. Laplace-transform theory and the concept of *bounded-input-bounded-output (BIBO)* stability. We outline in the next section some other analytical design techniques, which although requiring more mathematical background, have been introduced in some undergraduate textbooks and courses.

2. Some Analytical Design Methods

- *Mean-Square Design.* This approach is based on the minimization of the mean-square error

$$E\{e^2(t)\}$$

or equivalently an integral of the form

$$\int_{-\infty}^{\infty} |E(j\omega)|^2 \, d\omega \tag{1}$$

The analytical solution of the problem requires the *spectral factorization* of a polynomial, that is the factorization of a polynomial into the product of a stable polynomial and an anti-stable polynomial. This is the approach taken in one of the first books on analytical feedback design, i.e. the text of Newton, Gould, and Kaiser [9] published in 1957. This approach requires knowledge of stochastic processes and complex variables, but allows one to design feedback systems which can deal with random disturbances and control-effort constraints. The level of complex variable analysis required for this approach is beyond most undergraduate engineering programs in the USA.

- *State-space Design.* This approach is based on a state-space representation of the plant, i.e. a representation of the form

  $$\dot{x} = Ax + Bu, \quad y = Cx \tag{2}$$

  where x, y, and u represent the plant state, output, and input, respectively. A basic result here is that if the system is *controllable* there exists a state-feedback controller , $u(t) = -Kx(t)$, such that the poles (eigenvalues) of the closed-loop system can be located arbitrarily. State-space theory for feedback design was introduced by Kalman in the early sixties [8]. Many textbooks are now available on this approach, see for example [7].
  One state-space design methodology, which is especially well suited for multivariable feedback systems, is the so-called *linear-quadratic (LQ)* theory. In the LQ theory the problem is to find a state-feedback control law which minimizes an integral quadratic performance measure of the form

  $$V = \int_0^{\infty} (x'Qx + u'Ru) \, dt$$

  It can be shown that this problem can be reduced to a solution of a matrix Riccati equation. See, for example, [2]. However a rather extensive knowledge of matrices (linear algebra) is required for this theory. Outside the USA undergraduate students generally have a good linear algebra background, and indeed this unit on analytic design is often taught in the first control course. However in the USA only about 25 % of undergraduate engineering programs require a course in linear algebra (See reference [1]) so that state-space methods would be difficult to cover in a first control course in the USA. However state-space methods is an analytic approach which is often included in a second control course.

- H^∞ *Design.* Recently a new theory for analytical feedback system design has evolved, based on the minimization of a performance measure of the form

  $$V = \sup_{\omega} |E(j\omega)| \tag{3}$$

  where, for example, $E(s)$ may be the Laplace transform of the error signal $e(t)$. When $E(s)$ is analytic in the RHP, then the value of V given in equation (3) is also called the H^∞ *norm* of $E(s)$.
  This H^∞ theory for feedback system design was developed by Zames and Francis in 1983 [16]. An early book on the approach is the text of Vidyasagar [13]. This theory requires rather advanced concepts in complex variables and matrices, and is generally not included in introductory courses in feedback design. For single-input-single-output (SISO) systems, a more mathematically accessible version of this theory, based largely on interpolation methods, may be found in the monograph of Doyle et.al. [5]. Unfortunately this text is now out of print.

We propose an interpolation approach for a unit of analytic design to be included in a first control course. The mathematics for this approach should be available to most undergraduate engineering students in the USA. Basically all one needs are the concepts of transfer function, bounded-input-bounded-output (BIBO) stability, and “good” pole-zero cancellations. A “book supplement” monograph is under preparation which presents this approach to analytic design (See Dorato [4]).

3. Internal Stability

A key concept in the design of feedback control systems using analytical methods, and interpolation theory in particular, is that of *internal stability.* Unfortunately this is a concept which is often not discussed in introductory texts. The basic idea is to insure that the closed-loop system is stable, not only between the command input and the controlled output, but also between internal points. This is important because disturbance signals can arise at all

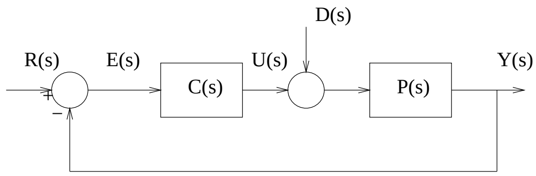

Figure 1: Block Diagram Of Feedback System

points in the closed-loop system. To define the concept, consider the feedback system shown in Figure 1.

Definition: The feedback system in Figure 1 is said to be *internally stable* if the three transfer functions:

$$E(s)/R(s) = \frac{1}{1+C(s)P(s)} \tag{4}$$

$$Y(s)/D(s) = \frac{P(s)}{1+C(s)P(s)} \tag{5}$$

$$U(s)/R(s) = \frac{C(s)}{1+C(s)P(s)} \tag{6}$$

are bounded-input-bounded-output (BIBO) stable.

Recall that a rational transfer function $G(s)$ is BIBO stable if and only if it is *proper* (the degree of the denominator polynomial is greater than or equal to the numerator polynomial), and its denominator polynomial is *Hurwitz* (all its roots have negative real-parts).

In the sequel we will let the "error" transfer function $E(s)/R(s)$ be denoted $S(s)$, and the "external" transfer function $Y(s)/R(s)$ be denoted $T(s)$. Note that $S(s)$ is also the *sensitivity function* for the closed-loop system, i.e.

$$S(s) = \frac{dT/T}{dG/G} = \frac{1}{1+C(s)P(s)} \tag{7}$$

Since $T(s) = 1 - S(s)$, the exterior transfer function $T(s)$ is often called the *complementary sensitivity function*.

Internal stability implies *external* stability (that is the stability of the transfer function $T(s) = Y(s)/R(s)$), but not conversely. For example, the PD compensator $C(s) = s+1$ externally stabilizes the plant $P(s) = 1/s^2$, however it does not internally stabilize the closed-loop system since the transfer function

$$U(s)/R(s) = \frac{C(s)}{1+C(s)P(s)} = \frac{s^3+s^2}{s^2+s+1}$$

is not proper, hence not BIBO stable. One bad consequence of internal instability in this particular case is that a step input reference signal $r(t)$ will result in an unbounded (impulse) control input $u(t)$.

Internal stability also guarantees that there are no "bad" pole/zero cancellations between controller $C(s)$ and plant $P(s)$. "Bad cancellations" are cancellations of *unstable* poles and zeros. An example of the bad effects of unstable pole/zero cancellation is to use the compensator $C(s) = (s-1)/(s+1)$ to "stabilize" the plant $P(s) = 1/(s-1)$. This compensator does yield the stable transfer $Y(s)/R(s) = 1/(s+2)$, however it results in the unstable transfer function

$$Y(s)/D(s) = \frac{P(s)}{1+C(s)P(s)} = \frac{(s+1)}{(s-1)(s+2)}$$

which means that a bounded disturbance signal $d(t)$ will result in an unbounded error signal $e(t)$.

4. Origins of The Interpolation Approach

One of the first discussions on what is now called "the interpolation approach" to control system synthesis, may be found in the classic text of Truxal [12], published in 1955. The basic idea presented there (which Truxal refers to as Guillemin's method), is to use the equation

$$C(s) = \frac{1}{P(s)}\frac{T(s)}{1-T(s)} \tag{8}$$

to design a compensator $C(s)$ which could be realized with an RC network. In those (pre-reliable op-amp) days, RC networks were considered the most practical way to electronically realize an analog compensator. The interpolation issue arises, for example, from equation (8) when one tries to avoid unstable pole zero cancellations. In particular if the plant has an unstable zero, i.e. a zero with positive real part, then, as can be seen from equation (8), the exterior transfer function $T(s)$ must "interpolate" to zero at the plant zero to avoid unstable pole/zero cancellation. This basic idea was expanded and applied to the design of digital control systems by Ragazzini and Franklin in their text [11], published in 1958.

5. Reduction of Feedback Design Problems to Interpolation Problems

We will focus on the problems of stability design with *stable* and *unstable* compensators. Other design problems e.g. gain-margin design, robust stabilization for unstructured plant perturbations, etc. may

be found in [4]. It is interesting to note that no existing introductory control text considers the basic question, *are there plants that cannot be stabilized with stable compensators*? Indeed most introductory texts are limited to trial-and-error techniques, where answering any existence question is very difficult.
To simplify the presentation it will be assumed that unstable poles and zeros are all *simple*. Also it is assumed that plant transfer functions have no unstable hidden modes and that all transfer functions are rational.

- *Stabilization with Stable Compensators.* The presentation here follows that in reference [13]. We assume that the plant is written as a ratio of two *stable* rational functions $N_p(s)$ and $D_p(s)$, i.e.

$$P(s) = \frac{N_p(s)}{D_p(s)} \tag{9}$$

Let the unstable zeros of the plant, including infinity, be denoted b_i. Consider now the compensator

$$C(s) = \frac{W(s) - D_p(s)}{N_p(s)} \tag{10}$$

If the function $W(s)$ is a BIBO stable function with a BIBO stable inverse (referred to a *BIBO-unit*) which interpolates to $D_p(s)$ at the unstable zeros of $N_p(s)$, i.e.

$$W(b_i) = D_p(b_i) \tag{11}$$

then the compensator given by (10) is *stable* and this stable compensator makes the closed-loop system internally stable. Internal stability is evident when $C(s)$ given in (10) is substituted into equations (4)-(6), resulting in

$$\begin{aligned} E(s)/R(s) &= \frac{D_p(s)}{W(s)} && (12)\\ Y(s)/D(s) &= \frac{N_p(s)}{W(s)} && (13)\\ U(s)/R(s) &= \frac{D_p(s)C(s)}{W(s)} && (14) \end{aligned}$$

The *existence condition* for this analytic design problem was first presented in reference [14] and may be stated as follows:

A stable internally stabilizing compensator exists for any plant where the number of poles between any pair of zeros on the non-negative real axis is even

This condition is referred to as the *parity-interlacing-property*, or the *p.i.p* condition. Most trial-and-error methods are based on the assumption that the compensator is stable. e.g. stable lag/lead. This critical result can save the designer a lot of time, yet is not to be found in existing introductory control texts.
In general interpolating with BIBO-units is not trivial. However for a a limited number of unstable poles and zeros, interpolation may be possible with a low-order unit.

- *Stabilization with Possibly Unstable Compensators.* To design with a compensator which may have to be unstable, consider the following compensator structure (See reference [16])

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)} \tag{15}$$

If this compensator is substituted into equations (4)-(6) on obtains,

$$\begin{aligned} E(s)/R(s) &= 1 - P(s)Q(s) && (16)\\ Y(s)/D(s) &= P(s)(1 - P(s)Q(s)) && (17)\\ U(s)/R(s) &= Q(s) && (18) \end{aligned}$$

¿From these equations it is clear the $C(s)$ given by (15) will internally stabilize the closed-loop system, if $Q(s)$ is any *BIBO stable function* which interpolates to $Q(a_i) = 0$ and $1 - P(a_i)Q(a_i) = 0$ at the unstable poles a_i of the plant $P(s)$. Interpolating with BIBO-stable functions can be reduced to solving a system of linear equations (See, for example, [4]), hence this is an easy interpolation problem. It can be shown that the compensator which results is never of degree greater that $n-1$ where n is the degree of the plant, and that a possibly unstable compensator (when *p.i.p* is not satisfied the compensator must be unstable) *always exists*!

6. Conclusions

The main conclusion of this paper is that it is possible to introduce a unit of analytic design in the first undergraduate control course here in the USA, using the interpolation approach. Some of the time spent on trial-and-error techniques can be reduced to make room for this unit. The interpolation approach also provides a transfer-function alternative to analytic state-space techniques which are generally covered in a second control course in the USA. For the same reason the interpolation approach may be of value outside the USA (where state-space methods are normally covered in a first course) to balance transfer-function methods with state-space methods.

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