On the Design of Non-Fragile Compensators via Symbolic Quantifier Elimination

P. Dorato*, C.T. Abdallah*, and D. Famularo**
* EECE Department
University of New Mexico
Albuquerque, NM 87131-1356, USA
** Dipartimento di Elettronica, Informatica e Sistemistica
Università degli Studi della Calabria
Contrada Arcavacata di Rende
Rende (CS) 87036, ITALY

Abstract

In this paper symbolic quantifier elimination methods are used to explore the *fragility* of feedback compensators, and to design feedback systems with *non-fragile* compensators. A compensator is said to be *fragile* if given variations in compensator parameters result in significant deterioration of feedback performance. The issue of fragility is important in understanding the level of acurracy required to implement a given compensator design.

1 Introduction

It is generally known that feedback control systems require accurate compensators, compared to relatively inaccurate plants. However, for some optimal designs, the accuracy may be so high that serious performance deterioration, including loss of stability, may occur even for very small perturbations in compensator parameters. The compensator in this case is said to be *fragile* [1]. The problem arises often in in optimal H^2 and H^∞ problems where the optimal compensator is of high order (order higher than the plant). Reference [1] includes a number of interesting examples of “fragile” compensators. The need for a “safety margin” for compensator parameters is also mentioned on page 75 of reference [2], in the context of robust feedback design.

The obvious solution to the fragility problem is to design with low-order fixed-structure compensators, sacrificing optimality. But this generally leads to nonlinear programming problems, with *non-convex* constraint functions. In reference [3] a “guaranteed-cost” approach is used to design non-fragile compensators with fixed-order compensators. In [3] the plant is assumed to be known, the performance objective is linear-quadratic, and the design objective is to minimize an upper bound on the linear-quadratic performance objective, (in order to “minimize” the loss of optimality due to compensator inaccuracies). As usual for fixed-order linear-quadratic guaranteed cost design, the sufficient conditions for optimality obtained in [3] involve a set of coupled Riccati equations.

In this paper we will explore some alternate approaches to non-fragile design, that are relevant to robust control problems where the design objectives can be reduced to *multivariate polynomial inequalitites*. As shown in [4], [5] and [7] many robust linear and nonlinear design problems can be reduced to problems of this type. In particular if the vector p denotes a set of uncertain plant parameter, the vector q denotes a set of compensator parameters (for a fixed-order compensator), and ω denotes angular frequency, then many robust feedback design problems in the frequency domain can be reduced to the satisfaction of inequalities of the form

$$F_i(p, q, \omega) > 0 \tag{1}$$

where F_i are multivariate polynomial functions (functions that are polynomials in a given variable when all the other variables are held fixed). Robust feedback design problems may be formulated by placing logic quantifiers of the "for all" ($\forall$) type and the "there exists" ($\exists$) type on Boolean formulas of the type

$$F_1(p, q, \omega) \wedge F_2(p, q, \omega) \wedge ... \tag{2}$$

Quantifier elimination theory allows one to elimate the quantifiers on the above Boolean function producing a quantifier-free Boolean fromula in the design vector q, which then may be used to select an non-fragile compensator operating point. We will explore symbolic quantifier elimination methods, as described as in [4] and [5], to design non-fragile compensators. The sofware package QEPCAD, [6], is used to symbolically eliminate quantifiers. This approach can only be used with problems of limited complexity. However it yields exact solutions for admissible compensator parameters. For more complex problems Bernstein polynomial branch-and-bound techniques, as described in [7], may be used to numerically eliminate quantifiers. Both these appraoches give a set of compensator-parameter values where robust design specifications are met. A non-fragile design is then selected by choosing a value of the design vector q in a suitable interior point of the admissible region.

2 Multivariate Polynomial Inequalities and Feedback Design

We assume the plant is defined by its tranfer function $G(s, \mathbf{p})$, and the compensator by its transfer function $C(s, \mathbf{q})$. We also assume a simple unity feedback structure. The closed-loop tranfer functions that will be of interest to us here is the "control effort" transfer function

$$T(s) = \frac{C(s, \mathbf{q})}{1 + C(s, \mathbf{q})G(s, \mathbf{p})} \tag{3}$$

Closed-loop stability may be tested by using the Routh-Hurwitz criterion on the closed-loop characteristic polynomial, i.e. on the numerator polynomial of the rational function

$$1 + C(s, \mathbf{q})G(s, \mathbf{p}) \tag{4}$$

If the parameter vectors components p_i and q_i appear polynomially in the transfer functions C and G, then the Routh-Hurwitz test will produce a set of multivariate polynomial inequalities (MPI's) in the variables p_i and q_i. In addition we will be interested in frequency domain design specifications of the form

$$|T(j\omega)| \leq \alpha_U, \forall \omega \tag{5}$$

If both sides of inequality (5 are squared and the fraction cleared, one once more obtains an MPI. The design problem is then reduced to the satisfaction a system of inequalities such as in [**?**], quantified by the $\forall$ logic operation over admissible ranges of p_i and ω. When the quantifiers are eliminated one obtain ranges of compensator parameter values, q_i, which guaranteed the satisfaction of design specification on robust stability and control effort.

In general a trade-off is required between non-fragility and design specifications. This trade-off is illustrated in the numerical example that follows.

3 Numerical Example

To show the effect of the trade-off between performance and fragility, we consider the following plant [4]

$$G(s, \mathbf{p}) = \frac{1}{s + p_1}, \; p_1 = \pm 1 \, .$$

This is an example of a plant which undergoes a catastrophic perturbation, and goes from a stable plant to an unstable plant. We are looking for a controller $C(s, \mathbf{q})$ which stabilizes the plant and satisfies the following design objectives:

1. zero steady state tracking error to a step command input;
2. the closed loop plant must exhibit an acceptable level α_U of control effort.

To satisfy the steady-state tracking error requirements, a PI controller is necessary

$$C(s, \mathbf{q}) = q_1 + \frac{q_2}{s}, \tag{6}$$

and, in order to test the fragility vs. performance characteristics of the controller (6), QE theory is used to explore how the admissibility region Ω in the parameter space (q_1, q_2) changes as the control effort level α_U is varied.
The design objectives are translated in the following polynomial inequalities:

- **Stability:** The characteristic polynomial has the following expression

  $$N(s) = s^2 + (q_1 + p_1)s + q_2$$

  and the stability requirements are expressed by

  $$F_1(\mathbf{p}, \mathbf{q}) = q_1 + p_1 > 0, \; F_2(\mathbf{p}, \mathbf{q}) = q_2 > 0 \, ;$$

- **Control Effort:** If α_U^2 is approximated by a ratio of two integers $\frac{n}{d}$, then the control-effort constraint leads to the inequality

  $$F_3(\mathbf{p}, \mathbf{q}, \omega) = (n - dq_1^2)\omega^4 + (n((p_1 + q_1)^2 - 2q_2) - d(q_1^2 + p_1^2 q_2^2))\omega^2 + (nq_2^2 - dq_2^2 p_1^2) \geq 0 \, .$$

If the control effort level α_U is fixed, the set of admissible parameters (q_1, q_2) which satisfies the desing requirements is given by the following logical formula

$$(\forall \omega)(\forall \mathbf{p}) \, [\mathbf{p} = 1 \wedge \mathbf{p} = -1] \Rightarrow [F_1(\mathbf{p}, \mathbf{q}) > 0 \wedge F_2(\mathbf{p}, \mathbf{q}) > 0 \wedge F_3(\mathbf{p}, \mathbf{q}, \omega) \geq 0] \tag{7}$$

and, in reference ([4]), the set of values of the control effort for which a solution to the problem exists was found to be

$$\alpha_U^2 > 4 \, .$$

The numerical experiment are based on the determination and the analysis of the admissibility region for four values of α_U^2, $(16, 9, 4.5, 4.1)$. QEPCAD software was used to obtain the following quantifier-free Boolean formula

$$\Omega = \Omega_1 \vee \Omega_2. \tag{8}$$

The closed form of the admissibility region as a set of *unquantified* polynominal inequalities in the (q_1, q_2) variables are given (shaded areas in Figures 1,2,3,4):

1. $(\alpha_U^2 = 16)$

$$\begin{array}{rcl} \Omega_1 & = & (\quad q_1 - 4 \leq 0 \quad \wedge \quad 5q_1 - 4 \geq 0 \quad \wedge \quad q_2 > 0 \quad \wedge \quad f_1(q_1, q_2) \leq 0 \quad) \\ \Omega_2 & = & (\quad 5q_1 - 4 \leq 0 \quad \wedge \quad q_2 > 0 \quad \wedge \quad f_2(q_1, q_2) \leq 0 \quad) \end{array} \tag{9}$$

where

$$\begin{array}{rcl} f_1(q_1, q_2) & = & q_2^2 + 32q_2 - 15q_1^2 + 32q_1 - 16 \\ f_2(q_1, q_2) & = & q_2^4 + 64q_2^3 + 30q_1^2q_2^2 + 64q_1q_2^2 + 32q_2^2 - 960q_1^2q_2 + 2048q_1q_2 - \\ & & 1024q_2 + 225q_1^4 - 960q_1^3 + 1504q_1^2 - 1024q_1 + 256 \end{array}$$

2. $(\alpha_U^2 = 9)$

$$\begin{array}{rcl} \Omega_1 & = & (\quad q_1 - 3 \leq 0 \quad \wedge \quad 2q_1 - 3 \geq 0 \quad \wedge \quad q_2 > 0 \quad \wedge \quad f_1(q_1, q_2) \leq 0 \quad) \\ \Omega_2 & = & (\quad 2q_1 - 3 \leq 0 \quad \wedge \quad q_2 > 0 \quad \wedge \quad f_2(q_1, q_2) \leq 0 \quad) \end{array} \tag{10}$$

where

$$\begin{array}{rcl} f_1(q_1, q_2) & = & q_2^2 + 18q_2 - 8q_1^2 + 18q_1 - 9 \\ f_2(q_1, q_2) & = & q_2^4 + 36q_2^3 + 16q_1^2q_2^2 + 36q_1q_2^2 + 18q_2^2 - 288q_1^2q_2 + 648q_1q_2 - \\ & & 324q_2 + 64q_1^4 - 288q_1^3 + 468q_1^2 - 324q_1 + 81 \end{array}$$

3. $(\alpha_U^2 = 4.5)$

$$\begin{array}{rcl} \Omega_1 & = & (\quad q_1 - 1 \leq 0 \quad \wedge \quad 2q_1^2 - 9 \leq 0 \quad \wedge \quad q_2 > 0 \quad \wedge \quad f_1(q_1, q_2) \leq 0 \quad) \\ \Omega_2 & = & (\quad q_1 - 1 \geq 0 \quad \wedge \quad q_2 > 0 \quad \wedge \quad f_2(q_1, q_2) \leq 0 \quad) \end{array} \tag{11}$$

where

$$\begin{array}{rcl} f_1(q_1, q_2) & = & 2q_2^2 + 18q_2 - 7q_1^2 + 18q_1 - 9 \\ f_2(q_1, q_2) & = & 4q_2^4 + 72q_2^3 + 28q_1^2q_2^2 + 72q_1q_2^2 + 36q_2^2 - 252q_1^2q_2 + 648q_1q_2 - \\ & & 324q_2 + 49q_1^4 - 252q_1^3 + 450q_1^2 - 324q_1 + 81 \end{array}$$

4. $(\alpha_U^2 = 4.1)$

$$\begin{array}{rcl} \Omega_1 & = & (\quad q_1 - 1 \geq 0 \quad \wedge \quad 10q_1^2 - 41 \leq 0 \quad \wedge \quad q_2 > 0 \quad \wedge \quad f_1(q_1, q_2) \leq 0 \quad) \\ \Omega_2 & = & (\quad q_1 - 1 \geq 0 \quad \wedge \quad q_2 > 0 \quad \wedge \quad f_2(q_1, q_2) \leq 0 \quad) \end{array} \tag{12}$$

where

$$\begin{array}{rcl} f_1(q_1, q_2) & = & 10q_2^2 + 82q_2 - 31q_1^2 + 82q_1 - 41 \\ f_2(q_1, q_2) & = & 100q_2^4 + 1640q_2^3 + 620q_1^2q_2^2 + 1640q_1q_2^2 + 820q_2^2 - 5084q_1^2q_2 + 13448q_1q_2 - \\ & & 6724q_2 + 961q_1^4 - 5084q_1^3 + 9266q_1^2 - 6724q_1 + 1681 \end{array}$$

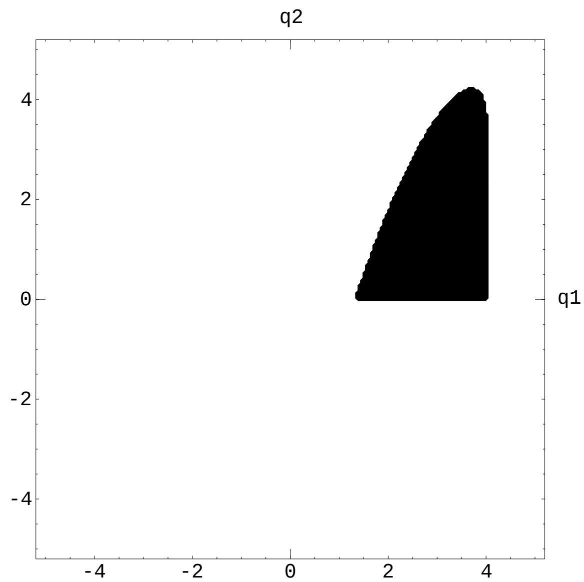

Figure 1: Admissibility region ($\alpha_U^2 = 16$)

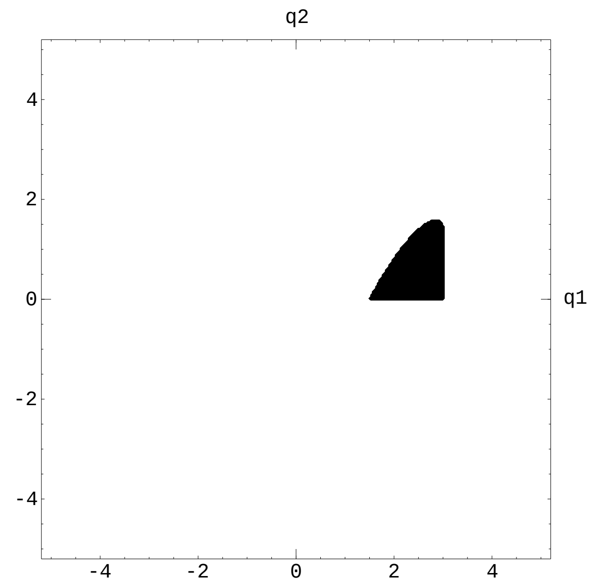

Figure 2: Admissibility region ($\alpha_U^2 = 9$)

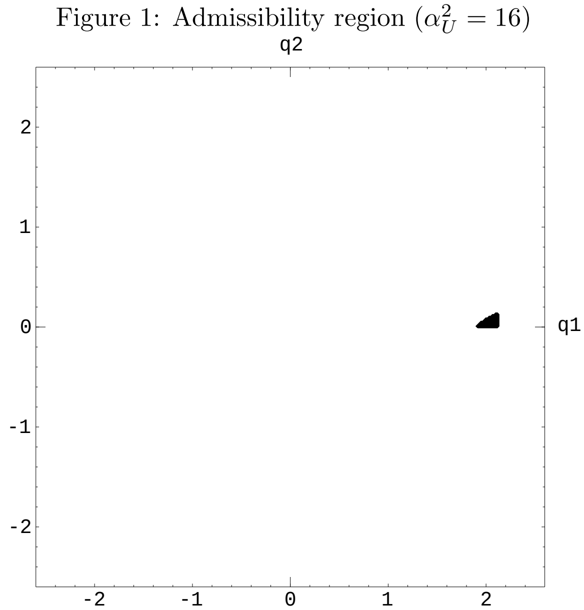

Figure 3: Admissibility region ($\alpha_U^2 = 4.5$)

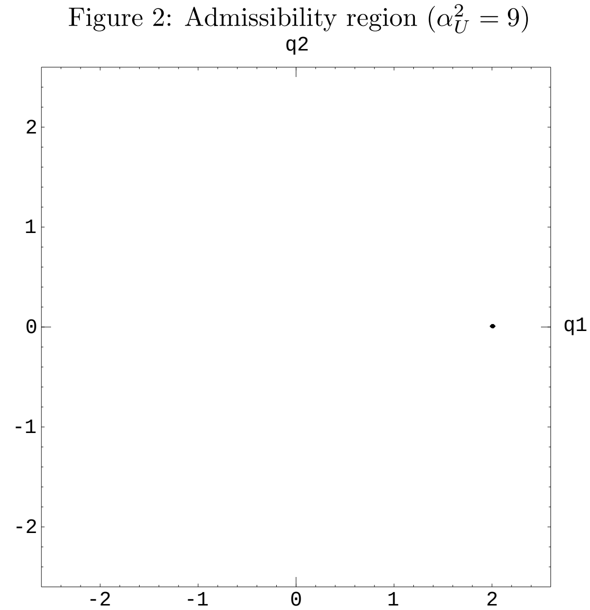

Figure 4: Admissibility region ($\alpha_U^2 = 4.1$)

It's easy to see that the effect of "pushing" α_U to its lower limit has its counterpart in the reduction of the admissibility region to a "point" in the (q_1, q_2) parameters space. When α_U^2 is equal to 4.5 and 4.1 the control effort performance is nearly optimal but the controller itself is *fragile* because tolerances in the parameters are not allowed due to the restricted admissibility area (Figures 3,4).

References

[1] L. H. Keel and S. P. Bhattacharyya, *Robust, fragile, or optimal?*, American Control Conference, Albuquerque, NM, June 4-6, 1997.

[2] J. Ackermann, *Robust Control*, Springer-Verlag, Berlin, 1993.

[3] W. M. Haddad and J. R. Corrado, *Non-fragile controller design via quadratic Lyapunov bounds*, Preprint, Georgia Institute of Technology, March 1997.

[4] P. Dorato, Wei Yang, and C. Abdallah, *Robust multi-objective feedback design by quantifier elimination*, Journal of Symbolic Computation, Vol. 24 (1997), pp. 153-159.

[5] P. Dorato, Wei Yang, D. Famularo, *Robust nonlinear feedback design via quantifer elimination theory*, 2nd IFAC Symposium on Robust Control Design, Budapest, Hungary, June 25-27, 1997.

[6] G. E. Collins and H. Hong, *Partial cylindrical algebraic decomposition for quantifier elimination*, Journal of Symbolic Computation, Vol. 12 (1991), pp. 299-328.

[7] S. Malan, M. Milanese, M. Taragna, *Robust analysis and design of control systems using interval arithmetics*, Proc. IFAC World Congress (Volume H), San Francisco, CA, June 30-July 5, 1996, pp. 25-30. Also accepted for publication in Automatica.