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NON-FRAGILE SYNTHESIS WITH STRUCTURED AND/OR UNSTRUCTURED PERTURBATIONS

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Abstract

This report discusses the synthesis of Non-fragile or resilient regulators for linear systems. The general framework for fragility is described using state-space methodologies, and the static state feedback case is examined in detail. We discuss the multiplicative and additive uncertainties case, and propose remedies of the fragility problem.

1 Introduction

The purpose of this report is to address and understand the effects of uncertainties in the implementation of regulators which optimize some performance and robustness criteria in linear systems theory. In the literature, there are different algorithms that give an answer to the following classical problem shown in Figure 1:

Given a linear plant with some uncertainties find a feedback controller which internally stabilizes the plant and satisfies some performance requirements,
We consider two classes of uncertainties to characterize our lack of knowledge about a linear plant:

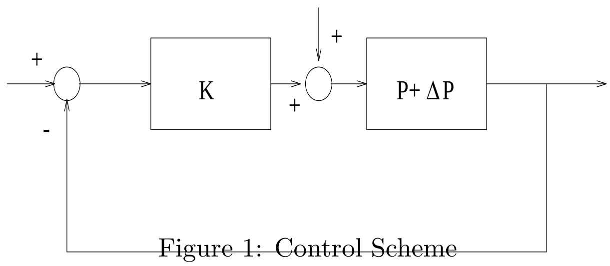

Figure 1: Control Scheme

- **Structured uncertainties**, which represent the effect of (generally) slowly time-varying parameters whose exact values are unknown and belong to an uncertainty set [2];
- **Unstructured uncertainties**, which represent the effect of unmodeled dynamics and in general are defined in terms of the $\mathcal{H}_\infty$ norm of a class of transfer functions [5].

Almost all the algorithms proposed in the literature do not consider the problems introduced by implementing uncertain controllers. We first remark that is is reasonable to consider only structured uncertainties in the controller since by design, one can choose the exact structure of the controller. The controllers obtained using most robust deign approaches are thus *optimal* if implemented exactly. There are however many reasons to believe that one can never exactly implement a compensator which theoretically meets all objectives. Moreover, it is easy to argue that even when exact implementation is possible, some tuning by the control engineer is required on the actual controller.

In a recent paper, Keel and Bhattacharyya [9] have shown that, in the case of unstructured uncertainties in the plant, and using $\mathcal{H}_\infty$, $\mathcal{H}_2$ and l_1 synthesis, the resulting controllers exhibit a poor stability margin [2]. This so-called "fragility" is displayed despite (or because of?) the fact that these controllers are optimal when implemented using their nominal parameters. Paper [9] ends with some considerations which attempt to overcome the fragility problem. Among the suggestions given, are the following:

1. Developing synthesis algorithms which take into account some structured uncertainties inside the controllers and searching for the "best" solution that guarantees a compromise between optimality and fragility,
2. Examining at the structure of the controller in order to parameterize it in a useful way (lower-order or fixed-structure controller).

In a preprint paper, Haddad and Corrado [8] address and solve the fragility problem by considering a *structured uncertain* dynamic compensator for a noise-driven linear plant. They obtain sufficient conditions by bounding the uncertainties in the controller using classical quadratic Lyapunov bounds [1]. The resulting controllers are proven to be "resilient" in the sense that even when they are not exactly implemented, stability and some measure of performance are guaranteed.

It is true that other authors have hinted at the problem of fragility [**?**] and that many critics have dismissed the issue, since robust controllers are not designed to be resilient. On the other hand, the problem is reminiscent of the LQG optimal controllers which were only useful when implemented on the exact plant, and had no robustness margins if the plant was uncertain. This lack of robustness was corrected using LQG/LTR [?]. In addition, even robust controllers will eventually have to be implemented on an actual system using digital hardware and should be resilient both to implementation errors (discretization, round-off errors, etc) and to tuning [?].

This report extends the ideas in the two papers [9, 8] and it tries to analyze the robust fragility problem by considering the combined effect of structured uncertainties both in the plant and in the compensator. The basic idea of the report is that instead of computing the controller as a single point in the parameters space, we look for a region using an *a priori* information. This is reminiscent of the designs of Ackermann [?] and of those in [?].

This report is organized as follows. In section 2, we outline the general non-fragile synthesis problem. In section 3, we consider first the case of simple static state feedback by allowing structured uncertainties in the feedback gain matrix. Multiplicative structured uncertainties schemes are then considered and numerical examples using Linear Matrix Inequality theory are given. Our conclusions and directions of future research are then given in section ??.

2 General Framework

Let us consider the following linear system

$$\left\{\begin{array}{rclcccc} \dot{x}(t) &=& A(t)x(t) &+& B^{(1)}(t)u(t) &+& B^{(2)}(t)w(t) \\ z(t) &=& C^{(1)}(t)x(t) &+& D^{(1)}(t)u(t) && \\ y(t) &=& C^{(2)}(t)x(t) &+& && D^{(2)}(t)w(t) \end{array}\right. \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{m_1}$ is the control input, $w(t) \in \mathbb{R}^{m_2}$ is an exogenous signal vector which collects bounded-norm disturbances, $y(t) \in \mathbb{R}^{p_2}$ is the output measurements vector, $z(t) \in \mathbb{R}^{p_1}$ is the output objective vector, and $A(t)$, $B^{(j)}(t)$, $C^{(j)}(t)$, $D^{(j)}(t)$, $j = 1, 2$ represent polytopic and/or affine uncertain system matrices. These uncertain matrices can be written in the form

$$\begin{array}{rcl} A(t) &=& A + \delta A(t) = A + \sum_{i=1}^{q} \alpha_i(t) A_i, \\ B^{(j)}(t) &=& B^{(j)} + \delta B^{(j)}(t) = B^{(j)} + \sum_{i=1}^{q} \alpha_i(t) B_i^{(j)},\ j = 1, 2 \\ C^{(j)}(t) &=& C^{(j)} + \delta C^{(j)}(t) = C^{(j)} + \sum_{i=1}^{q} \alpha_i(t) C_i^{(j)},\ j = 1, 2 \\ D^{(j)}(t) &=& D^{(j)} + \delta D^{(j)}(t) = D^{(j)} + \sum_{i=1}^{q} \alpha_i(t) D_i^{(j)},\ j = 1, 2 \end{array}$$

the matrices $A, A_i \in \mathbb{R}^{n\times n}$, $B^{(j)}, B_i^{(j)} \in \mathbb{R}^{n\times m_j}$, $C^{(j)}, C_i^{(j)} \in \mathbb{R}^{p_j\times n}$, $D^{(j)}, D_i^{(j)} \in \mathbb{R}^{p_j\times m_j}$ are known and the scalar coefficients $\alpha_i(t)$ represent unknown but slowly-varying coefficients whose values belong to an uncertainty interval

$$\underline{\alpha}_i \le \alpha_i(t) \le \overline{\alpha}_i. \tag{2}$$

Following [?], We look for a fixed-structure dynamic controller of the type

$$\left\{\begin{array}{rcl} \dot{\xi}(t) &=& F(t)\xi(t) + G(t)y(t) \\ u(t) &=& H(t)\xi(t) \end{array}\right. \tag{3}$$

where $\xi(t) \in \mathbb{R}^{n_c}$ $(n_c \le n)$ and $F(t)$, $G(t)$, $H(t)$ represent uncertain matrices of the form

$$\begin{array}{rcl} F(t) &=& F + \delta F(t), \\ G(t) &=& G + \delta G(t), \\ H(t) &=& H + \delta H(t), \end{array}$$

the matrices $F \in \mathbb{R}^{n_c \times n_c}$, $G \in \mathbb{R}^{n_c \times p}$, $H \in \mathbb{R}^{m_1 \times n_c}$ are unknown and the terms $\delta F(t), \delta G(t), \delta H(t)$ have different structures depending on the type of uncertainty (additive, multiplicative) we are faced with. In the following, we suppress the time dependence in the δ-terms. The closed-loop system has the following structure

$$\left\{ \begin{array}{lcl} \begin{pmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{pmatrix} & = & \mathcal{A} \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} + \mathcal{B} w(t) \\ z(t) & = & \mathcal{C} \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} \end{array} \right. \tag{4}$$

where

$$\mathcal{A} = \begin{bmatrix} A + \delta A & (B^{(1)} + \delta B^{(1)})(H + \delta H) \\ (G + \delta G)(C^{(2)} + \delta C^{(2)}) & F + \delta F \end{bmatrix} \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$$

$$\mathcal{B} = \begin{bmatrix} B^{(2)} + \delta B^{(2)} \\ (G + \delta G)(D^{(2)} + \delta D^{(2)}) \end{bmatrix} \in \mathbb{R}^{(n+n_c) \times m_2}$$

$$\mathcal{C} = \begin{bmatrix} (C^{(1)} + \delta C^{(1)}) & (D^{(1)} + \delta D^{(1)})(H + \delta H) \end{bmatrix} \in \mathbb{R}^{p_1 \times (n+n_c)}.$$

The problem can now be stated as follows:

Robust Fragility Design: *Find F, G, H matrices such that (4) is stable and some performance index $\mathcal{J}$ is minimum over all uncertainties $\delta A, \delta B^{(j)}, \delta C^{(j)}, \delta D^{(j)}, \delta F, \delta G, \delta H$, $(j = 1, 2)$.*

The index $\mathcal{J}$, depending on the type of optimization problem considered, may be expressed in one the following forms

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \operatorname{tr}\left(z(t) z^T(t)\right) \, dt \quad \text{(for } \mathcal{H}_2 \text{ optimization)} \tag{5}$$

$$\lim_{T \to \infty} \int_0^T \left(z^T(t) z(t) - \gamma^2 w^T(t) w(t)\right) \, dt \quad \text{(for } \mathcal{H}_\infty \text{ optimization)} \tag{6}$$

Mixed schemes $\mathcal{H}_2/\mathcal{H}_\infty$ [**?**] are also possible.

We are immediately faced with two issues when we consider the problem of robust non-fragile control design:

1. The dimension n_c of the controller and its relation to the fragility characteristics of the controller;
2. The presence in the closed loop matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of cross-coupling terms which make it difficult to apply convex analysis algorithms (LMI) to the solution of the problem.

Nevertheless, it is possible to solve simpler problems and to obtain analytical results which may help in understanding the effect of mixing uncertainties in the plant and in the controller. In the following section, we consider such a simple case, namely that the Full Information Static State Feedback case.

3 Static State Feedback

A special case of fragility concerns the synthesis of state feedback regulators for systems of type (1) where $A(t)$, $B(t)$, $C(t)$ represent polytopic and/or affine uncertainties (see [7]). Consider the simplified linear plant (1) where $B(t)$ and $C(t)$ are equal to the constant matrices B and C respectively,

$$\begin{aligned} \dot{x} &= (A + \sum_{i=1}^{q} \alpha_i A_i) x + Bu = (A + \delta A) x + Bu \\ y(t) &= C(t) x(t) \end{aligned} \tag{7}$$

It is useful to note that this model is similar to a stochastic differential equation with multiplicative noise [4]. The problem is then to find a state feedback compensator of the form $u = Kx$ which retains an acceptable closed-loop performance despite being implemented as

$$u = (K + \delta K)x \tag{8}$$

where K is unknown, and the term δK represents the drifting effect from the nominal solution. In the following section, we address this model in greater detail.

3.1 Multiplicative structured uncertainties

Let us suppose that the nominal state-feedback matrix K is a $p_1 \times n$, $(p_1 < n)$ matrix. If we allow drift from the nominal entries of the matrices K and represent each entry of the perturbed matrix as a multiplicative scalar uncertainty, we have

$$(K + \delta K) = \begin{bmatrix} k_{11}(1+\delta_{11}) & \dots & k_{1n}(1+\delta_{1n}) \\ \vdots & \ddots & \vdots \\ k_{p1}(1+\delta_{p1}) & \dots & k_{pn}(1+\delta_{pn}) \end{bmatrix} \tag{9}$$

It is in general difficult to decouple the effect of the disturbances δ_{ij} from the entries of the matrix K but, in the single input case, we obtain the following perturbation model

$$(K + \delta K) = \begin{bmatrix} k_1(1+\beta_1) & \dots & k_n(1+\beta_n) \end{bmatrix} \tag{10}$$

where β_j are scalar coefficients ($|\beta_j| \leq \tilde{\beta}_j < 1$). We can then write the controller as,

$$u = K(I_n + \sum_{k=1}^{n} \beta_k \Psi_k)x = K(I + \Delta)x \tag{11}$$

where Ψ_j are $n \times n$ rank-one matrices with entries equal to 1 located at the (j, j) position of the main diagonal with respect to the entries of the j-th column of the K matrix. Given the initial state $x(0)$, the performance index to be minimized is given by

$$J = \int_0^\infty \left(x^T C^{(1)T} C^{(1)} x + u^T D^{(1)T} D^{(1)} u \right) \, dt \,. \tag{12}$$

If we substitute into equation (12) the expression of the controller (11), we have that

$$J = \int_0^\infty \left(x^T C^{(1)T} C^{(1)} x + x^T K^T (I+\Delta) D^{(1)T} D^{(1)} (I+\Delta) K x \right) \, dt \,. \tag{13}$$

The index J can be easily bounded by noting the following matrix bound

$$(I+\Delta) D^{(1)T} D^{(1)} (I+\Delta) \leq \left(1 + \max_j \tilde{\beta}_j \right)^2 D^{(1)T} D^{(1)} = (1+\theta)^2 \, D^{(1)T} D^{(1)}$$

and this corresponds to consider the following guaranteed cost problem,

$$J \leq \mathcal{J} = \int_0^\infty \left(x^T C^{(1)T} C^{(1)} x + (1+\theta)^2 \, \hat{u}^T D^{(1)T} D^{(1)} \hat{u} \right) \, dt \tag{14}$$

where $\hat{u} = Ky$. Let us then examine the closed loop system:

$$\dot{x}(t) = \left(A + \sum_{i=1}^{q} \alpha_i A_i + BK \left(I_n + \sum_{j=1}^{n} \beta_j \Psi_j \right) \right) x(t). \tag{15}$$

The system matrix can be rewritten as

$$A(t) = A + BK + \sum_{i=1}^{q} \alpha_i A_i + \sum_{j=1}^{n} \beta_j BK\Psi_j$$

where the term $\sum_{j=1}^{n} \beta_j BK\Psi_j$ is given by

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \left(\beta_1 \begin{bmatrix} k_1 & 0 & \cdots & 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 & k_2 & \cdots & 0 \end{bmatrix} + \ldots + \beta_n \begin{bmatrix} 0 & 0 & \cdots & k_n \end{bmatrix}\right). \tag{16}$$

In this case it is difficult to obtain a simpler expression of (16) and the problem, even in this simpler case of single input is equivalent to the following static state output feedback problem [**?**]:

find K such that

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & (A + \sum_{i=1}^{q} \alpha_i A_i)\, x(t) + B\hat{u}(t) \\ \hat{u}(t) & = & Ky(t) \\ y(t) & = & \left(I + \sum_{j=1}^{n} \beta_j \Psi_j\right) x(t) \end{array} \right. , \tag{17}$$

and the index (14) is minimized.

We then obtain the following result:

Proposition 1 *If the coefficients β_j are slowly time-varying, the dynamic optimization problem expressed with the index (14), subject to the dynamic constraints (17), is equivalent to a guaranteed-cost full static state feedback problem.*

Proof: Let us consider the variable $y(t)$ in the equation (17) and consider the time derivative

$$\dot{y}(t) = \left(I + \sum_{j=1}^{n} \beta_j \Psi_j\right) \dot{x}(t)$$

we have

$$\dot{y}(t) = \left(I + \sum_{j=1}^{n} \beta_j \Psi_j\right) \left(A + \sum_{i=1}^{q} \alpha_i A_i\right) \left(I + \sum_{j=1}^{n} \beta_j \Psi_j\right)^{-1} y(t) + \left(I + \sum_{j=1}^{n} \beta_j \Psi_j\right) B\hat{u}(t)$$

the term

$$\left(I + \sum_{j=1}^{n} \beta_j \Psi_j\right)$$

is a diagonal and invertible matrix, and we can easily rewrite the closed-loop system in the y variable

$$\dot{y}(t) = \left(\sum_{i=1}^{n} \frac{1}{1+\beta_i} \hat{A}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\beta_j}{1+\beta_i} \tilde{A}_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{p} \frac{\beta_j}{1+\beta_i} \alpha_k \overline{A}_{ijk} \right) y(t) + \left(B + \sum_{j=1}^{n} \beta_j B_j\right) \hat{u}(t), \tag{18}$$

$$\hat{u}(t) = Ky(t)\ .$$

The index (14) has the following form

$$\mathcal{J} = \int_0^\infty \left(y^T \left(I + \sum_{j=1}^{n} \beta_j \Psi_j\right)^{-1} C^{(1)T} C^{(1)} \left(I + \sum_{j=1}^{n} \beta_j \Psi_j\right)^{-1} y + (1+\theta)^2\, \hat{u}^T D^{(1)T} D^{(1)} \hat{u} \right) dt\ . \tag{19}$$

The matrix

$$\left(I+\sum_{j=1}^{n}\beta_j\Psi_j\right)^{-1} C^{(1)T}C^{(1)} \left(I+\sum_{j=1}^{n}\beta_j\Psi_j\right)^{-1}$$

can be bounded by

$$\frac{1}{\left(1+\min_i\tilde{\beta}_i\right)^2}C^{(1)T}C^{(1)} = \frac{1}{(1+\eta)^2}C^{(1)T}C^{(1)}.$$

and the upper bound on the index becomes

$$\tilde{\mathcal{J}} = \int_0^\infty \left(\frac{1}{(1+\eta)^2}y^T C^{(1)T}C^{(1)}y + (1+\theta)^2\,\hat{u}^T D^{(1)T}D^{(1)}\hat{u}\right)\,dt\,. \tag{20}$$

subject to the dynamic bound (18). ■

Only when the entries of the state feedback vector K are perturbed in the same manner ($\beta_1 = \beta_2 = \ldots = \beta_n = \beta$) is it possible to treat the problem easily, resulting in the following static state feedback problem

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \left(A+\sum_{i=1}^{q}\alpha_i A_i\right)x(t) + (B+\beta B)\,\hat{u}(t) \\ \hat{u}(t) & = & Kx(t) \end{array}\right. \tag{21}$$

subject to the index (14). Even if the problem is as such, it is interesting to see how this type of uncertainty can affect the LQR performance of the system.

3.1.1 An Example

Let us consider the following mechanical system [7], also known as the "Benchmark Problem" where

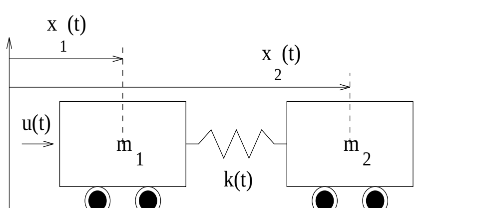

Figure 2: Benchmark Problem

1. $u(t)$ is the input and $w_i(t)$, $i = 1, 2$ are external disturbances that, which for LQR design, we assume equal to zero;
2. x_1, x_2 are the positions of the masses from reference points as shown in the figure;
3. the masses m_1, m_2 are equal to 1 in the appropriate units;
4. the stiffness is a slowly varying parameter whose values belong to the interval [0.5, 2].

The linear time-varying model which describes the behavior of the system is

$$\left\{\begin{array}{rcl} \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix} & = & \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k(t) & k(t) & 0 & 0\\ k(t) & -k(t) & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}(u(t)+w_1(t)) + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}w_2(t) \\ z(t) & = & \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + u(t) \end{array}\right. \tag{22}$$

It is easy to see that we can represent (22) as an affine uncertain model where the matrix $A(t)$ is given by

$$A(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + k(t) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = A_0 + pA_1$$

and the matrices B, C, D are constant. Using the LMI toolbox and the function `msfsyn` we performed a nominal $\mathcal{H}_2$ static state feedback synthesis. The guaranteed $\mathcal{H}_2$ performance was found to be 1.54 and the controller gain vector is given by

$$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} = \begin{bmatrix} -2.7917 & 1.7912 & -2.3651 & -0.1045 \end{bmatrix}. \tag{23}$$

Using the nominal controller, we generated an affine family of controllers according to the following rule

$$\tilde{K} = K + \alpha_1 \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

The uncertain terms α_i are defined over the interval

$$[-\delta\,|k_i|\,,\ \delta\,|k_i|] \tag{25}$$

where k_i represent the entries of the nominal vector K and δ is a parameter which express a "fraction" of the nominal value k_i. The fragility of the controller is tested by varying δ and checking for which value of δ the closed-loop system is not quadratically stable [3, 7] or, less conservatively, does not admit a parameter-dependent Lyapunov function [3, 6, 7]. For this particular system we obtained that δ is equal to 0.1 for the first case and 0.78 in the second. Now, fixing δ equal to 0.1 we considered the problem (21) with $-0.1 \leq \beta \leq 0.1$ and tried an LQR synthesis. We obtained a new "center" value for the K vector

$$K = \begin{bmatrix} -3.0930 & 2.0916 & -2.6365 & -0.0396 \end{bmatrix}$$

and the guaranteed $\mathcal{H}_2$ performance in this case is equal to 1.7.

The effect of this type of synthesis is now clear: to recover the quadratic stability property using an "uncertain" controller we pay the price of a worse guaranteed-performance with respect to the nominal case.

3.1.2 General Perturbation Model

Now, turning back to the general case (9), it can be seen that the formulation of the problem is more involved than the single input case. In fact, equation (9) can be rewritten in a compact form:

$$u = \left(K + \sum_{i=1}^{p} \sum_{j=1}^{q} \delta_{ij} \Psi_i^{(p)} K \Psi_j^{(n)} \right) x \tag{26}$$

where $\Psi_i^{(p)}$ and $\Psi_j^{(n)}$ are rank-one square matrices of p-th and n-th order with a "1" entry located at the i-th and j-th position of the main diagonal, respectively. Looking at the closed-loop system we have

$$\dot{x}(t) = \left(A + \sum_{i=1}^{p} \alpha_i A_i + B \left(K + \sum_{i=1}^{p} \sum_{j=1}^{q} \delta_{ij} \Psi_i^{(p)} K \Psi_j^{(n)} \right) \right) x(t) \tag{27}$$

and the closed-loop system matrix has the following complicated form:

$$A(t) = A + \sum_{i=1}^{p} \alpha_i A_i + BK + \sum_{i=1}^{p} \sum_{j=1}^{q} \delta_{ij} B_i K \Psi_j^{(n)} \tag{28}$$

where $B_i = B\Psi_i^{(p)}$. If we allow that entry of the controller matrix to vary independently from all others, we have the following problem:

Find K such that

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \left(A + \sum_{i=1}^{p} \alpha_i A_i\right) x(t) + Bu(t) + \sum_{i=1}^{p} \sum_{j=1}^{q} \delta_{ij} B_i \hat{u}_j(t) \\ u(t) & = & Kx(t) \\ \hat{u}_j(t) & = & Ky_j(t) \\ y_j(t) & = & \Psi_j^{(n)} x(t) \end{array}\right. \tag{29}$$

is stable and minimizes the index (12).

3.2 Additive structured uncertainties

Let us consider the "uncertain" controller

$$u = (K + \sum_{j=1}^{p} \beta_j K_j)x = (K + \delta K)x \tag{30}$$

and suppose that we allow *absolute* errors on the entries of K even if the entries of K are not known in advance. In this case, δK has the following form

$$\delta K = \sum_{i=1}^{q} \beta_i K_i, \tag{31}$$

where the K_i matrices are known and the coefficients β_i are in the following intervals,

$$\underline{\beta}_i \leq \beta_i \leq \overline{\beta}_i$$

represent uncertain terms. The problem is now to determine the matrices K_ii. A workable scheme may be the following:

1. Nominal synthesis by letting the uncertainty $\delta K = 0$ and considering the nominal LQR index (12),
2. Using the nominal K obtained from 1) to determine the uncertainty range beyond which stability property is lost using quadratic stability tests [3, 7] and less conservative robustness stability tests [6, 7],
3. Using the uncertainty range from 2), design a "Non-Fragile" compensator using a guaranteed-cost approach by bounding the following term

$$J = \int_0^\infty \left(x^T C^{(1)T} C^{(1)} x + x^T \left(K^T + \sum_{i=1}^{q} \beta_i K_i^T \right) D^{(1)T} D^{(1)} \left(K + \sum_{i=1}^{q} \beta_i K_i \right) x \right) dt \tag{32}$$

   the problem is how to bound the following matrix

$$\left(K^T + \sum_{i=1}^{q} \beta_i K_i^T \right) D^{(1)T} D^{(1)} \left(K + \sum_{i=1}^{q} \beta_i K_i \right)$$

   We guess that it is possible to find a proper quadratic bound $\mathcal{J}$ for the index J.

Let us consider the closed-loop system

$$\dot{x}(t) = (A + \delta A + BK + B\delta K)x(t) = (A + BK + \Delta)x(t), \tag{33}$$

where Δ is given by

$$\Delta = \sum_{i=1}^{p} \alpha_i A_i + \sum_{i=1}^{q} \beta_i B K_i \tag{34}$$

which is a new affine perturbation scheme with $p+q$ elements

$$A_1,\ A_2,\ \cdots\ A_p,\ BK_1,\ BK_2,\ \cdots\ BK_q\,.$$

This perturbation can be translated into its polytopic equivalent [3, 7] and the LQR/LMI problem amounts to the minimization of the trace of a positive definite matrix $\mathcal{P}$ subject to the following set of Linear Matrix Inequalities

$$\begin{bmatrix} Q\hat{A}_i^T + \hat{A}_i Q + Y^T B^T + BY & (CQ+DY)^T \\ CQ+DY & -I \end{bmatrix} \le 0,\ i=1,\ldots,2^{(p+q)} \tag{35}$$

where $\mathcal{P} = Q^{-1}$ and $KQ = Y$.

3.2.1 Guaranteed cost (Lyapunov parameter function) approach

Another way to solve this problem is the following: let us consider the closed-loop system (33) with the perturbation scheme (34). The uncertainty region can be easily transformed in the following form, taking into account the middle point of every uncertainty interval (2) and shifting in the proper manner every $\alpha_i(t)$

$$\mathcal{U} = \left\{ \Delta \in \mathbb{R}^{n\times n} \mid \Delta = \sum_{i=1}^{p} \tilde{\alpha}_i A_i + \sum_{j=1}^{q} \beta_j BK_j,\ |\tilde{\alpha}_i| \le \gamma_i,\ |\beta_j| \le \epsilon_j \right\}. \tag{36}$$

With this type of uncertainty region we can use well known results from Bernstein and Haddad [1]: we have that the optimal solution that takes into account the uncertainties is, for fixed Δ,

$$\tilde{K}_\Delta = -(D^{(1)T} D^{(1)})^{-1} B^{(1)T} P_\Delta \tag{37}$$

and P_Δ satisfies the following algebraic Riccati equation

$$(A+\Delta)^T P_\Delta + P_\Delta (A+\Delta) + C^{(1)T} C^{(1)} - P_\Delta B^{(1)} \left(D^{(1)T} D^{(1)} \right)^{-1} B^{(1)T} P_\Delta = 0 \tag{38}$$

rewriting this equation we have

$$A^T P_\Delta + P_\Delta A + C^{(1)T} C^{(1)} - P_\Delta B \left(D^{(1)T} D^{(1)} \right)^{-1} B^T P_\Delta + \Delta^T P_\Delta + P_\Delta \Delta = 0 \tag{39}$$

and the optimal value of the performance index is

$$J^*_\Delta = \mathrm{tr}\left(P_\Delta\right)\ .$$

Using well known results [10, 1] and slightly modifying some theorems we can bound the term

$$\Delta^T P_\Delta + P_\Delta \Delta$$

and we have the following proposition

Proposition 2 *Let $\mathcal{P}$ be the solution of the following equation*

$$A^T \mathcal{P} + \mathcal{P} A + C^T C - \mathcal{P} B (D^T D)^{-1} B^T \mathcal{P} + \Omega(\mathcal{P}) = 0 \tag{40}$$

then, $\mathcal{P}$ is an upper-bound to the solution P_Δ of equation (39) iff

$$\Delta^T \mathcal{P} + \mathcal{P}\Delta \le \Omega(\mathcal{P}) \tag{41}$$

and the couple

$$\left(\left(\Omega(\mathcal{P}) - (\Delta^T \mathcal{P} + \mathcal{P}\Delta)\right)^{1/2}, A+\Delta \right)$$

is detectable.

Proof: See [1] for the proof of slightly different but similar theorems.

It is straightforward to see that the function

$$\Omega(\mathcal{P}) = \sum_{i=1}^{p} \gamma_i \left(\alpha \mathcal{P} + \frac{1}{\alpha} A_i^T \mathcal{P} A_i \right) + \sum_{j=1}^{q} \epsilon_j \left(\alpha \mathcal{P} + \frac{1}{\alpha} (BK_j)^T \mathcal{P} (BK_j) \right)$$

where α is a strictly positive real parameter satisfying condition (41). With this linear bound we can solve a general Algebraic Riccati equation for a fixed α by minimizing the trace of $\mathcal{P}$, using well known numerical techniques [4].

4 Dynamic Feedback - $\mathcal{H}_2$ case

The purpose of this section is to generalize and extend the results of the static state feedback case, we begin with the formulation of the following simple problem: let us consider the following simplified version of system (1)

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \left(A + \sum_{i=1}^{q} \alpha_i(t) A_i\right) x(t) + B^{(1)} u(t) + B^{(2)} w(t) \\ y(t) & = & C^{(2)} x(t) + D^{(2)} w(t) \end{array} \right. \tag{42}$$

and we look for a dynamic controller of the type

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & F(1 + \delta_1(t))\xi(t) + G(1 + \delta_2(t))y(t) \\ u(t) & = & H(1 + \delta_3(t))\xi(t) \end{array} \right. \tag{43}$$

where $\delta_i(t)$ are uncertain parameters whose values belong to the open interval $(-1, 1)$. This scheme is representative of a simplified multiplicative uncertainty scheme: it is not general but it enables us to derive a nice formulation of the problem.

References

[1] D.S. Bernstein and W.H. Haddad: *Robust Stability and Performance via Fixed-Order Dynamic Compensation with Guaranteed Cost Bounds*; Math. Control Signals Systems (1990) 3, 139-163.

[2] S.P. Bhattacharyya, H. Chapellat and L.H. Keel: *Robust Control: The Parametric Approach*; Prentice-Hall, Englewood Cliffs, NJ, (1995).

[3] S. Boyd, L. EL Ghaoui, E. Feron and V. Balakrishnan: *Linear Matrix Inequalities in System and Control Theory*; SIAM, 1994.

[4] P. Dorato, C.T. Abdallah and V. Cerone: *Linear-Quadratic Control, An Introduction*; Prentice Hall, 1995

[5] J.C. Doyle, B.A. Francis and A.R. Tannenbaum: *Feedback Control Theory*; Macmillan Publishing Company, New York, (1992).

[6] P. Gahinet, P. Apkarian and M. Chilali: *Affine Parameter-Dependent Lyapunov Functions for Real Parametric Uncertainty*; Proc. Conf. Dec. Contr., 1994, pp. 2026-2031.

[7] P. Gahinet, A. Nemirovski, A.J. Laub and M. Chilali: *LMI Control Toolbox*; The Mathworks, 1995

[8] W.M. Haddad and J.R. Corrado: *Non-fragile controller design via quadratic Lyapunov bounds*; Preprint, Georgia Institute of Technology, March 1997.

[9] L.H. Keel and S.P. Bhattacharyya: *Robust, fragile, or optimal?*; Proceedings of the American Control Conference, Albuquerque, NM, June 4-6, 1997

[10] I.R. Petersen and C.V. Hollot: *A Riccati Equation Approach to the Stabilization of Uncertain Systems*; Automatica, Vol. 22, pp. 397-411, 1986.