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OUTPUT STABILIZABILITY

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Abstract

In this report, we provide algebraic tests to determine whether a linear Single-Input-Single-Output (SISO) system, is stabilizable with a constant output feedback.

1 Introduction

The problem of output stabilizability of linear systems remains one of the most challenging problems in systems theory. While it is true that many techniques exist to stabilize systems using only output measurements, the fundamental question of the existence of such controllers is still open. In other words, given a linear, time-invariant system (LTI), the existence of a constant output feedback that will stabilize the system can not in general be answered, short of using a root-locus or Nyquist approach that will actually answer the existence question by finding such a stabilizing controller. One might argue that with the advent of graphing software, the question is moot since one can answer the question graphically for almost any LTI, SISO system. It is however important to obtain an algebraic answer to the stabilizability question for many reasons. First, a constant output feedback is the simplest member of the hierarchy of fixed-structure controllers, and an answer to the constant output feedback stabilizability might provide an answer to the more general fixed-structure controllers, where a graphical approach is not available. Second, the algebraic conditions may provide the designer with a negative answer to the stabilizability question without actually solving the problem. Finally, these conditions will provide an alternate view at this classical problem, allowing us to consider the robust stabilizability problem in a future paper.

This report is organized as follows: The problem is stated in section 2, our main results are given in section 3, a numerical example is presneted in section 4, while our conclusions are given in section 5.

2 Problem Statement

We consider the problem of stabilizing the SISO continuous-time, linear, time-invariant system described by the transfer function

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^n + \cdots + b_{n-1} s + b_n}{s^n + \cdots + a_{n-1} s + a_n} \tag{1}$$

connected in the standard feedback configuration, with the output feedback compensator $u = -ky + r$, so that the closed-loop system is described by

$$\begin{aligned} T(s) &= \frac{kG(s)}{1 + kG(s)} \\ &= \frac{k(b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n)}{p(s,k)} \end{aligned} \tag{2}$$

where $p(s,k) = a(s) + kb(s)$. Let us decompose $p(s,k)$, $a(s)$ and $b(s)$ into their even and odd parts

$$\begin{aligned} p(s,k) &= p_e(s^2,k) + sp_o(s^2,k) \\ a(s) &= a_e(s^2) + sa_o(s^2) \\ b(s) &= b_e(s^2) + sb_o(s^2) \end{aligned} \tag{3}$$

The approach we consider is to determine first the jw axis crossings w_i of the roots of $p(s,k)$, solve for the corresponding gains k_i and then determine whether a particular crossing is from the Left-Half-Plane (LHP) to the Right-Half-Plane (RHP) or vice-versa. By keeping track of the number and the direction of crossings, we will be able to answer the stabilizability question for a given $G(s)$.

3 Main Results

Let us then consider the closed-loop characteristic equation $0 = p(s,k)$, which becomes along the jw axis

$$\begin{aligned} 0 &= a(jw) + kb(jw) \qquad (4) \\ &= a_e(-w^2) + jwa_o(-w^2) + k[b_e(-w^2) + jwb_o(-w^2)] \\ &= [a_e(-w^2) + kb_e(-w^2)] + jw[a_o(-w^2) + kb_o(-w^2)] \\ &= [a_R(-w^2) + kb_R(-w^2)] + [a_I(-w^2) + kb_I(-w^2)] \end{aligned}$$

where $x_R(jw) = x_e(jw)$ and $x_I(jw) = wx_o(jw)$. Therefore, setting both real and imaginary parts to zero, we can eliminate k and obtain

$$Y(-w^2) = a_R(-w^2)b_I(-w^2) - a_I(-w^2)b_R(-w^2) = 0 \tag{5}$$

Note that $Y(-w^2)$ is actually the negative of the numerator of the imaginary part of $b(jw)/a(jw)$, in other words

$$\begin{aligned} G(jw) &= R(jw) + jI(jw) \\ &= \frac{X(-w^2)}{D(jw)} + j\frac{-Y(-w^2)}{D(jw)} \end{aligned} \tag{6}$$

and note also that the roots of $Y(-w^2) = 0$ are exactly those frequencies at which $I(jw) = 0$ which are also those frequencies where the Nyquist plot of $G(jw)$ intersect the real axis. The positive real roots of equation (5) $w_i,\ i = 1, \cdots, m$ represent the jw axis crossings. We can then find the corresponding gains as

$$\begin{aligned} k_i &= -a(jw_i)/b(jw_i);\ i = 1, \cdots, m \\ &= -a_I(-w_i^2)/b_I(-w_i^2);\ i = 1, \cdots, m \\ &= -a_R(-w_i^2)/b_R(-w_i^2);\ i = 1, \cdots, m \end{aligned} \tag{7}$$

and order them as $k_1 < k_2 < \cdots < k_m$. Let us assume that $a(s)$ has at least one root in RHP. Otherwise, a small enough value of k which stabilizes $p(s, k)$ always exists. The closed-loop system will be stabilized if, at any w_i, all n roots are in the LHP. Let $x_R(jw) = x_e(jw)$ and $x_I(jw) = wx_o(jw)$. We then have the following results.

Lemma 1 *The output stabilizbility problem is solvable if and only if any of the m polynomials $p(s, k_s)$ is stable, where $k_{i-1} \leq k_s \leq k_i$; $i = 1, \cdots, m$ and $k_0 \leq k_1$.*

Proof: Obvious. ■

Lemma 2 *Suppose that $p(s)$ has a single root at $s = jw_i - \epsilon$, for a sufficiently small real $\epsilon > 0$. Then the argument of $p(jw)$ is a strictly increasing function of w at w_i, i.e., $\frac{\partial}{\partial w} arg\{p(jw)\} \mid_{w=w_i} > 0$.*

Proof: The proof can be obtained by writting $p(jw) = (jw + \epsilon - jw_i)R(jw)$, $R(jw_i - \epsilon) \neq 0$ then differentiating its argument. ■

We will next present a lemma and its proof for the special case where only one branch of the root locus crosses the jw axis at a particular k_i. The more general case where l roots cross the jw axis is discussed in lemma 5.

Lemma 3 *A complex conjugate pair crosses the jw axis as k increases*

1. *From the LHP to the RHP at $\pm jw_i$ if and only if*

$$\frac{\partial}{\partial w}[Y(-w^2)] \mid_{w=w_i} > 0$$

2. *From the RHP to the LHP at $\pm jw_i$ if and only if*

$$\frac{\partial}{\partial w}[Y(-w^2)] \mid_{w=w_i} < 0$$

Finally, the roots stay in one half-plane if

$$\frac{\partial}{\partial w}[Y(-w^2)] = 0; \quad \forall w$$

Proof: We will only prove case 1) for the case where one branch of the root locus crosses the jw axis at $k = k_i$. At the frequency w_i and $k = k_i - \epsilon$, for a small $\epsilon > 0$, we have a pair of complex conjugate roots in the LHP, but close to the jw axis. Then, by Lemma 2,

$$\frac{\partial}{\partial w} arg\{a(jw) + (k_i - \epsilon)b(jw)\} \mid_{w=w_i} > 0$$

In the following, we drop the explicit dependence on w and w_i, to obtain

$$\begin{aligned}
& \frac{\partial}{\partial w} arg\{a(jw) + (k_i - \epsilon)b(jw)\} \mid_{w=w_i} > 0 \\
\iff & \frac{\partial}{\partial w} Arctan\{\frac{a_I + k_i b_I - \epsilon b_I}{a_R + k_i b_R - \epsilon b_R}\} > 0 \\
\iff & [a_I' + (k_i - \epsilon)b_I'][a_R + (k_i - \epsilon)b_R] \\
& > [a_R' + (k_i - \epsilon)b_R'][a_I + (k_i - \epsilon)b_I] \\
\iff & [a_I' + (k_i - \epsilon)b_I'][-\epsilon b_R] - [a_R' + (k_i - \epsilon)b_R'][-\epsilon b_I] > 0 \\
\iff & -(a_I' + k_i b_I')b_R + (a_R' + k_i b_R')b_I - \epsilon(b_I' b_R - b_R' b_I) > 0
\end{aligned}$$

then, since ϵ is arbitrarly small, and using (7),

$$\begin{aligned}
& (a_R' + k_i b_R')b_I - (a_I' + k_i b_I')b_R > 0 \\
\iff & a_R' b_I - a_I b_R' - a_I' b_R + a_R b_I' > 0 \\
\iff & \frac{\partial}{\partial w}\left[a_R(jw)b_I(jw) - a_I(jw)b_R(jw)\right] \mid_{w=w_i} > 0
\end{aligned}$$

■ Note at this point that an interpretation of this lemma in terms of the Nyquist plot of $G(jw)$ is possible. It is possible to count the number of crossings of the $G(jw)$ with the real axis and note their directions (down or up). If there is a real axis region where there is a net of n up crossings, a real value of k exists for which the closed-loop $T(s)$ is stable. This will be further investigated when studying a robust version of our test.

Lemma 4 *A complex conjugate pair crosses the jw axis from the LHP to the RHP at $\pm jw_i$ as k increases if and only if*

$$k_i \frac{\partial}{\partial w} arg\{b(jw)/a(jw)\} \mid_{w=w_i} < 0$$

Proof: Consider

$$k_i \frac{\partial}{\partial w} \left[arg(b(jw)/a(jw)) \right] |_{w=w_i} > 0$$

In the following, we drop the explicit dependence on w and w_i, to obtain

$$\begin{aligned}
& k_i \frac{\partial}{\partial w} \left[arg(b(jw)/a(jw)) \right] |_{w=w_i} > 0 \\
\Longleftrightarrow \quad & k_i \frac{\partial}{\partial w} \left[arg\{ba^*\} \right] > 0 \\
\Longleftrightarrow \quad & k_i \frac{\partial}{\partial w} (\frac{-b_R a_I + b_I a_R}{b_I a_I + a_R b_R}) > 0 \\
\Longleftrightarrow \quad & k_i(-b_R a_I + b_I a_R)'(b_I a_I + a_R b_R) \\
& > k_i(b_I a_R + a_R b_R)'(-b_R a_I + b_I a_R)
\end{aligned}$$

but using (7),

$$\begin{aligned}
-b_R a_I + b_I a_R &= \frac{1}{k_i}(b_R b_I - b_R b_I) = 0 \\
b_I a_I + a_R b_R &= \frac{1}{k_i}(-a_I^2 - a_R^2)
\end{aligned} \tag{8}$$

therefore, (8) is satisfied if an only if

$$(b_I a_R - b_R a_I)' > 0 \tag{9}$$

which is condition 1) in Lemma 3. Therefore, the lemma is proven. ∎

Finally, we present the general result in the following lemma.

Lemma 5 *Suppose that l branches cross the jw axis at $w = w_i$. Let l_{LR} be the number of branches crossing from the LHP to the RHP as k increases and l_{RL} the number of branches crossing from the RHP to the LHP as k increases. Also let $m = l_{RL} - l_{LR}$ be the number of net crossings from RHP to LHP. Then, the following is true*

1. *m=-1 if and only if $Y(-w^2)$ is a strictly increasing function of w at $w = w_i$.*

2. *m=1 if and only if $Y(-w^2)$ is a strictly decreasing function of w at $w = w_i$.*

3. *m=0 if and only if $Y(-w^2)$ has a local maximum/minimum at $w = w_i$.*

Proof: Suppose l branches of the root locus cross the jw axis at $w = w_i$ and for $k = k_i$. Clearly then, we must have

$$p(s,k) = (s - jw_i)^l R(s) + (k - k_i) b(s)$$

where $R(jw_i) \neq 0$. Note that we have $p(s,k) = a(s) + kb(s)$ so that

$$a(s) = (s - jw_i)^l R(s) - k_i b(s) \tag{10}$$

Then, the function $Y(-w^2)$ in (5) can be written in the following form

$$Y(-w^2) = -j^{l-1}(w - w_i)^l (R_I b_I + R_R b_R), \ l \ odd \tag{11}$$
$$Y(-w^2) = j^l (w - w_i)^l (R_R b_I - R_I b_R), \ l \ even \tag{12}$$

From the expressions above, it is clear that the conditions on $Y(-w^2)$ at $w = w_i$ can be rewritten as

1. $m = -1$ if and only if l is odd and the polynomial $-j^{l-1}(R_I b_I + R_R b_R) \mid_{w=w_i} > 0$.
2. $m = 1$ if and only if l is odd and the polynomial $-j^{l-1}(R_I b_I + R_R b_R) \mid_{w=w_i} < 0$.
3. $m = 0$ if and only if l is even.

In order to determine what happens to the root locus in a neigborhood of jw_i, we need to determine those directions such that the root locus can exist at in the neigborhood of jw_i. In other words, we need to determine the locus of points s such that $p(s,k) = 0$, or such that the number

$$k - k_i = -\frac{(s - jw_i)^l R(s)}{b(s)} \tag{13}$$

is real. The root locus can then exist around jw_i in those directions where

$$Im\{(s - jw_i)^l R(s) b^*(s)\} = 0 \tag{14}$$

and

$$Re\{(s - jw_i)^l R(s) b^*(s)\} < 0, \ k > k_i \tag{15}$$
$$Re\{(s - jw_i)^l R(s) b^*(s)\} > 0, \ k < k_i \tag{16}$$

where $b^*(s)$ is the complex conjugate of $b(s)$. Now, in the neigborhood of jw_i, we can write

$$s = jw_i + \rho e^{j\Phi} \tag{17}$$

The problem then reduces to finding Φ such that (14) is satisfied as $\rho \to 0$. Substituting (17) in (14) and (15), this is equivalent to

$$(2p+1)\pi = l\Phi + arg\{R(jw_i)b^*(jw_i)\};\ p\ integer, \tag{18}$$

for $k > k_i$ and analogously,

$$(2p)\pi = l\Phi + arg\{R(jw_i)b^*(jw_i)\};\ p\ integer, \tag{19}$$

for $k < k_i$. Clearly, the net number of crossings from the RHP to the LHP is given by the difference between the number of different solutions to (19) lying in the interval $(-\pi/2, \pi/2)$ and those lying in the interval $(\pi/2, 3\pi/2)$. According to (19) this number can be either -1 or 1 for l odd, and 0 when l is even. This latter conclusion leads to the proof of case 3) in the Lemma so we will hereafter concentrate on the l odd case. Let us designate $\phi = arg\{R(jw_i)b^*(jw_i)\}$. The solutions to (19) are given by

$$\Phi = 2p\pi/l + \phi/l;\ p\ integer, \tag{20}$$

Now the number l_{RL} of different values of Φ in the interval $(-\pi/2, \pi/2)$ satisfying (20) is given by

1. When $l = 4r+1$, r integer, and $\phi \in (-\pi/2, \pi/2)$, $l_{RL} = 2r+1$.
2. When $l = 4r+3$, r integer, and $\phi \in (-\pi/2, \pi/2)$, $l_{RL} = 2r+1$.
3. When $l = 4r+1$, r integer, and $\phi \in (\pi/2, 3\pi/2)$, $l_{RL} = 2r$.
4. When $l = 4r+3$, r integer, and $\phi \in (\pi/2, 3\pi/2)$, $l_{RL} = 2r+2$.

And since $m = 2l_{RL} - l$, these cases imply in turn that $m = 1$ in cases 1) and 4) and $m = -1$ in cases 2) and 3).
In order to complete the proof of this Lemma, first note that

$$\phi \in (-\pi/2, \pi/2) \Longleftrightarrow (R_I b_I + R_R b_R) > 0 \tag{21}$$
$$\phi \in (\pi/2, 3\pi/2) \Longleftrightarrow (R_I b_I + R_R b_R) < 0 \tag{22}$$

Now, conditions 2) and 3) (which are equivalent to $m = -1$) can be written in a more compact form as

$$-j^{l-1}(R_I b_I + R_R b_R) > 0 \tag{23}$$

and, similarly, conditions 1) and 4) (equivalent to $m = 1$) as

$$j^{l-1}(R_I b_I + R_R b_R) > 0 \tag{24}$$

This completes the proof. ∎

4 Numerical Example

Consider the polynomials

$$a(s) = s^7 - 3.25s^6 - 9s^5 - 39.375s^4 - 70.75s^3 - 79.5s^2 - 88s - 48 \quad (25)$$
$$b(s) = s^7 - 7.5s^6 - 25s^5 - 85.75s^4 - 149.5s^3 - 167s^2 - 160s - 80 \quad (26)$$

By means of the Routh-Hurwitz algorithm it is easy to check that the open loop plant $p(s) = b(s)/a(s)$ has 4 poles in the LHP and 3 poles in the RHP. In order to check whether $p(s)$ can be stabilized with a constant gain, we construct the polynomial $Y(-w^2)$ as in (5) and compute its positive real roots. These roots happen to be $\{0, 1.2180, 2, 2, 2.5732\}$. For each of these values of w_i we compute k_i using (7), and obtain the set $\{-0.6, 1.7408, -0.5, -0.5470\}$. Once these values are ordered, the local behaviour of $Y(-w^2)$ is studied at the corresponding w_i either by explicitely calculating its derivative or by examining the plot of $Y(-w^2)$. Table (4) summarizes the set of k_i, w_i as well as the corresponding sign of the derivative $\frac{\partial Y(-w^2)}{\partial w}$. Starting with $k = 0$ (where there

k_i	-0.6	-0.5470	-0.5	1.7408
w_i	0	2.5732	2	1.2180
$\frac{\partial Y(-w^2)}{\partial w}$	+	+	0	-

are 3 roots in the RHP) we see that at $k = -0.5$ there are no net crossings ($m = 0$), at $k = -0.5470$ a pair of complex conjugate roots crosses from RHP to LHP (thus leaving 1 root in RHP) and at $k = -0.6$ there is one real root ($w = 0$) crossing from RHP to LHP. Therefore, for $k < -0.6$ the plant can be stabilized.
In Figure 1 the function $Y(-w^2)/T(w)$ is plotted in the interval $[0, 5]$, where $T(w)$ is a suitably chosen polynomial such that $T(w) > 0,\ w \geq 0$ and the ratio $Y(-w^2)/T(w)$ is adequately bounded for graphical purposes. In addition, the roots of $Y(-w^2)$ are labeled with the corresponding values of k.

5 Conclusions

In this report we have provided algebraic conditions for the stabilizability of SISO systems with constant gains. The conditions are simple, testable, and may be extended to the robust stabilizability problem as will be reported on in a future paper.

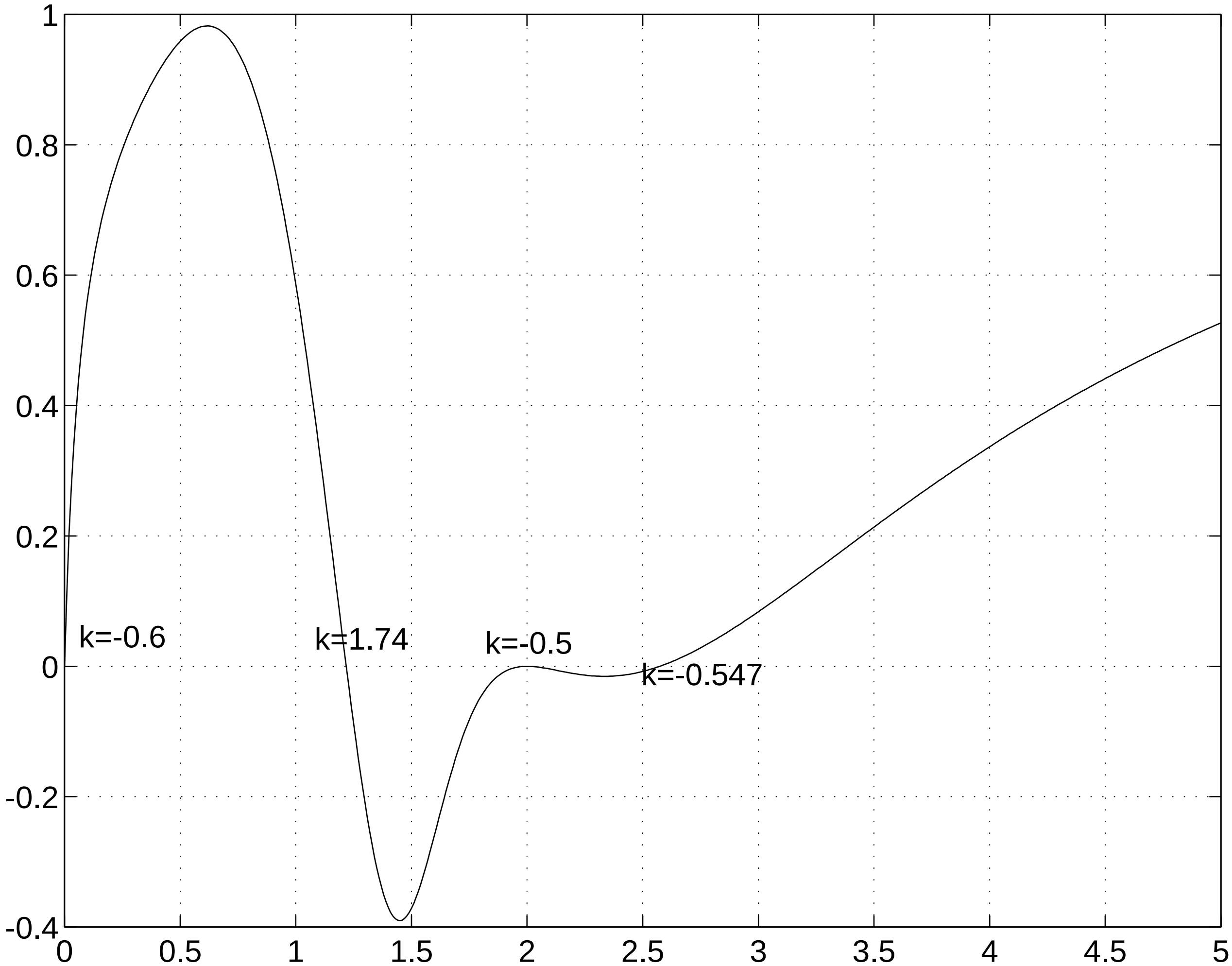

Figure 1: $Y(-w^2)/T(w)$