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Applications of Quantifier Elimination Theory to Control Theory

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Abstract

In this paper we show how a number of interesting linear control system analysis and design problems can be reduced to Quantifier Elimination (QE) problems. We assume a fixed structure for the compensator, with design parameters q_i. The problems considered are problems that currently have no general solution, e.g. the output stabilization problem, the simultaneous stabilization problem, the robust multiobjective problem, etc. However, the problems must be of modest complexity if existing QE software packages are to produce answers. The software package QEPCAD is used to solve some numerical design examples.

Chapter 1

Introduction

As will be demonstrated in this paper, many interesting control system design and analysis problems can be reduced to *Quantifier Elimination* or QE problems. In particular, for linear time-invariant systems, important control issues such as *stability, robust stability*, and *robust performance* can be reduced to systems of multivariable-polynomial inequalities with logic quantifiers such as "there exits" ($\exists$) and "for all" ($\forall$). For simplicity we will refer to multivariable-polynomials as simply *polynomials* in the sequel. Typically the variables in the polynomials are real variables that come from plant (controlled system) and compensator (controller) parameters. The final design objective is to obtain quantifier-free formulas for the compensator parameters or, for the existence problem, to obtain a "true" or "false" output when compensator parameters are quantified with the existence quantifier QE methods are especially attractive for control problems where there are no general analytical design algorithms and where, for practical reasons, one would like to have the simplest possible compensator. An example of this is the static output-feedback stabilization problem, which we will refer to as simply the *output-feedback stabilization problem.* This is the problem of finding a "zero-order" compensator, or what is commonly referred to *simple proportional feedback*, such that the closed-loop system is stabilized. Proportional feedback is the simplest possible type of feedback that can be used, yet this problem remains an open analytical problem. Indeed one of the first attempts to use QE methods, also referred to as *Decision Methods* to solve control system design problems, was to solve the output-feedback stabilization problem, as reported in the paper by Anderson et. al. [2] in 1975. Unfortunately at the time the available QE algorithms were very complex and no software was available for computer solutions. Since then, improved QE algorithms have been developed, see for example [8, 9], and software packages have been written to implement the new algorithms, for example the software package QEPCAD *Quantifier Elimination by Partial Cylindrical Algebraic Decomposition* package [12, 13]. It seems appropriate then to re-examine the application of QE methods to control system design problems.

This paper is organized as follows. Section 2 contains a discussion of the control problems we studied using QE. It is divided into stabilization problems and performance problems. Section 3 contains an overview of the mathematical theory of QE, with special emphasis on issues arising in control problems. Examples of using QE to solve control problems are given in chapter 4. Our conclusions and directions for future research are included in chapter 5.

Chapter 2

Control Design Problems

In this chapter, we review a few control problems divided into two categories: The first contains problems where the main objective is to stabilize one or a family of systems. The second category contains in addition to stabilizability, some performance requirements. In the sequel, we refer to Figure 2.0.1 for the closed-loop block diagram used in control theory. The functions $C(s,q)$ and $G(s,p)$ are the Laplace-transform transfer functions of the compensator and the plant, respectively. We assume here that our system is linear, time-invariant, and lumped, so that these two functions are rational functions in the transform variable s. The plant may also be characterized in state-space form, i.e.

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x \tag{2.0.1}$$

with the usual relation $G(s,p) = C(p)[sI - A(p)]^{-1}B(p)$.

2.1 Stability

For feedback control systems, as shown in Figure 2.0.1, the first design problem one must address is that of stability of the closed-loop system. Stability requires that all the zeros of the numerator polynomial (closed-loop characteristic polynomial) of the rational function $1 + C(s,q)G(s,p)$ have *negative real parts.* If the closed loop characteristic polynomial is denoted $f(s) = a_0 s^n + a_1 s^{n-1} + ... + a_n, \;\; a_0 > 0$ the Liénard-Chipart criterion [11] then states that all the roots of $f(s) = 0$ have negative real parts if and only if

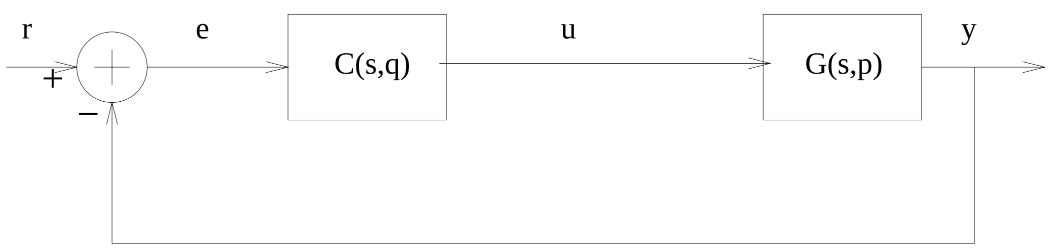

Figure 2.0.1: Block Diagram Of Feedback System

$a_n > 0, a_{n-2} > 0, \cdots; \Delta_1 > 0, \Delta_3 > 0, \cdots$ where Δ_i is the Hurwitz determinant of order i, i.e.

$$\begin{aligned} \Delta_i &= \det \begin{bmatrix} a_1 & a_3 & a_5 & \cdots \\ a_0 & a_2 & a_4 & \cdots \\ 0 & a_1 & a_3 & \cdots \\ 0 & a_0 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & a_i \end{bmatrix} \\ a_k &= 0; \quad k > n \end{aligned} \tag{2.1.2}$$

See section 13, chapter XV of reference [11] for more details. We list next four stabilizability problems which will later be framed as QE problems, and solved using the software package QEPCAD.

Problem 1: Output Feedback Stabilization
The output feedback problem is probably the most important open question in control engineering. Simply stated, the problem is as follows: Given the linear, time-invariant system (2.0.1), find a static output feedback $u(t) = -Ky(t)$ so that the closed-loop system has some desirable characteristics, or determine that such a feedback does not exist. The problem is important in its own right, but also because many other problems, e.g. fixed-order compensation, are reducible to some variation of it. A recent survey of the state of the art into this problem may be found in [19]. For multi-input-multi-output (MIMO) systems, this problem has no analytical solution, even when all the plant parameter values are known.

Problem 2: Simultaneous Stabilization
The problem of stabilizing n different plants is a longstanding problem in the robust control literature. The problem is stated as one of finding conditions for the existence and a method of designing one controller to stabilize a set of n plants: $G_1,\ G_2,\ \ldots,\ G_n$, or determining that such a controller can not exist. The controller should be exhibited once its existence conditions are satisfied. The problem is relevant in applications where the plant is only known to belong to a set of n different plants, or where the failure of sensors or actuators will drastically change the plant from its current description. More recently, the problem has been studied in conjunction with the problem of stabilizing a nonlinear plant, which is linearized about n operating points. The simultaneous-stabilization has no analytical solution if $n \geq 2$.

Problem 3: Robust Stabilization
In many situations, the mathematical description of a physical plant is not exactly known. In special but important situations, the uncertainty of the plant is in the values of its parameters, which are assumed to appear as polynomials in the coefficients of s in the plant transfer function $G(s,p)$ or as polynomial functions in the entries of the A, B, and C matrices. For example

$$G(s,p) = \frac{a_{n-1}(p)s^{n-1} + \cdots + a_0(p)}{b_n(p)s^n + \cdots + b_0(p)}$$

where the a_i, b_i are polynomials in the entries of the vector p. or

$$A(p) = A + \alpha_1(p)A_1 + ... + \alpha_n(p)A_n \tag{2.1.3}$$

where A_i are known matrices, and $\alpha_i(p)$ are polynomial in the entries of the vector p. The robust stabilization problem is to determine if a compensator vector q exists which stabilizes the closed-loop system, given the uncertainty in plant vector p; and when one does exist to compute a value for q, or characterize the set of vectors q that preserve stability. It has been shown that for the transfer function model, using Kharitonov theory, that some problems of this type can be reduced to the problem of simultaneous stabilization of a finite collection of plants. See, for example chapter 11 in [3] and [5].

2.2 Performance

In general, stability is but one of many requirements of a closed-loop control system. We will focus here on performance measures specified in the frequency domain, since measures of this type fit the QE problem formulation very nicely.

Problem 4: Frequency Domain Multiobjective Design
With respect to the feedback system of Figure 2.0.1, some typical performance objectives to be met (simultaneously) are represented by the inequalities

- **Tracking Error**
$$|S(j\omega)| \leq \delta, \;\; 0 < \omega \leq \omega_1$$
- **Control Effort**
$$|W(j\omega)| < \epsilon, \;\; \text{for all } \omega$$

where the function $S(s)$ represents the transfer function between the command signal r and the error signal e, and the function $W(s)$ represents the transfer function between the command signal r and the control signal u. These two functions may be computed from

$$S(s) = \frac{1}{1 + C(s,q)G(s,p)}, \;\; W(s) = \frac{C(s,q)}{1 + C(s,q)G(s,q)} \tag{2.2.4}$$

By computing the magnitude squared and clearing fractions, these two specifications can be reduced to polynomial inequalities in the variables q_i, p_i, and ω, with integer coefficients if δ^2 and ϵ^2 are rational numbers. For *robust* performance the various performance specifications must be met for all admissible plant vectors p. For *nominal* performance, the specification must be met only for a fixed vector p. For QE to apply, the entries of the vector p must be rational numbers.

In the next chapter, we review the QE problem as it relates to control, and discuss the availability of software packages for the purpose of solving QE problems.

Chapter 3

QE Methods and QEPCAD Software

In this chapter, we review the general QE problem and introduce the software package QEPCAD which we use to solve our control problems. A more detailed treatment may be found in [20, 4].

Given the set of polynomials with integer coefficients $P_i(X, Y), \quad 1 \leq i \leq s$ where X represents a k dimensional vector of quantified real variables and Y represents a l dimensional vector of unquantified real variables, let $X^{[i]}$ be a block of k_i quantified variables, Q_i be one of the quantifiers $\exists$ (there exists) or $\forall$ (for all), and let $\Phi(Y)$ be the quantified formula

$$\Phi(Y) = (Q_1 X^{[1]}, ..., Q_w X^{[w]}) F(P_1, ..., P_s), \tag{3.0.1}$$

where $F(P_1, ..., P_s)$ is a quantifier free *Boolean* formula, that is a formula containing the Boolean operators $\wedge$ (and) and $\vee$ (or), operating on *atomic predicates* of the form

$$P_i(Y, X^{[1]}, ..., X^{[w]}) \quad \geq \quad 0 \tag{3.0.2}$$
$$P_i(Y, X^{[1]}, ..., X^{[w]}) \quad > \quad 0 \tag{3.0.3}$$
$$P_i(Y, X^{[1]}, ..., X^{[w]}) \quad = \quad 0 \tag{3.0.4}$$

We can now state the general quantifier elimination problem

General Quantifier Elimination Problem: Find a quantifier-free Boolean formula $\Psi(Y)$ such that $\Phi(Y)$ is true if and only if $\Psi(Y)$ is true. In control problems, the unquantified variables are generally the compensator parameters, represented by the parameter vector $Y = q$, and the quantified variables are the plant parameters, represented by the plant parameter vector p, and the frequency variable ω. Uncertainty in plant parameters are characterized by quantified formulas of the type $\forall(p_i) \;\; [\underline{p_i} \leq p_i \leq \overline{p_i}]$ where $\underline{p_i}$ and $\overline{p_i}$ are rational numbers. *The quantifier-free formula $\Psi(q)$ then represents a characterization of the compensator design.*
An important special problem is the QE problem with no unquantified variables (free variables), i.e. $l = 0$. This problem is referred to as the *General Decision Problem.*

General Decision Problem: With no unquantified variables, i.e. $l = 0$, determine if the quantified formula given in 3.0.1 is true or false.

The general decision problem may be applied to the problem of *existence* of compensators that meet given specifications, in which case an "existence" quantifier is applied to the compensator parameter q.

Algorithms for solving general QE problems were first given by Tarski [20] and Seidenberg [18], and are commonly called Seidenberg-Tarski decision procedures. Tarski showed that QE is solvable, but his algorithm and later modifications are exponential in the size of the problem. Researchers in Control Theory have been aware of Tarski's results and their applicability to Control problems since the 1970's but the tedious operations made the technique very limited [2]. Later, Collins [8] introduced a theoretically more efficient QE algorithm that uses a cylindrical algebraic decomposition (CAD) approach. However, this algorithm was not capable of effectively handling nontrivial problems. More recently Hong [12], Collins and Hong [9], Hong [13] have introduced a significantly more efficient partial CAD QE algorithm. In our work we use the Hong's implementation of the CAD algorithm called QEPCAD. The CAD algorithm always completely solves any QE problem. However, the computational cost is extremely high. Our experience (see [14, 15]) indicates that QEPCAD can always solve, in a few seconds on a large workstation, most textbook examples. It can also solve some significantly harder problems and a few non-trivial problems. However the problem with this general implementation of CAD quantifier elimination is that its time complexity is double exponential which means that for many problems, current computer resources are not sufficient. It is very important to simplify the QE problem as much as possible before using QEPCAD.

Chapter 4

Control System Design Examples

The general control structure we use was shown in Figure 2.0.1. The controller $C(s,q)$ is designed so that the closed-loop system satisfies some design requirements in the face of significant uncertainties in the description of $G(s,p)$. This leads to the question of robust control design. First and foremost however, the controller should be chosen to make the closed-loop system stable. The examples in this section illustrate the application of QE methods to stability problems.

4.1 Stability

As noted in the introduction, stability is for linear time-invariant systems is generally checked by a criterion such as the Liénard-Chipart criterion, applied to a closed-loop characteristic polynomial.

4.1.1 Robust Stability Analysis Example

Here we consider the problem from [21] which deals with robust stability analysis for systems with nonlinearly correlated parametric uncertainties. The closed-loop system, with given state-feedback $u = -Kx$, has a system matrix given by

$$A - BK = \begin{bmatrix} 1.5 & 3.25 \\ -1.5 & -2.25 \end{bmatrix} + p_1 \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} + p_2 \begin{bmatrix} -0.5 & -1.5 \\ 0.5 & 1.5 \end{bmatrix} + p_1 p_2 \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \tag{4.1.1}$$

with plant parameters p_1, p_2 constrained to the uncertainty set

$$B(\rho) = \{0 \le p_1 \le \rho,\ 0 \le p_2 \le \rho\}$$

and the task is to find maximal ρ for which the system is stable for all p_1, p_2 from $B(\rho)$.

The characteristic polynomial of the matrix (4.1.1) is

$$s^2 + (p_1 p_2 - p_1 - p_2 + 0.75)s + (-p_1 - 0.5p_2 + 1.5)\ .$$

For a second oder polynomial stability requires only that all the coefficients have the same sign. In this case,

$$-2p_1 - p_2 + 3 > 0,\ 4(p_2 - 1)p_1 - 4p_2 + 3 > 0\ .$$

After making substitutions $p_i = \rho\omega_i, i = 1, 2$ we have formulated the quantifier elimination problem which solves this example as

$$\forall\omega_1, \forall\omega_2,\ -2\rho\omega_1 - \rho\omega_2 + 3 > 0,\ 4(\rho\omega_2 - 1)\rho\omega_1 - 4\rho\omega_2 + 3 > 0\ .$$

The QEPCAD code eliminates the quantified variables $\omega_i, i = 1, 2$ and give us the quantifier free formula equivalent to the above quantified formula

$$2\rho - 1 < 0$$

in 1.033 s. So the result is that if $\rho < 0.5$ then our system is stable for all parameters from $B(\rho)$ (compare this precise result with several numerical testing in [21]).

4.1.2 Output Feedback Stabilization Example

Consider the static output feedback example in Anderson et.al. [2] where the Tarski-Seidenberg theory was applied manually. We have the plant

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u;\ \ y = \begin{bmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{bmatrix} x \tag{4.1.2}$$

with the static output feedback, $u = -Kx$ where $K = [q_1, q_2] = [v, w]$ so that the closed-loop system matrix is

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -w & 13 + 5v - w & -v \end{bmatrix} \tag{4.1.3}$$

with a closed-loop characteristic polynomial $s^3 + vs^2 + (w - 5v - 13)s + w$. The Liénard-Chipart criterion gives us the conditions

$$v > 0, \quad w > 0, \quad -5v^2 - 13v + vw - w > 0 \tag{4.1.4}$$

for the polynomial to be stable, i.e. to have all roots with negative real part. The solution of the inequalities (4.1.4) can be stated as the quantifier elimination problem. We have stated it as

$$\exists w,\ (v > 0) \wedge (w > 0) \wedge (-5v^2 - 13v + vw - w > 0).$$

The QEPCAD code eliminated w gave us the unquantified equivalent formula

$$v - 1 > 0 \wedge v > 0$$

in 0.034 seconds (compared to about 2 pages in [2]). Now we can choose $v > 1$, e.g $v_0 = 2$, and the inequalities (4.1.4) give us restrictions on w, namely for v_0, we obtain that $w > 0 \wedge w - 46 > 0$ so that one parameterization of stabilizing controllers is, $v_0 = 2, w > 46$.

4.1.3 Simultaneous Stabilization Example

The problem here is to find a (stable) compensator which simultaneously stabilize three different plants with transfer functions

$$\begin{aligned} G_1(s) &= \frac{2 - s}{(s^2 - 1)(s + 2)}, \\ G_2(s) &= \frac{2 - s}{s^2(s + 2)}, \\ G_3 s) &= \frac{2 - s}{(s^2 + 1)(s + 2)}. \end{aligned} \tag{4.1.5}$$

This problem is taken form [1] where it is shown, using certain sufficient conditions, that the problem can be solved using a third-order compensator. We wish to use QE methods to explore the possibility of simultaneously stabilizing these three plants with a lower order compensator. So the problem is to explore first and second order (stable) compensators $C(s)$ such that the numerators of all the transfer functions $Q_i = 1 + CP_i,\ ,i = 1, 2, 3$ are stable.

First-order compensator: As a first attempt we have chosen the stable compensator of the form

$$C = \frac{as + b}{s + d} \tag{4.1.6}$$

with $d > 0$. In this case the numerators of Q_i are

$$\begin{aligned} f_1(s) &= s^4 + s^3(d+2) + s^2(-a+2d-1) + s(2a-b-d-2) + 2(b-d), \\ f_2(s) &= s^4 + s^3(d+2) + s^2(-a+2d) + s(2a-b) + 2b, \\ f_3(s) &= s^4 + s^3(d+2) + s^2(-a+2d+1) + s(2a-b+d+2) + 2(b+d), \end{aligned}$$

and we need to choose such a, b, d so that the polynomials Φ_i are stable. By using the Liénard-Chipart criterion, we obtain a set of inequalities in parameters a, b, d . After simplification, these inequalities become

$$\begin{aligned} -2a^2d - 8a^2 + abd + 6ab + 5ad^2 + 14ad + 8a - b^2 - 4bd^2 - 13bd - 10b &> 0 \\ -ad - 4a + b + 2d^2 + 4d &> 0 \\ b - d &> 0 \\ 2a - b - d - 2 &> 0 \\ -a + 2d - 1 &> 0 \\ -2a^2d - 8a^2 + abd + 6ab + 4ad^2 + 8ad - b^2 - 4bd^2 - 12bd - 8b &> 0 \\ -2a^2d - 8a^2 + abd + 6ab + 3ad^2 + 2ad - 8a - b^2 - 4bd^2 - 11bd - 6b &> 0 \end{aligned} \tag{4.1.7}$$

The quantifier elimination problem stated with these inequalities connected by logical conjunction and quantified by $\exists a, \exists b, \exists d$ was solved by the QEPCAD program. The result given by QEPCAD was "false" which means that there does not exist any real a, b, d for which inequalities (4.1.7) hold. So we have proved that there does not exist any stable first-order compensator (4.1.6) which stabilize simultaneously all three plants. It should be noted that while the application of the stability criterion for fourth-order polynomials is straightforward, an direct analysis of all the inequalities in (4.1.7) is non-trivial.

Second-order compensator with three parameters: In order to reduce the number of parameters to be considered we assume a second-order compensator of the form

$$C(s) = \frac{A(s+B)^2}{(s+D)^2} \tag{4.1.8}$$

with $D > 0$, for stability of the compensator. The Liénard-Chipart criterion gives us after some inequality simplification $A > 0, B > 0$ and

$$P_1 > 0 \wedge P_2 > 0 \wedge P_3 > 0 \wedge P_4P_5 > 0 \wedge A > 0 \wedge BP_6P_7 > 0 \wedge P_8P_9 > 0 \tag{4.1.9}$$

where

$$P_1 = AB^2 - D^2, \tag{4.1.10}$$

$$P_2 = -AB + A + D^2 - D - 1, \tag{4.1.11}$$

$$P_3 = AB - AD - 2A + D^3 + 4D^2 + 4D, \tag{4.1.12}$$

$$
\begin{aligned}
P_4 &= AB - 2A - BD^2 - 4BD - 4B + 2D^2 + 5D + 2, && (4.1.13)\\
P_5 &= AB^3 - AB^2D - 4AB^2 + 2ABD + 4AB + 2BD^3 + 5BD^2 + 2BD - D^3 \\
&\quad -4D^2 - 4D, && (4.1.14)\\
P_6 &= AB - 2A - BD^2 - 4BD - 4B + 2D^2 + 4D, && (4.1.15)\\
P_7 &= AB^2 - ABD - 4AB + 2AD + 4A + 2D^3 + 4D^2, && (4.1.16)\\
P_8 &= AB - 2A - BD^2 - 4BD - 4B + 2D^2 + 3D - 2, && (4.1.17)\\
P_9 &= AB^3 - AB^2D - 4AB^2 + 2ABD + 4AB + 2BD^3 + 3BD^2 - 2BD + D^3 \\
&\quad +4D^2 + 4D. && (4.1.18)
\end{aligned}
$$

By further "hand" simplification of the inequality (4.1.9) may be written

$$P_1 > 0 \wedge P_2 > 0 \wedge P_3 > 0 \wedge P_5 > 0 \wedge P_8 > 0 \wedge A > 0 \wedge B > 0 \wedge D > 0 \qquad (4.1.19)$$

where we have added the inequality on D. QEPCAD was able to solve the existence problem, but only after 2 hours of CPU time. The QEPCAD output indicated that a solution did indeed exist, so that a second order stable stabilizing compensator does exist. To find a particular compensator and to reduce the CPU time require to find a solution, the following supplementary techniques were used.
Note here that all polynomials in (4.1.19) are linear in A which allows us to apply Weispfenning's method [22] to the elimination of $\exists A$ from (4.1.19). The result of the elimination has been simplified using REDLOG [23] package to eliminate equations (we are interested only in solutions A_0, B_0, D_0 for which exists neighborhood of this point such that all point in the neighborhood are also solution and equations do not have such solutions) from the result and put it into a disjunction normal form of six conjunctions. Each of the conjunctions has been simplified by our INEQ package and further simplified manually. After eliminating the variable A, we obtain the Boolean formula

$$
\begin{aligned}
& D > 0 \wedge B > 0 \wedge (D^2 - D - 1)B^2 - BD^2 + D^2 > 0 \wedge \\
& (D+2)^2B^2 + BD - D^2 - 2D > 0 \wedge \qquad (4.1.20)\\
& -((D^2 + 4D + 2)B - (D^2 + 3D + 1)B^2 - D^2 - 2D) > 0 \wedge \\
& (2D-1)(D+2)B^2 - (D+2)^2B^3 + BD^2 - 2D^2 > 0.
\end{aligned}
$$

The QEPCAD package solved the quantifier elimination problem $\exists D$ in (4.1.20) with the result $1 < B < 2$. We chose $B = 3/2$, substituted into (4.1.20) and from the plot of the resulting polynomials in D we have chosen $D = 15$ (any $D > 13.01$ would work fine). These values of B, D are then substituted into (4.1.19) which gives us a system of linear inequalities in A. This system was then simplified by the package INEQ which resulted in $100 < A < 119$ from which we chose $A = 110$. Thus a particular second-order compensator is given by $A = 110, B = 3/2, D = 15$.
It can be shown that the quantifier-free formula in (4.1.20) is equivalent to the points above the curve in Figure (4.1.3).

4.2 Performance

As noted in chapter 2, multiple frequency domain design objectives involving constraints on magnitude of transfer functions, can be reduced to QE problems with the $\forall$ quantifier on the the frequency variable ω. The following example illustrates a problem of this type.

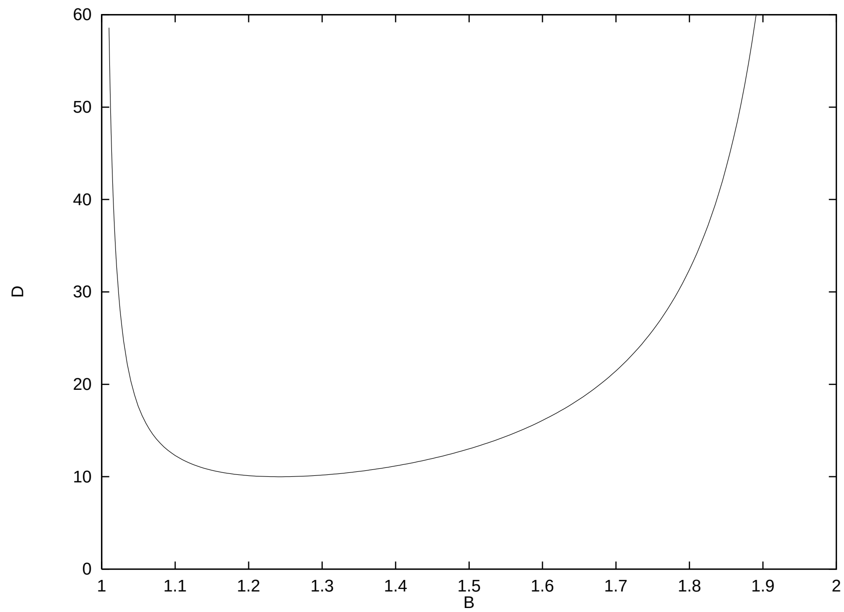

Figure 4.1.1: The set of acceptable values of B, D

4.2.1 Robust Multiobjective Design Example

The problem considered here is a simplified version of the problem in [10, 17]. The plant is assumed to be an unstable first order system with transfer function

$$G(s,p) = \frac{p_1}{1 - s/p_2}, \quad 0.8 \le p_1 \le 1.25 \tag{4.2.21}$$

with simple output feedback $C(s,q) = q_1$. The tracking error bound (See chapter 2) is assumed to be given by $\delta = 0.1$, with $\omega_1 = 10$, and the control effort bound is given to be $\epsilon = 20$. To solve the robust multiobjective problem we must satisfy the following inequalities for *all* $16 \le 20p_i \le 25$. Note that the constraints on p_i have been re-written in terms of integers to satisfy the requirements of the QE theory, i.e. all numerical polynomial coefficients must be integers.

Robust stability: The stability criterion for the first order closed-loop characteristic polynomial is simply,

$$p_2(1 + p_1 q_1) < 0 \tag{4.2.22}$$

Tracking error: If the magnitude-squared of $S(j\omega)$ is computed, and the denominator polynomial is cleared we obtain the condition,

$$99\omega^2 + (p_2)^2(100(1 + p_1 q_1)^2 - 1) > 0, \quad 0 \le \omega \le 10 \tag{4.2.23}$$

Control effort: With the same computations as for the tracking error, we obtain for control effort the condition,

$$(400 - q_1^2)\omega^2 + (p_2)^2(400(1 + p_1 q_1)^2 - q_1^2), \text{all real } \omega \tag{4.2.24}$$

The direct solution from QEPCAD, with no preliminary simplifications, yields the quantifier-free formula

$$\Psi(q_1) = [(q_1 + 20 \ge 0) \wedge (q_1 + 2 \le 0] \vee [(8q_1 + 11 < 0) \wedge (q_1 + 2 \ge 0)] \tag{4.2.25}$$

From (4.2.25) one obtains the following parameterization of output-feedback compensation which satisfies the robust multiobjective problem.

$$-20 \le q_1 < -1.375$$

```
[33 CDC -Milanese et.al]
(q1,p1,p2,w1,w2)
1
(A p1)(A p2)(A w1)(A w2)
[
[16 <= 20 p1 /\ 20 p1 <= 25 /\
 16 <= 20 p2 /\ 20 p2 <= 25 /\ 0 <= w1 /\ w1 <= 10 ]
==>
[p2 (1 + p1 q1) < 0 /\
99 w1^2 + p2^2 (100 (1 + p1 q1)^2 - 1 ) > 0 /\
(400 - q1^2) w2^2 + p2^2 (400 (1 + p1 q1)^2 - q1^2) > 0
]
].
go
go
go
go
```

Table 4.2.1: Input File for QEPCAD

To illustrate QEPCAD syntax we list for this particular example, the QEPCAD input file in Table 4.2.1 and the output file in Table 4.2.1. Note that the first step in defining the inputs is to list all the variables, e.g. $(q_1, p_1, p_2, w1, w2)$, with the un-quantified variables listed first. The un-quantified variables are identified by listing the number of such variables in the variable list. In this particular example the number one is listed since there is only one un-quantified variable, q_1. Note also that the variable ω is specified as two separate variable $w1$ and $w2$, since the inequalities involving ω are for different ranges of ω.

```
=======================================================

                Quantifier Elimination
                          in
            Elementary Algebra and Geometry
                          by
      Partial Cylindrical Algebraic Decomposition

              Version 12 (Interactive)
                    September 1993

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                 with contributions by
                  George E. Collins
                  Jeremy R. Johnson
                 Mark J. Encarnacion

=======================================================
Enter an informal description  between '[' and ']':
[33 CDC -Milanese et.al]Enter a variable list:
(q1,p1,p2,w1,w2)Enter the number of free variables:
1
Enter a prenex formula:
(A p1)(A p2)(A w1)(A w2)
[
[16 <= 20 p1 /\ 20 p1 <= 25 /\
 16 <= 20 p2 /\ 20 p2 <= 25 /\ 0 <= w1 /\ w1 <= 10 ]
==>
[p2 (1 + p1 q1) < 0 /\
99 w1^2 + p2^2 (100 (1 + p1 q1)^2 - 1 ) > 0 /\
(400 - q1^2) w2^2 + p2^2 (400 (1 + p1 q1)^2 - q1^2) > 0
]
].

=======================================================

Before Normalization >
go

Before Projection (w2) >
go

Before Choice >
go

Before Solution >
go
```

```
=======================================================

An equivalent quantifier-free formula:

[ [ q1 + 20 >= 0 /\ q1 + 2 <= 0 ] \/ [ 8 q1 + 11 < 0 /\ q1 + 2 >= 0 ] ]

In other words,

[ [ P_1,1 >= 0 /\ P_1,19 <= 0 ] \/ [ P_1,7 < 0 /\ P_1,19 >= 0 ] ]

where
P_1,1 = q1 + 20
P_1,19 = q1 + 2
P_1,7 = 8 q1 + 11

========================================================================

=====================  The End  =======================

-----------------------------------------------------------------------------
0 Garbage collections, 0 Cells and 0 Arrays reclaimed, in 0 milliseconds.
1729490 Cells in AVAIL, 2000000 Cells in SPACE.

System time: 15384 milliseconds.
System time after the initialization: 11467 milliseconds.
-----------------------------------------------------------------------------
```

Table 4.2.2: Output File for QEPCAD

Chapter 5

Conclusions

Quantifier-elimination theory provides a method for the solution of control system design problems using simple fixed-structure compensators. While the currently available software limits the application of QE theory to modest size problems, the theory is valuable for the following reasons:

- One can solve problems for which there is no general analytical theory, e.g. problems of output feedback, simultaneous stabilization, robust multiobjective design, etc.
- From a practical point of view it is important to keep the compensator as simple as possible, since some on-line tuning of compensator parameters is almost always required.
- Even modest size practical problems cannot be effectively computed by hand because of the large number of inequalities involved.

Finally it should be noted that discretization methods, both deterministic and random (Monte Carlo), where all parameters are discretized, are alternate approaches to the solution of the class of problems consider here, i.e. problems of modest complexity with no analytical solutions. However QE methods have the advantage of leaving no "holes" in the parameter space in the deterministic discretization case, or of yielding "probability-one" results in the random discretization case.

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