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FROM NYQUIST TO KHARITONOV: ROBUST CONTROLLERS

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ABSTRACT

This paper contains the proofs of many recent extreme-point stability results. The main contribution of the paper is to relate the extreme-point results to classical methods such as the root locus, Bode diagrams and Nyquist criterion, and then, to use these results in robustly stabilizing a family of single-input-single-output (SISO) systems.

1 INTRODUCTION

The stability of uncertain polynomials is an important problem in the analysis and design of robust control systems. The problem was elegantly solved in the continuous-time case and for independent coefficients by the celebrated *Kharitonov theorem* [**?**], which reduces the stability test of a family of polynomials to that of 4 special members of that family. Later, the *Edge Theorem* [**?**] generalized *Kharitonov's theorem* to the case of interdependent coefficients and $\mathcal{D}$-stability.

Since the appearance of the Edge Theorem, a great deal of effort has been made to improve its efficiency. Two approaches have been considered, both them trying to avoid the sweep required for each exposed edge, thus reducing the combinatorical explosion and the computational burden. Bialas [**?**] developed a test for checking the convex combination of two stable polynomials based on the computation of the eigenvalues of a matrix. In [**?**] and [**?**], for example, the frequencies at which some edge's polynomial becomes marginally stable are determined. These frequencies depend only on the extreme polynomials and allow to test a finite number of polynomials.

The other approach (which is used in the present paper), is termed the *Extreme-point* approach and is based on placing conditions on the perturbation of the coefficients which provide a stability test involving only the extremes of the edge. This approach boils down to finding "convex directions" [**?**] and [**?**] for varying polynomials so that the stability of the vertices of an edge is sufficient to guarantee the stability of the whole edge. This approach has found immediate applications in the synthesis of robust controllers where a reduced number of interval plants has to be considered. Examples of this work can be found in [**?**], [**?**], and [**?**].

One of the main objective of the paper is to bring together some of the new robust stability methodologies and some classical control concepts such as root locus, Bode diagrams, and Nyquist criterion. In fact, we present some simple proofs of the new concepts using these classical methods and compare them to the available more powerful (but complicated) proofs which we present in the appendices. The paper is then part of a trend of revival in using classical methodologies to study the stability of structured but uncertain systems [?]. The closest papers (that we are aware of) to our research are [?] where similar graphical tests have been used to study the robust stability of Time-Delay systems, and [?] where a more general problem was solved. The difference between this paper and [?] is that our proofs are much simpler (although more limited), and our results are then used in the synthesis of robust controllers. This synthesis is accomplished by calling on some H_∞ results which provide a stabilizing controller of 2 plants, then using the extreme-point results to guarantee the stability of a family of plants. Many results have recently appeared to incorporate Kharitonov-like techniques in designing robust controllers. Unfortunately, all of these results either require a fixed structure controller, and/or reduce the problem to that of simultaneously stabilizing a minimum of 4 and as many as 16 plants [?]. It is however known that the simultaneous stabilization of more than 2 plants is an open problem [?]. Our approach is then to stabilize 2 extreme plants while guaranteeing the stability of a large family of plants (posiibly infinite) using our results.

The paper is organized as follows. Section 2 presents some preliminaries and classical interpretations of the extreme-point stability results. In Section 3, we present some new and simple proofs of known results, using whenever possible the classical interpretations of Section 2. In Section 4, we discuss the simultaneous stabilization problem. Finally, our conclusions are presented in Section 5.

2 PRELIMINARIES AND CLASSICAL INTERPRETATIONS

Our results will be a consequence of the following lemma which is a restatement of a special zero-exclusion principle and appears in different forms in the literature, see for example [?]. The proof is presented because it illustrates a change of variable which will allow us to interpret the result using classical concepts. A similar change of variable was used in [?].

Lemma 1 *Let*

$$P(s,\lambda) \ = \ P_0(s) + \lambda P_1(s) \tag{1}$$

be a family of polynomials of constant degree (i.e. its degree is constant regardless of the variations in λ), where $\lambda \in [\lambda_{min}, \lambda_{max}]$ and where $P_0(s)+\lambda_{min}P_1(s)$

and $P_0(s)+\lambda_{max}P_1(s)$ are stable. Then, $P(s,\lambda)$ is stable for all $\lambda \in [\lambda_{min}, \lambda_{max}]$ if and only if:

$$arg\{P_0(jw) + \lambda_{min}P_1(jw)\} \neq arg\{P_0(jw) + \lambda_{max}P_1(jw)\} + (2l+1)\pi, \quad (2)$$

$\forall w \in [0,\infty)$ *and all integers l.*

Proof:
Sufficiency: Let us write $k = \frac{\lambda-\lambda_{min}}{\lambda_{max}-\lambda}$ and note that k goes between 0 and ∞ as λ varies. Then, let us rewrite

$$\begin{aligned} P(s,\lambda) &= P_0(s) + \lambda P_1(s) \\ &= P_0(s) + \frac{k\lambda_{max} + \lambda_{min}}{1+k} P_1(s) \end{aligned} \quad (3)$$

Therefore,

$$\begin{aligned} 0 &= P(s,\lambda) \\ &= [P_0(s) + \lambda_{min}P_1(s)] + k[P_0(s) + \lambda_{max}P_1(s)] \end{aligned} \quad (4)$$

Note that this form will allow a root locus-type analysis of our result by writing

$$0 = 1 + k\frac{P_0(s) + \lambda_{max}P_1(s)}{P_0(s) + \lambda_{min}P_1(s)} \quad (5)$$

where both the numerator and denominator are stable polynomials. Let us proceed by contradiction and assume that for all integers l and $\forall w \in [0,\infty)$, (2) is satisfied, but that $P(s,\lambda_0)$ is unstable for some $\lambda_0 \in [\lambda_{min}, \lambda_{max}]$, or some $k_0 \in [0,\infty)$. Using a root-locus argument, If $P(s,\lambda_0)$ is to have at least one root in RHP, and since both $P(s,\lambda_{min})$ and $P(s,\lambda_{max})$ are stable, there must exist at least one frequency $w_0 \in R^+$ at which

$$0 = [P_0(jw_0) + \lambda_{min}P_1(jw_0)] + k_0[P_0(jw_0) + \lambda_{max}P_1(jw_0)] \quad (6)$$

which, because neither $P_0(jw_0) + \lambda_{max}P_1(jw_0)$ nor $P_0(jw_0) + \lambda_{min}P_1(jw_0)$ can be zero, leads to

$$\begin{aligned} k_0 &= -[\frac{P_0(jw_0) + \lambda_{min}P_1(jw_0)}{P_0(jw_0) + \lambda_{max}P_1(jw_0)}] \\ &= -\frac{P_m(jw_0)}{P_M(jw_0)} \end{aligned} \quad (7)$$

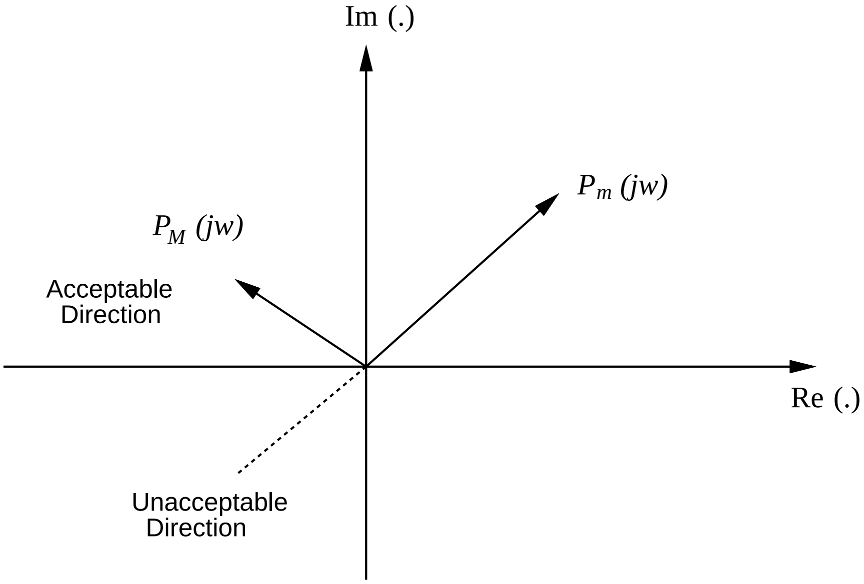

Figure 1: The Argument Condition

See Figure 1. Therefore, since k_0 is a positive number, we must have

$$arg\{P_0(jw_0) + \lambda_{min}P_1(jw_0)\} = arg\{P_0(jw_0) + \lambda_{max}P_1(jw_0)\} + (2l+1)\pi \quad (8)$$

which is a contradiction.

Necessity: Now assume that $P(s, \lambda)$ is stable for all $\lambda \in [\lambda_{min}, \lambda_{max}]$. Therefore,

$$0 \neq [P_0(jw) + \lambda_{min}P_1(jw_0)] + k[P_0(jw) + \lambda_{max}P_1(jw)], \ \forall w \in R^+ \quad (9)$$

which then leads $\forall w \in [0, \infty)$ to

$$arg\{P_0(jw) + \lambda_{min}P_1(jw_0)\} \neq arg\{P_0(jw) + \lambda_{max}P_1(jw)\} + (2l+1)\pi,$$

$$\triangle$$

Note 1: Condition (2) may be restated in different ways which will be useful in the sequel. For example, if we let $P_i(jw) = p_i(w) + jq_i(w)$, where $p_i(w)$ is the real part of $P_i(jw)$ and $q_i(w)$ is its imaginary part, then

$$\begin{aligned} (2) \quad \Leftrightarrow \quad & -[p_0(w) + \lambda_{min}p_1(w)] \neq k[p_0(w) + \lambda_{max}p_1(w)], \ \mathit{or} \\ & -[q_0(w) + \lambda_{min}q_1(w)] \neq k[q_0(w) + \lambda_{max}q_1(w)] \qquad (10) \end{aligned}$$

$\forall w \in [0, \infty)$, and $\forall k \in [0, \infty)$.

$\triangle$

We can in fact discard the case where $k = 0$ because it will lead to the instability of $P_0(jw) + \lambda_{min} P_1(jw)$. Therefore, in the sequel, we can let $k \in (0, \infty)$.

Note 2: If (2) is violated, there will exist some frequency w_0 and some gain k_0 such that

$$\begin{aligned}
p_0(w_0) &= -[\frac{k_0\lambda_{max} + \lambda_{min}}{1 + k_0}]p_1(w_0) \\
q_0(w_0) &= -[\frac{k_0\lambda_{max} + \lambda_{min}}{1 + k_0}]q_1(w_0) \\
\frac{k_0\lambda_{max} + \lambda_{min}}{1 + k_0}p_1(w_0) &= k_0[p_0(w_0) + \lambda_{max}p_1(w_0)] = -p_0(w_0) \\
\frac{k_0\lambda_{max} + \lambda_{min}}{1 + k_0}q_1(w_0) &= k_0[q_0(w_0) + \lambda_{max}q_1(w_0)] = -q_0(w_0) \qquad (11)
\end{aligned}$$

$\triangle$

Note 3: The root locus interpretation has been suggested by [?] in a similar setting. In fact, consider again equation (4)

$$0 = 1 + k\frac{P_0(s) + \lambda_{max}P_1(s)}{P_0(s) + \lambda_{min}P_1(s)}$$

where the open-loop system

$$G(s)H(s) = \frac{P_0(s) + \lambda_{max}P_1(s)}{P_0(s) + \lambda_{min}P_1(s)} \qquad (12)$$

is stable and minimum-phase. Note also that there is an equal number of poles and zeros so no root-locus branches may escape to infinity. As k increases from 0 to ∞, the root locus of the closed-loop system will start from the roots of $P_0(s) + \lambda_{min}P_1(s)$ and end at the roots of $P_0(s) + \lambda_{max}P_1(s)$. It is obvious that instability will occur if the root locus crosses (or touches) the jw axis. It is also obvious that if the locus crosses into the right half plane (RHP) it will eventually have to cross back into the LHP.

$\triangle$

Note 4: Yet another interpretation of Lemma 1 is afforded by considering the phase Bode plots of $P_0(jw) + \lambda_{min}P_1(jw)$ and $P_0(jw) + \lambda_{max}P_1(jw)$. Assuming without loss of generality that both $P_0(j0) + \lambda_{min}P_1(j0)$ and $P_0(j0) +$

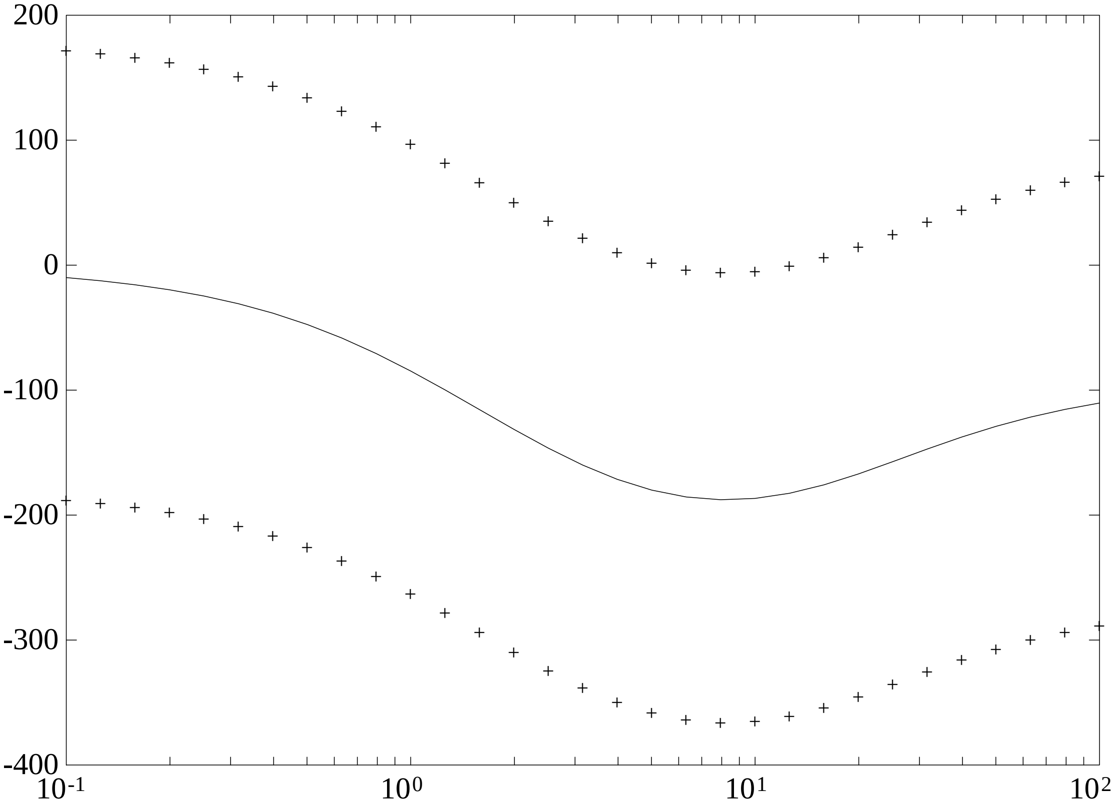

Figure 2: $\pm\pi$ envelop around the phase of $P_0(s) + \lambda_{max} P_1(s)$

$\lambda_{max} P_1(j0)$ are positive, the condition (2) of the lemma translates into the phase plot of $P_0(jw) + \lambda_{min} P_1(jw)$ never crossing the $\pm\pi$ envelop around the phase of $P_0(jw) + \lambda_{max} P_1(jw)$ as shown in Figure 2. This is a very easy test to use given $P_0(s)$, λ_{min}, λ_{max}, and $P_1(s)$ and was described in [**?**] in a very general setting.

$\triangle$

We can also study the stability of the closed-loop system using the Nyquist criterion and Positive-Realness (PR) concepts which we define next.

Definition 1 *An exactly-proper real rational function $T(s)$ which is not identically zero is analytic positive-real or APR if*

1. *$T(s)$ is analytic in the closed right half plane, i.e. in the region $Re(s) \geq 0$, and*
2. *$Re[T(jw)] \geq 0$ for all $w \in R$ at which $Im[T(jw)] = 0$.*

$\triangle$

Therefore, an APR function can never cross the negative real axis, but as opposed to an SPR function it can cross into the LHP. It is obvious that an SPR function is APR but that the converse is not true. We can now prove the following lemma.

Lemma 2 *Let*

$$P(s, \lambda) = P_0(s) + \lambda P_1(s) \tag{13}$$

be a family of polynomials of constant degree. Then, the family is stable for all $\lambda \in [\lambda_{min}, \lambda_{max}]$ if and only if

$$G(s)H(s) = \frac{P_0(s) + \lambda_{max} P_1(s)}{P_0(s) + \lambda_{min} P_1(s)} \tag{14}$$

is APR.

Proof:
Sufficiency: Assume that $G(s)H(s)$ is APR. Then, both $P_0(s)+\lambda_{max}P_1(s)$ and $P_0(s) + \lambda_{min}P_1(s)$ are stable, and $Re[G(jw)H(jw)] \geq 0$ for all w. Therefore, $Re[1 + kG(jw)H(jw)] = 1 + kRe[G(jw)H(jw)] \geq 1, \ \forall k > 0$. That of course guarantees the stability of closed-loop characteristic equation $1+kG(s)H(s) = 0$ and thus of $P(s, \lambda)$ for all $\lambda \in [\lambda_{min}, \lambda_{max}]$.

Necessity: Note first that the stability of the extremes $P_0(s) + \lambda_{min}P_1(s)$ and $P_0(s)+\lambda_{max}P_1(s)$ is obviously necessary. Consider again the open-loop transfer function $G(s)H(s)$ evaluated at $s = jw$

$$\begin{aligned} G(jw)H(jw) &= \frac{P_0(jw) + \lambda_{max}P_1(jw)}{P_0(jw) + \lambda_{min}P_1(jw)} \\ &= 1 + (\lambda_{max} - \lambda_{min}) \frac{P_1(jw)}{P_0(jw) + \lambda_{min}P_1(jw)} \end{aligned}$$

Let us plot the Nyquist plot of $G(jw)H(jw)$ and note that for the stability of the closed-loop characteristic equation $1 + kG(s)H(s)$ for all $k \geq 0$, the Nyquist plot of $G(jw)H(jw)$ should not intersect the negative real axis, or that $Re[G(jw)H(jw)] \geq 0$ for all w where $Im[G(jw)H(jw)] = 0$, which makes $G(s)H(s)$ APR by Definition 1.

△

Note 5: According to the proof of Lemma 2, the plot of

$$\frac{(\lambda_{max} - \lambda_{min})P_1(jw)}{P_0(jw) + \lambda_{min}P_1(jw)}$$

should cross the real axis to the right of the -1 point. Note that because of the stability of $P_0(s) + \lambda_{max}P_1(s)$ we are already guaranteed 0 encirclements of $-1/(\lambda_{max} - \lambda_{min})$ by $P_1(jw)/[P_0(jw) + \lambda_{min}P_1(jw)]$ in the clockwise direction. The condition for the stability of $P(s, \lambda)$ for all $\lambda \in [\lambda_{min}, \lambda_{max}]$ requires more: $-1/(\lambda_{max} - \lambda_{min})$ should be to the left of the left-most real axis crossing by $P_1(jw)/[P_0(jw) + \lambda_{min}P_1(jw)]$.

△

Note 6: Note that if $G(s)H(s)$ is SPR then, the stability of $P(s, \lambda)$ is deduced, but that the necessity of SPR is not in general required. In fact, an SPR function has an angle contained in the region $[-\pi/2, \pi/2]$ for all w which leads to the stability of $P(s, \lambda)$ using Lemma 1 or Note 4.

△

Recently, a related result was proven in [**?**], where it was shown that the SPRness of the exactly-proper $a(s)/b(s)$ is equivalent to the stability of

$$P(s, \lambda) = (1 - \lambda)a(s) + j\lambda b(s); \;\; \lambda \in [0, 1]$$

3 OLD RESULTS, NEW PROOFS

We next prove some recent results as special cases of the above lemmas. Note that we may present 2 different proofs of a particular result, in order to contrast our approach with previous ones.

Corollary 1 *Let*

$$P(s, \lambda) \;\; = \;\; P_0(s) + \lambda P_1(s)$$

be a family of polynomials of constant degree, where $\lambda \in [\lambda_{min}, \lambda_{max}]$ *and where any of the following conditions hold*

1. $P_1(s)$ *is all even [**?**]*

2. *$P_1(s)$ is all odd [?]*

3. *$\frac{\partial arg\{P_1(jw)\}}{\partial w} \leq 0,\ \forall w \in R^+$ [?].*

4. *$P_1(s) = E(s)A(s)$ where $E(s)$ is all even and $A(s)$ is antistable.*

5. *$P_1(s) = O(s)A(s)$ where $O(s)$ is all odd and $A(s)$ is antistable.*

6. *$P_1(s) = s^m(as + b)P_2(s)$ where $a, b \in R$, m is a nonnegative integer and $P_2(s)$ is all odd or all even [?].*

7. *$P_1(s) = A(s)s^m(as + b)P_2(s)$ where $A(s)$ is antistable, $a, b \in R$, m is a nonnegative integer and $P_2(s)$ is all odd or all even [?].*

Then, $P(s, \lambda)$ is stable for all $\lambda \in [\lambda_{min}, \lambda_{max}]$ if and only if both $P_0(s) + \lambda_{min}P_1(s)$ and $P_0(s) + \lambda_{max}P_1(s)$ are stable.

Proof: Note that necessity is trivial, so we will show the sufficiency for each case.

1. Suppose $P_1(s)$ is all even, a plot of $P_1(jw)$ in the complex plane will then lie on the real axis. Therefore, the argument of $G(jw)H(jw)$ satisfies

   $$-\pi/2 < arg\{P_0(jw) + \lambda_{max}P_1(jw)\} - arg\{P_0(jw) + \lambda_{min}P_1(jw)\} < \pi/2$$

   Which along with the stability of the extremes guarantees that $G(jw)H(jw)$ is APR, and therefore the stability of $P(s, \lambda)$.

2. If $P_1(s)$ is all odd, its plot will lie on the jw axis which will violate the argument condition (8) for all w. We present a second proof which uses Lemma 2. Note that an odd $P_1(s)$ contributes a constant phase of $\pm\pi/2$ to the argument of $P_1(jw)/[P_0(jw)+\lambda_{min}P_1(jw)]$. Since $P_0(jw)+\lambda_{min}P_1(jw)$ is Hurwitz, its argument is always increasing with frequency so that the polar plot of $P_1(jw)/[P_0(jw) + \lambda_{min}P_1(jw)]$ will loop in the clockwise direction as w increases from 0 to ∞ as shown in Figure 3 for $\lambda_{max} - \lambda_{min} = 1$. Therefore, by Note 5, we see that for 0 encirclements of the $-1/(\lambda_{max} - \lambda_{min})$ point, no encirclements are allowed or that $-1/(\lambda_{max} - \lambda_{min})$ is to the left of $Re[P_1(jw)/(P_0(jw) + -\lambda_{min}P_1(jw))]$ for all w, which then satisfies the conditions of Lemma 2 and guarantees the stability of $P_0(s) + \lambda P_1(s)$.

3. We will prove this claim two ways, the first uses arguments similar to those of [?] and is given in Appendix A, and the second uses the

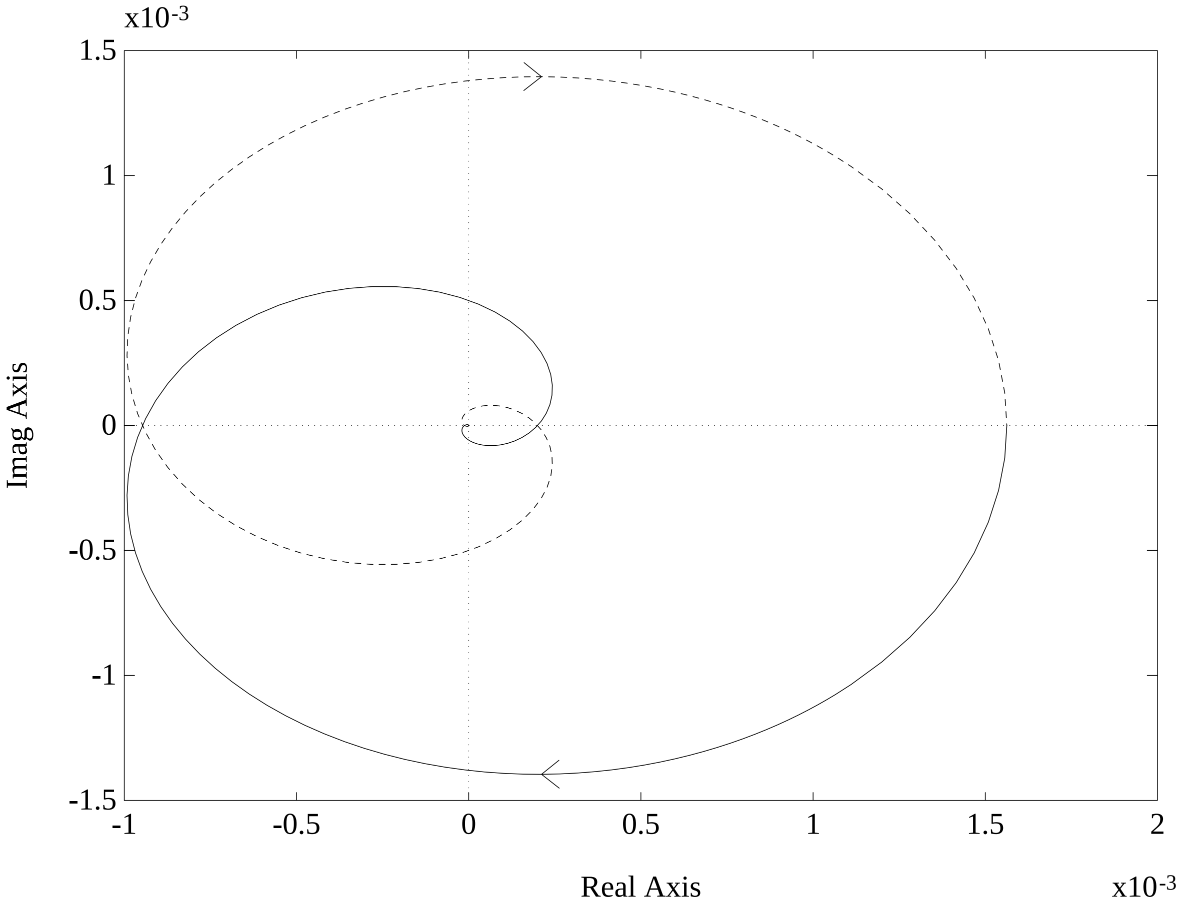

Figure 3: The Polar Plot of $P_1(jw)/(P_0(jw) + \lambda_{min} P_1(jw))$ for Decreasing Phase

Nyquist criterion approach of Lemma 2 and Note 5 and is presented next: Let $arg\{P_i(jw)\} = \theta_i(w)$, and let $arg\{P_0(jw) + \lambda_{min} P_1(jw)\} = arg\{P_m(jw)\} = \theta_m(w)$. Since we assume that $\partial\theta_1(w)/\partial w \leq 0$ and since $\partial\theta_m(w)/\partial w > 0$ for all $w \geq 0$, the plot $P_1(jw)/(P_0(jw) + \lambda_{min} P_1(jw))$ has a decreasing phase. It then starts on the positive real-axis when $w = 0$ and proceeds to loop in the clockwise direction as w goes to ∞ as shown in Figure 3. The proof then follows along the same lines as the proof in step 2.

4. Consider the antistable polynomial $A(s)$. Its argument is then a non-increasing function of w since all its roots are in the Right Half Plane (RHP). By multiplying $A(s)$ by $E(s)$, we introduce a constant argument of 0 or 180 degrees. Therefore, we can write

$$arg\{P_1(jw)\} = arg\{A(jw)\} + c$$

where $c = 0$ or 180 degrees. Then using the argument condition in step 3 we have that $\partial\theta_1(w)/\partial w \leq 0$ for any w, and the desired result follows.

5. This is similar to the last proof with $c = \pm 90$ degrees.

6. We will treat first the case where $m = 0$. The only difficulty occurs when a and b are of the same sign, since otherwise, $P_1(s)$ is the product of an antistable first-order polynomial and an all-even or all-odd polynomial as studied in steps 4 and 5. Let us assume that all coefficients of $P_0(s) + \lambda_{min} P_1(s)$ are positive. Again, we present 2 different proofs, the first of which is in Appendix B. The second proof relies on ideas from the Bode plot and Nyquist criterion. Note first that $P_1(s)$ has an increasing phase with frequency but that the total phase variation is $\leq \pi/2$ as w goes from 0 to ∞. let us then examine the plot of $P_1(jw)/[P_0(jw) + \lambda_{min} P_1(jw)]$ as suggested in the proof of Lemma 2. The phase of that plot will be decreasing as w increases except for the effect of the term $as + b$ in $P_1(s)$. Note however that $P_1(jw)/(P_0(jw) + \lambda_{min} P_1(jw))$ should encircle the $-1/(\lambda_{max} - \lambda_{min})$ zero times in the clockwise direction because of the stability of $P_0(s) + \lambda_{max} P_1(s)$. Assume therefore that the polar plot of $P_1(jw)/(P_0(jw) + \lambda_{min} P_1(jw))$ is as shown in Figure 4 and Figure 5, where $\lambda_{max} - \lambda_{min} = 1$ since this is the only way that both $P_0(s) + \lambda_{min} P_1(s)$ and $P_0(s) + \lambda_{max} P_1(s)$ are stable while allowing for an unstable $P_0(s) + \lambda P_1(s)$. Note for example that this plot can not cross the jw axis back into the RHP because that will imply an angular increase greater than the $\pi/2$ afforded by $b + jwa$. However, by considering the corresponding Bode plot of the phase of $P_1(jw)/[P_0(jw) + \lambda_{min} P_1(jw)]$

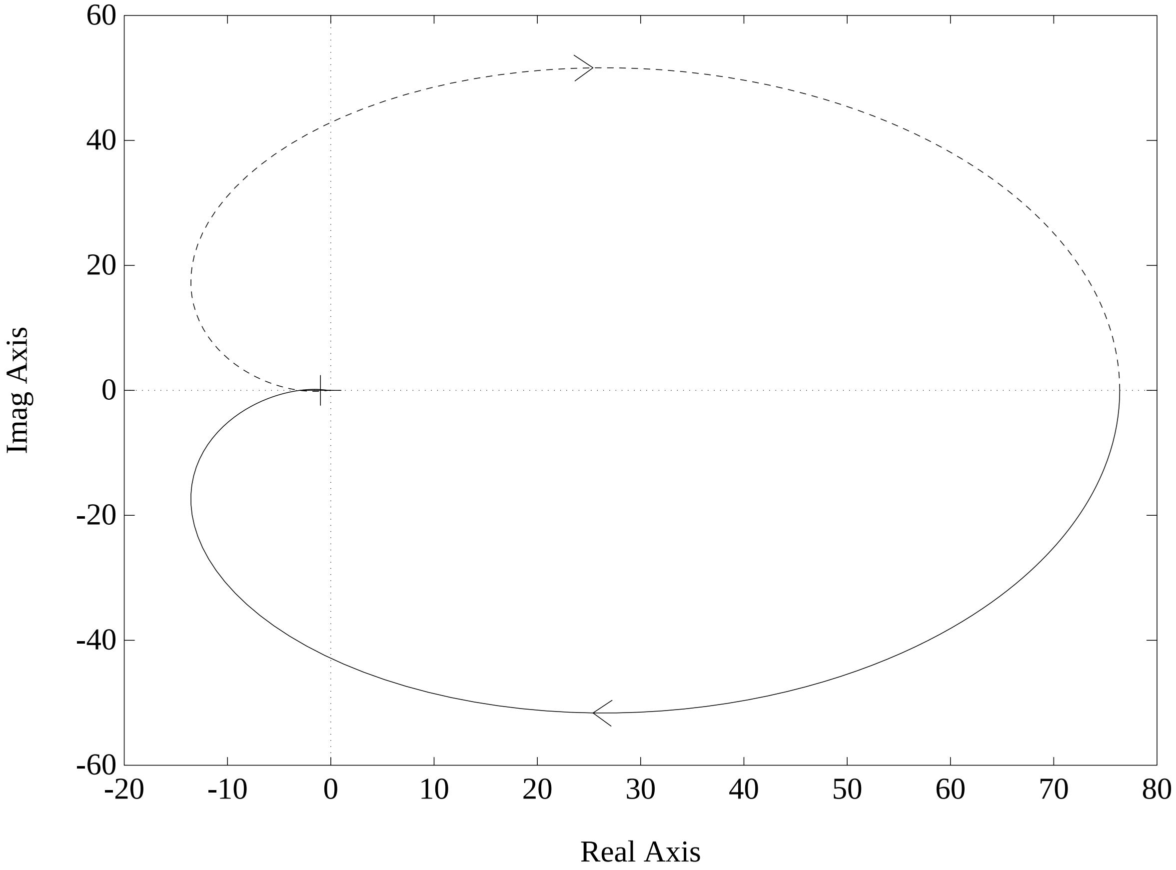

Figure 4: Large View of Polar Plot for $G(jw)H(jw)$

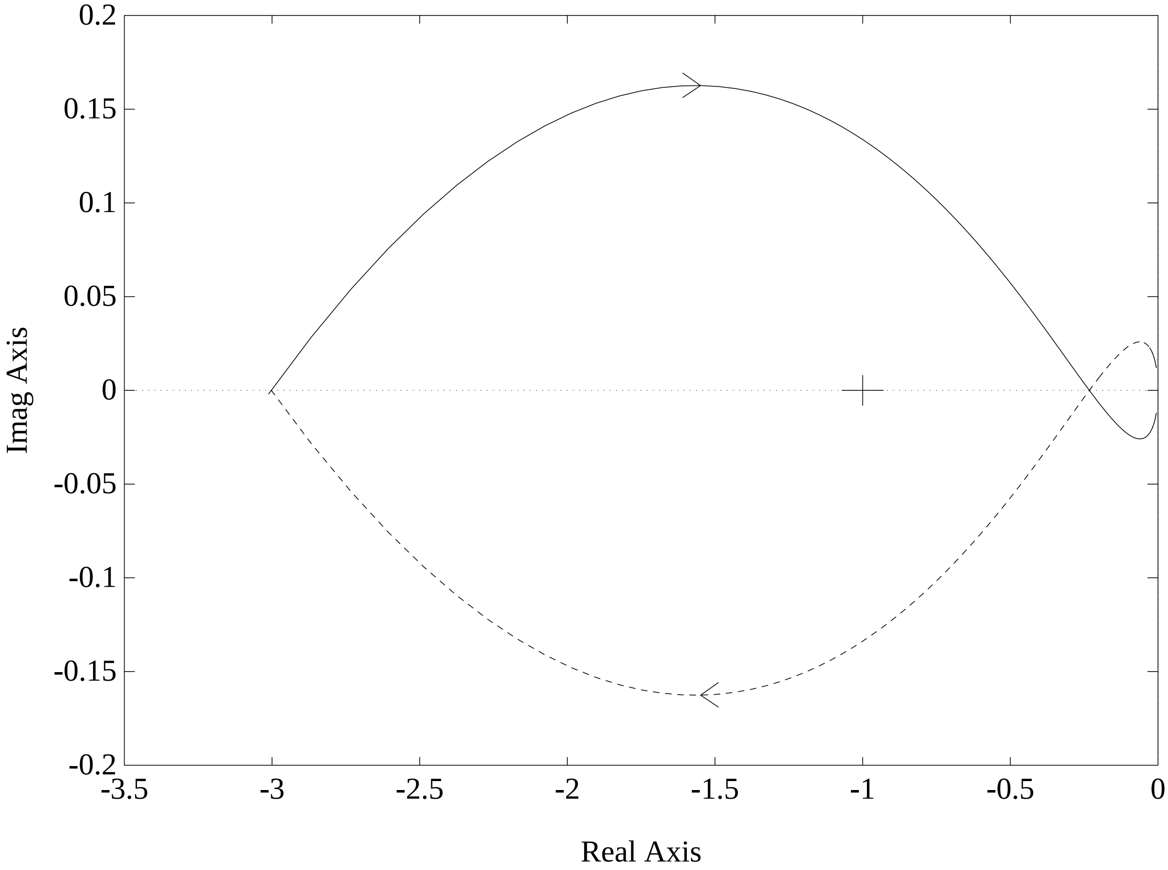

Figure 5: A Closer Look at The Nyquist Plot of $G(jw)H(jw)$

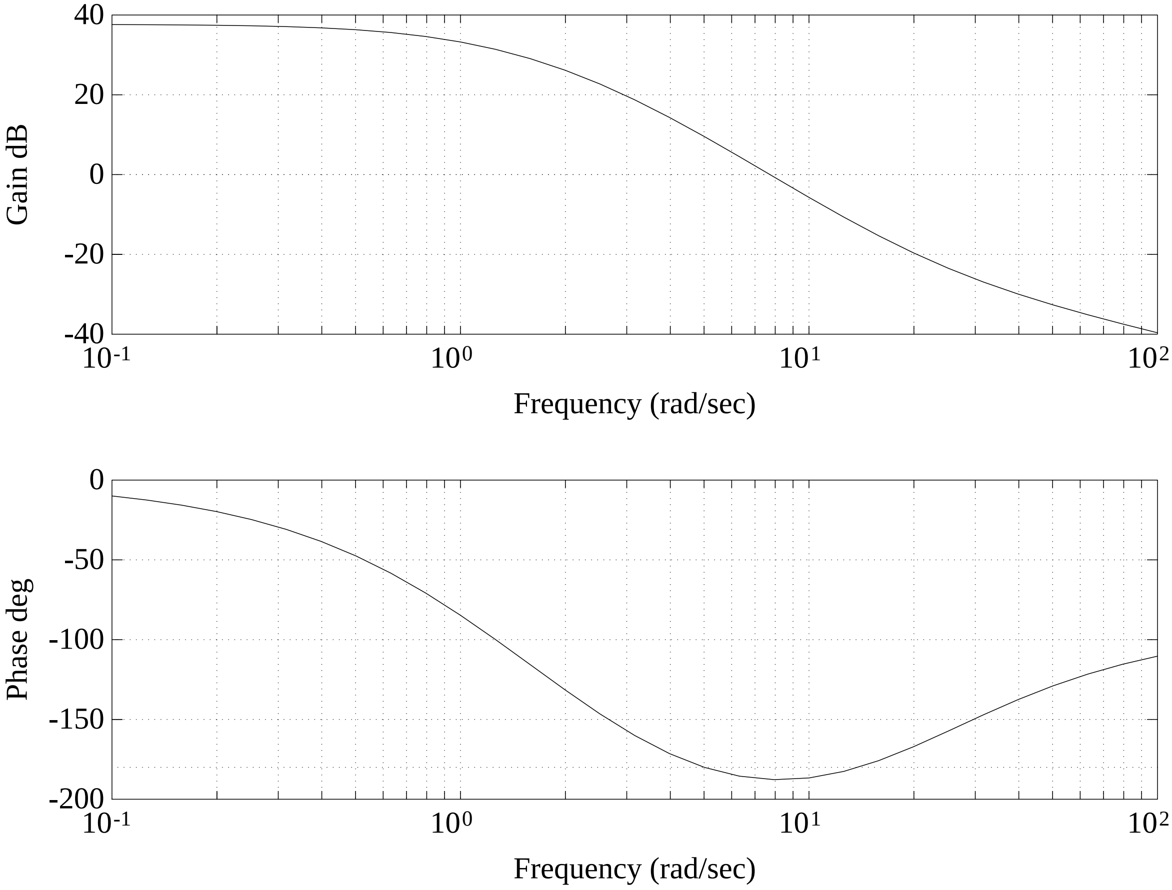

Figure 6: The Bode Plot for the System in Figure 4 and 5

as shown in Figure 6, we see that such polar plot is impossible because it would imply the presence of at least 2 zeros (or 2 roots of $P_1(s)$), in order to cross the -180 degrees line back and forth. That is clearly impossible since only the term $as + b$ contributes to the slope of the argument of $P_1(jw)$. Note that the proof when $P_1(s) = s^m(as + b)P_2(s), m > 0$ is now easily obtained from the above result, by letting $P_3(s) = s^m P_2(s)$ so that $P_3(s)$ is all-even or all-odd depending on whether $P_2(s)$ and m are even or odd.

7. Now consider the case where $P_1(s) = A(s)s^m(as + b)P_2(s)$, and let $\bar{A}(s)$ be a polynomial whose roots are the mirror image of the roots of $A(s)$ about the jw axis. $\bar{A}(s)$ is thus Hurwitz. Note then that $P(s, \lambda)$ is

Hurwitz if and only if $\bar{A}(s)P(s,\lambda)$ is Hurwitz. Now consider

$$\begin{aligned}\bar{P}(s,\lambda) &= \bar{A}(s)P(s,\lambda)\\ &= \bar{A}(s)P_0(s)+\lambda\bar{A}(s)A(s)s^m(as+b)P_2(s)\end{aligned}$$

and note that $\bar{A}(s)A(s)s^mP_2(s)$ is either all-even or all-odd since $\bar{A}(s)A(s)$ is all-even. Also note that $\bar{P}(s,\lambda_{min})$ and $\bar{P}(s,\lambda_{max})$ are Hurwitz, if and only if the corresponding $P(s,\lambda_{min})$ and $P(s,\lambda_{max})$ are. The result is then obvious from the proof of step 6.

△

Next, we present a simple proof of the original Kharitonov Theorem.

Corollary 2 Kharitonov Theorem: *Let a family of monic n-th degree polynomials be given by*

$$p(s)=s^n+a_{n-1}s^{n-1}+\cdots+a_0$$

where $\underline{a}_i\leq a_i\leq\bar{a}_i,\ i=0,1,\cdots n-1$. *Then, the family is Hurwitz if and only the following four polynomials are Hurwitz,*

$$\begin{aligned}p_{11}(s) &= \underline{a}_0+\underline{a}_1s+\bar{a}_2s^2+\bar{a}_3s^3+\underline{a}_4s^4+\underline{a}_5s^5+...\\ p_{12}(s) &= \underline{a}_0+\bar{a}_1s+\bar{a}_2s^2+\underline{a}_3s^3+\underline{a}_4s^4+\bar{a}_5s^5+...\\ p_{22}(s) &= \bar{a}_0+\bar{a}_1s+\underline{a}_2s^2+\underline{a}_3s^3+\bar{a}_4s^4+\bar{a}_5s^5+...\\ p_{21}(s) &= \bar{a}_0+\underline{a}_1s+\underline{a}_2s^2+\bar{a}_3s^3+\bar{a}_4s^4+\underline{a}_5s^5+...\end{aligned}$$

Proof: Note first that necessity is obvious so we will concentrate on proving sufficiency. In fact, define the 4 polynomials as in [**?**]

$$\begin{aligned}g_1(s) &= \underline{a}_0+\bar{a}_2s^2+\underline{a}_4s^4+...\\ g_2(s) &= \bar{a}_0+\underline{a}_2s^2+\bar{a}_4s^4+...\\ h_1(s) &= \underline{a}_1s+\bar{a}_3s^3+\underline{a}_5s^5+...\\ h_2(s) &= \bar{a}_1s+\underline{a}_3s^3+\bar{a}_5s^5+...\end{aligned}$$

Then, note that $p_{kl}(s)=g_k(s)+h_l(s)$. Also note that for all $w\geq 0$, and for any polynomial $p(s)$ in the family,

$$\begin{aligned}g_1(jw) &\leq Re[p(jw)]\leq g_2(jw)\\ -jh_1(jw) &\leq Im[p(jw)]\leq -jh_2(jw)\end{aligned}$$

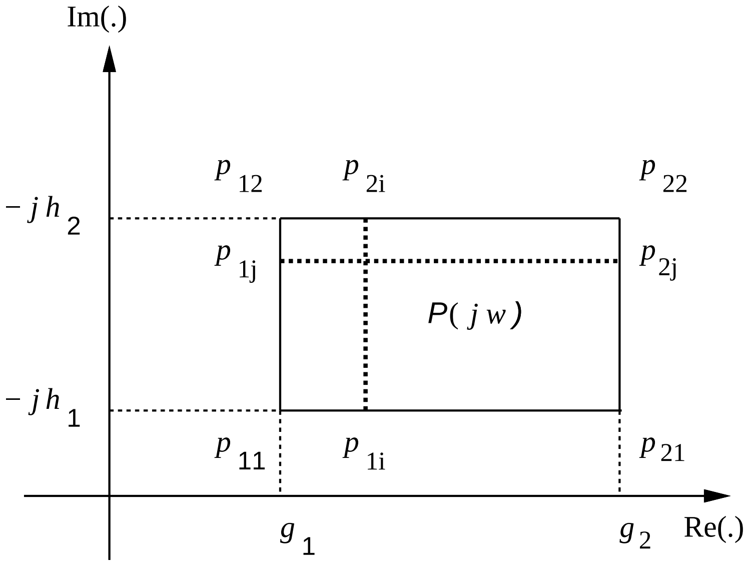

Figure 7: The Level Rectangle

Therefore, we see that for each w, we have a *level rectangle* [?] whose four corners are given by the 4 polynomials in the corollary as shown in Figure 7. In fact, we can write for all $p(jw) \in P(jw)$ because of the geometry of the set $P(jw)$

$$\begin{aligned} Re[p(jw)] &= g_1(jw) + \lambda[g_2(jw) - g_1(jw)];\ \lambda \in [0,1] \\ Im[p(jw)] &= -jh_1(jw) + \mu[-jh_2(jw) + jh_1(jw)];\ \mu \in [0,1] \end{aligned}$$

so that

$$\begin{aligned} p(jw) &= g_1(jw) - jh_1(jw) + \lambda[g_2(jw) - g_1(jw)] \\ &\quad + \mu[-jh_2(jw) + jh_1(jw)] \\ &= p_1 1(jw) + \lambda[p_{21}(jw) - p_{11}(jw)] + \mu[p_{12}(jw) - p_{11}(jw)] \end{aligned}$$

Therefore, if there were an unstable polynomial in the family, such a polynomial will be equal to zero at some w_0, λ_0, μ_0, i.e.

$$0 = p_1 1(jw_0) + \lambda_0[p_{21}(jw_0) - p_{11}(jw_0)] + \mu_0[p_{12}(jw_0) - p_{11}(jw_0)]$$

Now, consider for example the convex combination

$$p_b(s) = p_{11}(s) + \lambda[p_{21}(s) - p_{11}(s)]$$

where $\lambda \in [0, 1]$. Note that $p_{21}(s) - p_{11}(s)$ is all-even. Therefore, by step 1 of Corollary 1, $p_b(s)$ is stable when both extremes are. We use the same approach on the convex combination

$$p_t(s) = p_{12}(s) + \lambda[p_{22}(s) - p_{12}(s)]$$

to guarantee the stability of $p_t(s)$. Then, consider the convex combination

$$\begin{aligned} p(s) &= p_b(s) + \mu[p_t(s) - p_b(s)] \\ &= p_{11}(s) + \lambda[p_{21}(s) - p_{11}(s)] + \mu[p_{12}(s) - p_1 1(s)] \end{aligned}$$

which guarantees that $p(jw) \neq 0$ for all $w \in R$ all $\lambda \in [0, 1]$ and all $\mu \in [0, 1]$. Therefore, we have a contradiction unless all polynomials in the family are stable.

△

This proof shows that Kharitonov's theorem, although extermely powerful, may be obtained as a consequence of Nyquist criterion (thus the title of the paper).

4 Controller Design

Next, we use the previous results to design controllers that will stabilize a family of plants by simply stabilizing two extreme plants. We first review the design of controllers that will make a minimum-phase plants SPR, and thus APR.

4.1 APR Using Feedback

The question addressed in this section is to find conditions on (2.1) or (2.2) so that a feedback controller will render the closed-loop system APR. The result of Theorem 1 appeared in [**?**] for the case of a SISO continuous-time plant and a static output feedback, i.e. $u = -\gamma K y + K r$. Our proof however, appeared first in [**?**]. Let the open-loop system be

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned} \tag{15}$$

or in the transfer function notation

$$P(s) = \frac{Y(s)}{U(s)} = c(sI - A)^{-1} b$$

The closed-loop system is then given by

$$\begin{aligned} \dot{x} &= (A - \gamma bKc)x \; + \; bKr \\ y &= cx \end{aligned} \tag{16}$$

or in the frequency-domain

$$Y(s) = \frac{KP(s)}{1 + \gamma KP(s)} R(s) \tag{17}$$

We present a simple frequency domain proof to show the existence of K and γ that will render the closed-loop system $T(s)$ SPR, which for our purposes, means APR with $\text{Re}\{T(jw)\} > 0$ for all w.

Theorem 1 *Let system (15) be stabilizable and detectable and let its relative degree be $n^* = 1$. Then there exists a K and a positive scalar γ such that the closed-loop system (16) is SPR, if and only if $P(s)$ is minimum phase.*

Proof: *Sufficiency:* Consider the closed-loop transfer function

$$T(s) \; = \; \frac{KP(s)}{1 + \gamma KP(s)}$$

or

$$T(s) \; = \; \frac{1}{\gamma + \frac{1}{KP(s)}}$$

Since $P(s)$ is minimum phase with a relative degree $n^* = 1$, its inverse $P^{-1}(s)$ will be given by

$$P^{-1}(s) \; = \; sL \; + \; P_1(s)$$

where $P_1(s)$ is proper and stable, and $L \neq 0$. In fact, $cb \neq 0$ and $L = 1/cb$. On the other hand, since $P(s)$ is minimum phase, $P_1(s)$ cannot have any poles in Re $(s) \geq 0$. It is now obvious that $T(s)$ will be stable if and only if $W(s) = [\frac{1}{KP(s)} + \gamma]$ has no zeros in $\text{Re}(s) \geq 0$. Let K be given by

$$K \; = \; \frac{1}{cb}$$

then

$$W(s) \; = \; s \; + \; cbP_1(s) \; + \; \gamma$$

and

$$Re[W(jw)] \;=\; Re[cbP_1(jw)] \;+\; \gamma$$

Since $P_1(jw)$ has no poles on the jw axis, $Re[W(jw)]$ may be made positive-definite by a large enough positive scalar γ. This then implies that $W(s)$ is SPR. Since $T(s)$ is the inverse of $W(s)$, it is also SPR [**?**].

Necessity: Suppose now that a nonzero K and a γ were found to make the closed-loop system $T(s)$ SPR and that $n^* = 1$. Then

$$W(s) \;=\; [\gamma + \frac{1}{KP(s)}] = T^{-1}(s)$$

is also SPR. Writing $P^{-1}(s)$ as $P^{-1}(s) \;=\; sL \;+\; P_1(s)$ with $L \;=\; 1/cb$ we get

$$W(s) \;=\; \frac{P_1(s)}{K} + \frac{s}{Kcb} \;+\; \gamma$$

Since $W(s)$ is SPR, $P_1(s)$ must be stable, hence $P(s)$ must be minimum-phase.

$\triangle$

This result indicates that with the given assumptions on $P(s)$, static output feedback can always be found to stabilize the closed-loop system $T(s)$. Moreover, $T(s)$ can also be made SPR to give the desired robustness against passive uncertainties. It can also be seen that a dynamic output feedback compensator will not relax the conditions of the theorem since output compensation can not move the open-loop zeros nor change the relative degree of the plant. The choice of $K \;=\; 1/cb$ in the proof of the theorem is not unique. In fact, it is sufficient to choose $K \;=\; q/cb$ where q is any positive number. Next, note that the condition $cb \neq 0$ (or that $P(s)$ has a relative degree $n^* \;=\; 1$), also reveals that the system (15) has an inverse obtained by cascading one differentiator and a dynamical system [**?**]. Note that the inverse system given in the proof of the last theorem may be written in state-space as

$$\begin{aligned} \dot{x} &= [A - \frac{bcA}{cb}]x \;+\; \frac{b}{cb}\dot{y} \\ u &= -\frac{cA}{cb}x \;+\; \frac{1}{cb}\dot{y} \end{aligned} \tag{18}$$

Now recall that the invertibility of the system (15) may still be inferred even though $cb = 0$. In fact, a sufficient condition for the inverse to exist is that the first nonzero term in the sequence, $d, cb, cAb, cA^2b, ..., cA^{n-1}b$, be nonzero [**?**]. It is then obvious that for a nonzero d, the condition for $T(s)$ to be SPR is that $P(s)$ be minimum phase, i.e. an exactly-proper, minimum-phase transfer function may be made SPR with a static output feedback. In fact, for the SISO case where $d \neq 0$, we can always make the closed-loop system APR with the choice of

$$u = -\gamma y + r \tag{19}$$

with the choice of $\gamma \geq \gamma_m$ where γ_m is the smallest number that will make the closed-loop system SPR.

4.2 Robust Stabilization

Consider the system

$$P(s) = \frac{n_1 n_c + d_1 d_c}{n_2 n_c + d_2 d_c} \tag{20}$$

where n_1/d_1 and n_2/d_2 have been stabilized with the same compensator n_c/d_c. Note that $P(s)$ is the ratio of the 2 closed-loop polynomials which we will guarantee to be Hurwitz by choosing a simultaneously stabilizing compensator n_c/d_c. Also, note that $P(s)$ is exactly-proper under our assumptions. It would be extremely useful if we can choose n_c/d_c so that $P(s)$ is APR, because then, we may consider the numerator and denominator of $P(s)$ as 2 extreme polynomials and guarantee the stability of the line joining them. Unfortunately, we have as of yet to find a synthesis methodology to guarantee the stabilization of n_1/d_1 and n_2/d_2 while making the ratio of their closed-loop characteristic polynomials APR. Therefore, assume that $P(s)$ is our new open-loop system and use the compensator $u = -\gamma y + r$. See Figure 8. We can then see that with the results of Section 4.1 that a large enough γ, the closed-loop system given by

$$T(s) = \frac{n_1 n_c + d_1 d_c}{n_2 n_c + d_2 d_c + \gamma(n_1 n_c + d_1 d_c)} \tag{21}$$

may be made APR. Therefore, both its numerator and denominator are Hurwitz for $\gamma \geq \gamma_m$. Let us then apply the results of Section 3 and study the stability of

$$\lambda[n_1 n_c + d_1 d_c] + (1-\lambda)[n_2 n_c + d_2 d_c + \gamma(n_1 n_c + d_1 d_c)] \tag{22}$$

The stability of this polynomial is assured for any $\lambda \in [0, 1]$ by Lemma 2 since the ratio of the 2 extreme polynomials in (21) is APR. It remains then to show

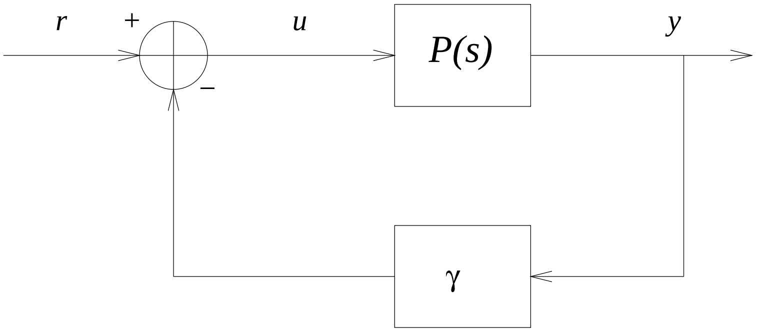

Figure 8: An APR closed-loop system When $P(s)$ is minimum-phase with $n^* = 0$

that we can stabilize n_1/d_1 and n_2/d_2 with the same compensator n_c/d_c. We therefore call on the well-known simultaneous-stabilization result for 2 plants as given for example in [**?**]. In order to state the main result, consider the 2 plants with the same number of poles

$$\begin{aligned} P_1(s) &= \frac{n_1(s)}{d_1(s)} = \frac{N_1(s)}{D_1(s)} \\ P_2(s) &= \frac{n_2(s)}{d_2(s)} = \frac{N_2(s)}{D_2(s)} \end{aligned}$$

where $N_i(s) = n_i(s)/d(s)$ and $D_i(s) = d_i(s)/d(s)$ with $d(s)$ a stable polynomial of degree equal to that of $d_1(s)$ and $d_2(s)$. Then, if n_i/d_i is coprime we can write the Bezout identity over the set of stable transfer functions

$$N_i X_i + D_i Y_i = 1 \tag{23}$$

and define

$$\begin{aligned} N &= N_2 D_1 - N_1 D_2 \\ D &= N_2 X_1 + D_2 Y_1 \\ P &= \frac{N}{D} \end{aligned} \tag{24}$$

We then have the following theorem [**?**].

Theorem 2 *The 2 plants P_1 and P_2 are simultaneously stabilizable with the same compensator C if and only if P has an even number of real poles between every pair of real zeros in $Re\{s\} \geq 0$*

In fact, the stabilizing compensator is given by

$$R = \frac{U - D}{N}$$

$$C = \frac{X_1 + D_1 R}{Y_1 - N_1 R}$$

where U is any minimum-phase, stable transfer function, such that $U(\alpha_i) = D(\alpha_i)$ for all α_i in $\mathrm{Re}\{s\} \geq 0$ where $N(\alpha_i) = 0$. Note that $U(s)$ is not unique, so that other design objectives may be satisfied by a particular choice of $U(s)$. Then, given the 2 plants P_1 and P_2 we can check to make sure that they are simultaneously stabilizable. Moreover, the comepenstaor which will simultaneously stabilize the 2 plants will, by our results also stabilize the family of plants

$$\begin{aligned} G(s) &= \frac{[\lambda + (1-\lambda)\gamma]n_1 + (1-\lambda)n_2}{[\lambda + (1-\lambda)\gamma]d_1 + (1-\lambda)d_2} \\ &= \frac{\lambda n_1 + (1-\lambda)(\gamma n_1 + n_2)}{\lambda d_1 + (1-\lambda)(\gamma d_1 + d_2)} \end{aligned}$$

where $\gamma \geq \gamma_m$ and $\lambda \in [0, 1]$. See Figure 9.

△

Of course, it may so happen that a particular choice of $U(s)$ will make $P(s)$ APR so that the γ feedback is unnecessary and the same compensator $C(s)$ will stabilize the family

$$G(s) = \frac{\lambda n_1 + (1-\lambda)n_2}{\lambda d_1 + (1-\lambda)d_2}$$

Eventually, we would like to choose $U(s)$ so that $P(s)$ is APR, a goal that is still unachieved at the present time. Also, note that the choice of $u = -\gamma y + \gamma r$ will lead us to similar result where

$$G(s) = \frac{\gamma n_1 + (1-\lambda)n_2}{\gamma d_1 + (1-\lambda)d_2} \tag{25}$$

5 CONCLUSIONS

In this paper, we have presented new proofs for some recent extreme-point results using classical frequency-domain concepts. This methodology has led

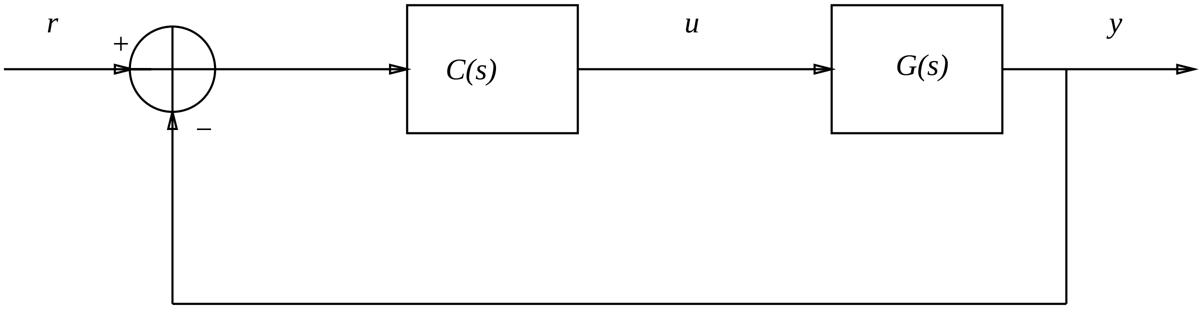

Figure 9: A Robustly-Stable closed-loop system for $G(s) = \frac{\lambda n_1+(1-\lambda)(\gamma n_1+n_2)}{\lambda d_1+(1-\lambda)(\gamma d_1+d_2)}$

us to a simple proof of Kharitonov's original theorem. In addition, a characterization of the ratio of the extreme polynomials in terms of an APR transfer function provided us with a controller synthesis for a family of uncertain systems. The method does not place any requirements on the compensator's order which is still parametrized by a stable, minimum-phase transfer function (a unit) $U(s)$. This parametrization will provide us with a design freedom that may be used to satisfy other performance objectives as will be discussed in a future paper.

APPENDIX A

Proof of 3 in Corollary 1: Consider $P_i(jw) = p_i(w) + jq_i(w)$ so that $tan[\theta_i(w)] = q_i(w)/p_i(w)$. Then, it can be shown that

$$\frac{\partial \theta_i(w)}{\partial w} = \frac{p_i(w)\dot{q}_i(w) - \dot{p}_i(w)q_i(w)}{p_i^2(w) + q_i^2(w)}$$

where $\dot{q}_i(w) = \partial q_i(w)/\partial w$. Let us apply the above to

$$P(jw) = P_0(jw) + \lambda_{max} P_1(jw)$$

Then, letting $\theta(w) = arg\{P(jw)\}$, we have

$$\begin{aligned} \frac{\partial \theta(w)}{\partial w} &= \frac{[p_0(w) + \lambda_{max} p_1(w)][\dot{q}_0(w) + \lambda_{max} \dot{q}_1(w)]}{[p_0(w) + \lambda_{max} p_1(w)]^2 + [q_0(w) + \lambda_{max} q_1(w)]^2} \\ &- \frac{[q_0(w) + \lambda_{max} q_1(w)][\dot{p}_0(w) + \lambda_{max} \dot{p}_1(w)]}{[p_0(w) + \lambda_{max} p_1(w)]^2 + [q_0(w) + \lambda_{max} q_1(w)]^2} \end{aligned}$$

Now, let us assume that a certain w_0, and a corresponding k_0 exist for instability to occur and let us evaluate the above expression at w_0 using the expressions in Note 2,

$$\begin{aligned} \frac{\partial \theta(w)}{\partial w} |_{w=w_0} &= k_0^2 \frac{p_0(w)\dot{q}_0(w) - q_0(w)\dot{p}_0(w)}{p_0^2(w) + q_0^2(w)} |_{w=w_0} \\ &+ (1+k_0)^2 \frac{p_1(w)\dot{q}_1(w) - q_1(w)\dot{p}_1(w)}{p_1^2(w) + q_1^2(w)} |_{w=w_0} \\ &- k_0(1+k_0) \frac{p_0(w)\dot{q}_0(w) - q_0(w)\dot{p}_0(w)}{p_0^2(w) + q_0^2(w)} |_{w=w_0} \\ &- k_0(1+k_0) \frac{p_1(w)\dot{q}_1(w) - q_1(w)\dot{p}_1(w)}{p_1^2(w) + q_1^2(w)} |_{w=w_0} \end{aligned}$$

which then leads to

$$\begin{aligned} \frac{\partial \theta(w)}{\partial w} |_{w=w_0} &= -k_0 \frac{p_0(w)\dot{q}_0(w) - q_0(w)\dot{p}_0(w)}{p_0^2(w) + q_0^2(w)} |_{w=w_0} \\ &+ (1+k_0) \frac{p_1(w)\dot{q}_1(w) - q_1(w)\dot{p}_1(w)}{p_1^2(w) + q_1^2(w)} |_{w=w_0} \\ &= -k_0 \frac{\partial \theta_0(w)}{\partial w} |_{w=w_0} + (1+k_0) \frac{\partial \theta_1(w)}{\partial w} |_{w=w_0} \end{aligned}$$

Noting that the left hand side is positive (because $P(s)$ is Hurwitz) and that the right hand side can never be positive if $\partial\theta_1(w)/\partial w \leq 0$ and if $P_0(s)$ is Hurwitz, we have a contradiction.

APPENDIX B

Proof of 6 in Corollary 1: This proof is similar to that in [**?**] but simplifies some of the steps. Let us find $\partial\theta_1(w)/\partial w$ for $P_1(s) = (as+b)P_2(s)$, given that

$$tan[\theta_1(w)] = \frac{aw}{b} + c$$

where c is a constant angle. We can then write

$$(1 + tan^2[\theta_1(w)])\frac{\partial\theta_1(w)}{\partial w} = \frac{a}{b}$$

so that,

$$\begin{aligned} \frac{\partial\theta_1(w)}{\partial w} &= \frac{ab}{b^2 + a^2w^2} \\ &= \frac{sin[\theta_1(w)]cos[\theta_1(w)]}{w} \\ &= \frac{sin[2\theta_1(w)]}{2w} \end{aligned}$$

Now consider a Hurwitz polynomial

$$Q(s) = \prod_{i=1}^{m/2} [(s + a_i)^2 + b_i^2] \prod_{j=1}^{n-m} [s_j + c_j]$$

where $a_i > 0$, $b_i > 0$, $c_j > 0$, and m is even. Let

$$\begin{aligned} tan[\theta_i(w)] &= \frac{w - b_i}{a_i} \\ tan[\beta_i(w)] &= \frac{w + b_i}{a_i} \\ tan[\gamma_i(w)] &= \frac{w}{c_i} \end{aligned}$$

Then $\theta(w) = arg[Q(jw)] = \sum_{i=1}^{m/2}[\theta_i + \beta_i] + \sum_{i=1}^{n-m}\gamma_i$. Repeating the same derivation as before we obtain

$$\begin{aligned}
\frac{\partial\theta_i(w)}{\partial w} &= \frac{sin[2\theta_i(w)]}{2w} + \frac{b_i cos^2[\theta_i(w)]}{a_i w} \\
\frac{\partial\beta_i(w)}{\partial w} &= \frac{sin[2\beta_i(w)]}{2w} - \frac{b_i cos^2[\beta_i(w)]}{a_i w} \\
\frac{\partial\gamma_i(w)}{\partial w} &= \frac{sin[2\gamma_i(w)]}{2w}
\end{aligned}$$

Consider then the corresponding phases of a complex conjugate root $-\alpha_i \pm j\beta_i$. We then have

$$\begin{aligned}
\frac{\partial\theta_i(w)}{\partial w} + \frac{\partial\beta_i(w)}{\partial w} &= \{\frac{sin[2\theta_i(w)]}{2w} + \frac{sin[2\beta_i(w)]}{2w}\} \\
&\quad + \frac{b_i}{a_i w}\{cos^2[\theta_i(w)] - cos^2[\beta_i(w)]\} \\
&= \{\frac{sin[2\theta_i(w)]}{2w} + \frac{sin[2\beta_i(w)]}{2w}\} \\
&\quad + \frac{b_i}{a_i w}\{\frac{1}{a_i^2 + (w-b_i)^2} - \frac{1}{a_i^2 + (w+b_i)^2}\} \\
&= \{\frac{sin[2\theta_i(w)]}{2w} + \frac{sin[2\beta_i(w)]}{2w}\} \\
&\quad + \frac{4b_i^2}{a_i[a_i^2 + (w-b_i)^2][a_i^2 + (w+b_i)^2]} \\
&> \{\frac{sin[2\theta_i(w)]}{2w} + \frac{sin[2\beta_i(w)]}{2w}\}
\end{aligned}$$

Now, we can add the contributions of all angles and write

$$\begin{aligned}
\frac{\partial\theta(w)}{\partial w} &= \sum_{i=1}^{m/2}\{\frac{\partial\theta_i(w)}{\partial w} + \frac{\partial\beta_i(w)}{\partial w}\} + \sum_{i=1}^{n-m}\frac{\partial\gamma_i(w)}{\partial w} \\
&> \sum_{i=1}^{m/2}\{\frac{sin[2\theta_i(w)]}{2w} + \frac{sin[2\beta_i(w)]}{2w}\} + \sum_{i=1}^{n-m}\frac{sin[2\gamma_i(w)]}{2w} \\
&> \frac{sin[2\sum_{i=1}^{m/2}\{\theta_i(w) + \beta_i(w)\} + 2\sum_{i=1}^{n-m}\gamma_i(w)]}{2w} = \frac{sin[2\theta(w)]}{2w}
\end{aligned}$$

Let us apply the above result at the frequency w_0 using $Q(s) = P_0(s)$ and $Q(s) = P_0(s) + \lambda_{max}P_1(s)$. Note first that if instability is to occur at w_0, then

$$\begin{aligned}
\theta_1(w_0) &= \theta_0(w_0) + (2l+1)\pi \\
&= \theta(w_0)
\end{aligned}$$

Then, using the last equality of step 3, we have

$$\frac{\partial\theta(w)}{\partial w}\mid_{w=w_0}= -k_0\frac{\partial\theta_0(w)}{\partial w}\mid_{w=w_0}+(1+k_0)\frac{\partial\theta_1(w)}{\partial w}\mid_{w=w_0}$$

so that

$$\begin{aligned}\frac{\partial\theta(w)}{\partial w}\mid_{w=w_0}+k_0\frac{\partial\theta_0(w)}{\partial w}\mid_{w=w_0} &= (1+k_0)\frac{\partial\theta_1(w)}{\partial w}\mid_{w=w_0}\\ &= (1+k_0)\frac{sin[2\theta_1(w_0)]}{2w_0}\end{aligned}$$

Note however that $sin[2\theta_1(w_0)] = sin[2\theta_0(w_0)] = sin[2\theta(w_0)]$ so that

$$\frac{\partial\theta(w)}{\partial w}\mid_{w=w_0}+k_0\frac{\partial\theta_0(w)}{\partial w}\mid_{w=w_0}> (1+k_0)\frac{sin[2\theta_1(w_0)]}{2w_0}$$

leading to a contradiction.