Robust, Non-Fragile Controller Synthesis Using Model-Based Fuzzy Systems: A Linear Matrix Inequality Approach

by

Ali Jadbabaie

BS Electrical Engineering, Sharif University of Technology
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THESIS

Submitted in Partial Fulfillment of the
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To my parents who have been a constant source of inspiration, motivation, and support.

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Abstract

The purpose of this research is to establish a systematic framework to design controllers for a class of uncertain linear and nonlinear systems. Our approach utilizes a certain type of fuzzy systems that are based on "Takagi-Sugeno" fuzzy models to approximate nonlinear systems. We will show that the resulting fuzzy model has a structure very similar to popular models used in robust control. Therefore, we use a robust control methodology to design controllers. In other words, this thesis tends to narrow the gap between two active areas of research in control theory, namely, robust control and fuzzy control.

Since its introduction in control theory, fuzzy control has been legitimately questioned about the mathematical justification of the promising results it tends to provide, and to the best knowledge of the author, there are few results which have succeeded in providing such justification. This is perhaps due to the fact that fuzzy control systems are based on linguistic

models rather than mathematical equations, and without a mathematical model, stability and other theoretical issues become harder to study. This might be the main reason why model-based fuzzy systems based on "Takagi-Sugeno" fuzzy models are becoming popular, since with a mathematical model being present, stability and performance issues can be addressed.

Utilizing convex optimization methods, we attempt in this thesis to fill the gap between model-based fuzzy control and robust control, by blending the latest advances in both of these areas and using some new results obtained in this research.

Specifically, we address the issue of *controller fragility*, which has been brought up in the control literature quite recently. Briefly, controller fragility can be described as the sensitivity of the controller to variations in controller parameters. We show that the proposed method leads to the synthesis of controllers which are not only *robust* with respect to uncertainty in the plant dynamics, but also *non-fragile* towards a special form of uncertainty in the controller parameters. Several theorems are presented with their proofs, and these are followed by a series of benchmark numerical examples adopted from the literature.

The results in almost all of the studied cases turn out to be quite promising. However, the author by no means claims that the methods proposed in this research are the best way to deal with all control systems, but the simplicity of the methods makes them a good alternative for controlling a class of uncertain linear, and nonlinear plants.

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Chapter 1

Introduction

Over the past two decades, several researchers in the control community have come up with different techniques for designing linear time-invariant control systems that are *robust* and *optimal*. Such control systems have the ability to tolerate and cope with uncertainty in the dynamics of the plant, and are optimal with respect to a given performance measure. Since no real-life plant is completely linear and time-invariant, the robustness property of the controller makes it possible to handle some unmodeled time variations and nonlinearities.

Despite the success of robust control theory in dealing with a wide class of control problems, researchers have been looking for new and revolutionary ideas to replace the existing methods and solve problems not addressed by the current robust controllers. Among these revolutionary ideas, *fuzzy logic control* is probably one of the most popular, and at the same time a controversial one.

1.1 Fuzzy Logic History

Fuzzy set theory was first introduced in a seminal paper by Lotfi Zadeh [1] published in the rather obscure journal of *Information and Control.* Zadeh, an electrical engineer by training, was one of the leading authorities in control theory in the 1950's and early 1960's. However, during the process of writing a book on linear systems with Charles Desoer in 1963, he noticed that in spite of the richness of the existing mathematical theory of control, we have been able to deal with a very special case of systems that are linear and time invariant, or nonlinear but with a specific property [2, 3]. He traced this problem back to the Aristotlian notion of absolute truth and falsehood, and generalized such notion to the case of partial truth and partial membership in a set.

During the 1965-1971 period, he extended the theory and managed to attract the attention of one of the most prominent applied mathematicians of our time, the late Richard Bellman. In 1972, Zadeh wrote another seminal paper titled, "*A rationale for fuzzy control*" [4] in which he pointed out the use of fuzzy logic in control, and predicted that in the future, fuzzy logic control will play a major role in control theory. It did not take too long for the first use of fuzzy logic in control to appear. In 1974, Mamdani and his associates used a fuzzy logic controller to control the temperature in a rotary cement kiln [9, 10]. After that, more applications of fuzzy logic in control were presented by researchers all over the world. By the late 1980's, with the advent of fuzzy chips, fuzzy logic was well on its way to become a billion dollar industry in Japan alone. Despite its success in Japan, however, fuzzy logic and its applications in control have been faced with reluctance in the US. The most probable reason being that although the theory resulted in promising practical results, it lacked basic theoretical verifications of such concepts as stability and robustness which classical control theoreticians were used to. This was due to the lack of mathematical models of the system and the controller. When a mathematical model is not available, it makes little sense to talk about stability or any other structural properties. Several researchers have tried to come up with stability conditions for fuzzy systems and have reported some success [5, 6, 7, 8]. However, the primary goal of fuzzy logic is to develop an alternative to mathematical modeling for systems which either lack a proper mathematical model because it is either too ill-defined, or the model is so complicated that it is of no practical use.

1.2 Mamdani Versus Takagi-Sugeno Controllers

In this section, we give a brief description of the two popular methods used in fuzzy logic control. The first one, known as *Mamdani* type fuzzy models are systems based on fuzzy **If ... Then** rules with linguistic fuzzy sets in both antecedents and consequents. This type of fuzzy systems is named after E. H. Mamdani, who was the first researcher to use fuzzy logic [10].

A different type of fuzzy systems was introduced in 1985 by Takagi and Sugeno [11], and later by Sugeno and Kang [12]. This approach is similar to the Mamdani type model in the sense that they are both described by **if ... Then** rules and that their antecedents have linguistic fuzzy sets. However, Takagi-Sugeno (T-S for short from now on) models differ in the consequents which are represented by analytic dynamical or algebraic equations. This type of fuzzy modeling is very simple. The system dynamics are written as a set of fuzzy implications which characterize local models in the state space. The main feature of a T-S fuzzy model is that it expresses the local dynamics of each fuzzy rule by a linear dynamical model. The overall fuzzy model is achieved by a blending of these rules.

This type of modeling was shown able to approximate nonlinear systems quite efficiently [13]. Since the local models are linear, linear control methodology can be used to design local controllers, and by blending the local controllers, we obtain a global controller for the system. Throughout this thesis, we will focus on Takagi-Sugeno fuzzy systems.

1.3 Controller Fragility

Fragility is an issue brought up quite recently in the robust control literature [14, 15]. While robust control has been able to handle uncertainty in the plant dynamics, it has always been assumed that since the controller is not given, and it is rather tailor-made by the designer, it can be implemented with any required degree of accuracy. However, the examples given in [14, 15] suggest otherwise. These examples show that infinitely small perturbations in the controller parameters might result in the instability of the closed-loop system. This property is known as *fragililty*. In this thesis, we will show that we can design stabilizing Takagi-Sugeno controllers which are *non-fragile* or *resilient*, i.e., they are able to tolerate uncertainty in the controller parameters.

1.4 Overview

This section describes the thesis outline. Takagi-Sugeno fuzzy systems will be discussed in Chapter 2. In Chapter 3, we introduce Linear Matrix Inequalities (LMIs), and some of the existing numerical methods for their solution. We also deal with LMI formulation of stability conditions for T-S fuzzy systems and a brief discussion about the approximation accuracy of these systems. Chapter 4 extends the previous results to the case of dynamic output feedback, and we introduce the notion of *fuzzy observers*. All results are given both in continuous-time and discrete-time. In chapter 5 we develop a guaranteed-cost framework to guarantee performance, in addition to stability, for both continuous-time and discrete-time systems. We introduce the notion of controller fragility in Chapter 6 and will show that resilient controllers can be designed within the framework of T-S fuzzy models. In the remaining parts of Chapter 6, we present a scheme for the design of robust and resilient controllers for systems with polytopic uncertainties using guaranteed-cost bounds. In all cases, we present simulations of benchmark systems adopted from the literature. Finally we present our conclusions and discuss some future research directions in this area in Chapter 7.

Chapter 2

Takagi-Sugeno Fuzzy Systems

In this chapter, we give an introduction to Takagi-Sugeno (T-S) fuzzy systems. We assume that the reader is familiar with basic concepts of fuzzy set theory. For further details on basics of fuzzy set theory, the interested reader is referred to [16, 17].

There has recently been a rapidly growing interest in using Takagi-Sugeno fuzzy models to approximate nonlinear systems. This interest relies on the fact that dynamic T-S models are easily obtained by linearization of the nonlinear plant around different operating points. Once the T-S fuzzy models are obtained, linear control methodology can be used to design local state feedback controllers for each linear model. Aggregation of the fuzzy rules results in a generally nonlinear model, but in a very special form called Polytopic Linear Differential Inclusions (PLDIs) that we will discuss in detail in chapter 3. This approach is also similar to gain scheduling control [18], since a different linear model is used based on the position of the state variable in state space. This has led some researchers to call this method *fuzzy gain scheduling* [19].

2.1 Continuous Time Takagi-Sugeno Models

2.1.1 Open Loop T-S Models

A continuous-time T-S model is represented by a set of fuzzy **If** $\cdots$ **Then** rules written as follows :

i^{th} Plant Rule: **IF** $x_1(t)$ is M_{i1} and $\ldots, x_n(t)$ is M_{in} **THEN** $\dot{x} = A_i x$

where $x \in \mathrm{I\!R}^{n\times 1}$ is the state vector, $i = \{1, \cdots, r\}$, r is the number of rules, M_{ij} are input fuzzy sets, and the matrices $A_i \in \mathrm{I\!R}^{n\times n}, \mathrm{B_i} \in \mathrm{I\!R}^{n\times m}$.

Using singleton fuzzifiers, max-product inference, and weighted average deffuzifiers [16, 17], the aggregated fuzzy model is given as follows:

$$\dot{x} = \frac{\sum_{i=1}^{r} w_i(x)(A_i x)}{\sum_{i=1}^{r} w_i(x)} \tag{2.1}$$

where w_i is defined as

$$w_i(x) = \prod_{j=1}^{n} \mu_{ij}(x_j) \tag{2.2}$$

and μ_{ij} is the membership function of jth fuzzy set in the ith rule. Now, defining

$$\alpha_i(x) = \frac{w_i(x)}{\sum_{i=1}^{r} w_i(x)} \tag{2.3}$$

we can write (2.1) as

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(x) A_i x \tag{2.4}$$

where $\alpha_i > 0$ and

$$\sum_{i=1}^{r} \alpha_i = 1$$

The interpretation of equation (2.4) is that the overall system is a fuzzy blending of the implications. It is evident that the system (2.4) is generally nonlinear due to the nonlinearity of the membership functions. In the next section, we present sufficient conditions based on Lyapunov stability theory for the stability of open-loop system (2.4). The following theorem is due to Sugeno and Tanaka [20]:

Theorem 1 *The continuous-time T-S system (2.4) is globally asymptotically stable if there exists a common positive definite matrix $P > 0$ which satisfies the following inequalities:*

$$A_i^T P + PA_i < 0 \quad \forall\, i = 1, \cdots, r \tag{2.5}$$

where r is the number of T-S rules.

Proof: We introduce the quadratic Lyapunov function candidate

$$V = x^T P x \tag{2.6}$$

To show that the T-S system is globally asymptotically stable, we need to show that the derivative of the above Lyapunov function along the trajectory of the system (2.4) is negative definite. With a straightforward calculation, it can be shown that

$$\dot{V} = x^T[\sum_{i=1}^{r} \alpha_i (A_i^T P + P A_i)]x \tag{2.7}$$

multiplying each inequality in (2.5) by α_i and keeping in mind that $\alpha_i > 0$ for all values of i, we can easily achieve the following

$$\dot{V} < 0$$

therefore, the system in (2.4) is asymptotically stable. To show global asymptotic stability, we note that V is positive definite everywhere, $\dot{V}$ is negative definite everywhere, and also V is radially unbounded. ∎

Remark 1 *Note that the above theorem is a sufficient condition for asymptotic stability, i.e., it is possible for a T-S system to be asymptotically stable, but that a common positive P does not exist.*

Remark 2 *The conditions in above theorem guarantee quadratic asymptotic stability, i.e., stability provable by a quadratic Lyapunov function. The T-S system might be asymptotically stable, without being quadratically asymptotically stable .*

Remark 3 *It can be shown that [21] the non existence of a positive definite solution to (2.5) is equivalent to finding $Q_0, \cdots, Q_r$, not all zero, such that*

$$Q_0 \geq 0, \ldots, Q_r \geq 0 \qquad Q_0 = \sum_{i=1}^{r}(Q_i A_i^T + A_i Q_i)$$

To illustrate the above remarks, we present the following counter-example:

Example 1 *Consider the T-S model described by the following two matrices:*

$$A_1 = \begin{bmatrix} -100 & 0 \\ 0 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 8 & -9 \\ 120 & -18 \end{bmatrix} \tag{2.8}$$

from Remark 3, this T-S system is not quadratically stable if there exist $Q_0 \geq 0, Q_1 \geq 0$ and $Q_2 \geq 0$ not all zero, such that :

$$Q_0 = A_1 Q_1 + Q_1 A_1^T + A_2 Q_2 + Q_2 A_2^T$$

It can be verified that the matrices

$$Q_0 = \begin{bmatrix} 5.2 & 2 \\ 2 & 24 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.1 & 3 \\ 3 & 90 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 2.1 & 1 \\ 1 & 1 \end{bmatrix}$$

satisfy the conditions of Remark 3. However, the piecewise quadratic Lyapunov function

$$V(x) = \max\{x^T P_1 x, x^T P_2 x\}, P_1 = \begin{bmatrix} 14 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

proves that the T-S system described by the two matrices A_1 and A_2 is asymptotically stable. (See [21] for more details). In the next chapter, we will show how to search for a common positive definite Lyapunov matrix P using a Linear Matrix Inequality approach.

2.1.2 T-S Controllers and Closed-Loop Stability

In the previous section, we discussed the open-loop T-S fuzzy systems as well as sufficient conditions for the stability of the open-loop system. Now, we introduce the notion of the Takagi-Sugeno controller in the same fashion as the T-S system. The controller consists of fuzzy **If ... Then** rules. Each rule is a local state-feedback controller, and the overall controller is obtained by the aggregation of local controllers. A generic non-autonomous T-S plant rule can be written as follows

i^{th} Plant Rule: IF $x_1(t)$ is M_{i1} and $\ldots, x_n(t)$ is M_{in} THEN $\dot{x} = A_i x + B_i u$

The overall plant dynamics can be written as

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(x)(A_i x + B_i u) \tag{2.9}$$

in the same fashion, a generic T-S controller rule can be written as:

i^{th}Controller Rule: IF $x_1(t)$ is M_{i1} and $\ldots x_n(t)$ is M_{in} THEN $u = -K_i x$

The overall controller, using the same inference method as before, would be

$$u = -\sum_{i=1}^{r} \alpha_i(x) K_i x \tag{2.10}$$

where, α_is are defined in (2.3). Note that we are using the same fuzzy sets for the controller rules and the plant rules. substituting (2.10) in (2.9), and keeping in mind that

$$\sum_{i=1}^{r} \alpha_i = 1$$

we can write the closed-loop equation as follows:

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(x)\alpha_j(x)(A_i - B_i K_j)x \tag{2.11}$$

The following theorem presents sufficient conditions for closed-loop stability [22].

Theorem 2 *The closed-loop Takagi-Sugeno fuzzy system (2.11) is globally asymptotically stable if there exists a common positive definite matrix P which satisfies the following Lyapunov inequalities:*

$$\begin{aligned} (A_i - B_i K_i)^T P + P(A_i - B_i K_i) &< 0 \quad \forall\, i = 1, \cdots, r \\ G_{ij}^T P + P G_{ij} &< 0 \; i < j \leq r \end{aligned} \tag{2.12}$$

where G_{ij} is defined as

$$G_{ij} = A_i - B_i K_j + A_j - B_j K_i \quad i < j \leq r \tag{2.13}$$

Proof: The proof is similar to the open-loop case. Again we choose the quadratic Lyapunov function candidate

$$V = x^T P x, \quad P > 0$$

To complete the proof, we note that by multiplying the first set of inequalities in (2.12) by α_i and the second set of inequalities by $\alpha_i \alpha_j$, and adding up all the inequalities, we obtain the derivative of Lyapunov function V along the trajectory of the closed-loop system (2.12). Since $\dot{V} < 0$, and also V is radially unbounded, the closed-loop system is globally asymptotically stable. ■

Although this theorem and the previous one present a sufficient condition for stability, finding a common positive definite matrix P both in the open-loop and the closed-loop case is by no means trivial. Several researchers have tried to come up with heuristic methods to find a common positive definite matrix P, but little success was reported. That is why these theorems were found to have little use in practice. In the next chapter, we present the recent developments in solving Linear Matrix Inequalities, and we will show that the sufficient conditions can be checked very easily using convex optimization methods, that solve LMIs in a numerically tractable fashion. Complete details are given in chapter 4. Before moving to the next chapter, we present the discrete-time counterparts of the Takagi-Sugeno fuzzy systems and the stability theorems presented so far.

2.2 Discrete-Time Takagi-Sugeno Fuzzy Systems

The discrete time case of the T-S fuzzy systems is quite similar to the continuous-time version. The T-S model is again made up of fuzzy **If ... Then** rules with fuzzy sets in the antecedents and discrete-time dynamical or algebraic equations in the consequents. As in the previous section, our attention is focused on dynamical T-S models. A generic rule of the open-loop discrete-time T-S system can be written as:

i^{th} Plant Rule: IF $x_1(k)$ is M_{i1} and $\ldots, x_n(k)$ is M_{in} THEN $x(k+1) = A_i x(k)$

The aggregated model would be:

$$x(k+1) = \sum_{i=1}^{r} \alpha_i(x) A_i x(k) \tag{2.14}$$

where, α_i's are defined in (2.3). The stability of the system (2.14) can be checked using the discrete-time Lyapunov equation. We have the following theorem to check the stability [20, 23].

Theorem 3 *The T-S fuzzy system (2.14) is globally asymptotically stable, if there exist a common positive definite matrix P that satisfies the following inequalities:*

$$A_i^T P A_i - P < 0 \quad i = 1, \cdots, r \tag{2.15}$$

The proof can be found in [23]. ■

We can define the non-autonomous discrete-time T-S system in the same fashion as the continuous-time. The non-autonomous discrete-time T-S system can be written as:

$$x(k+1) = \sum_{i=1}^{r} \alpha_i(x)(A_i x(k) + B_i u(k)) \tag{2.16}$$

We define the discrete-time T-S controller as a set of fuzzy implications. A generic implication can be written as

i^{th} Controller Rule: IF $x_1(k)$ is M_{i1} and $\ldots x_n(k)$ is M_{in} THEN $u(k) = -K_i x(k)$

where $K_i \in \mathbb{R}^{m\times n}$. The over all controller will be

$$u(k) = -\sum_{i=1}^{r} \alpha_i(k) K_i x(k) \tag{2.17}$$

Replacing (2.17) in (2.16) we obtain the following closed-loop equation

$$x(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i \alpha_j (A_i - B_i K_j) x(k) \tag{2.18}$$

Sufficient conditions for the stability of the closed-loop can be expressed as the following theorem [22].

Theorem 4 *The closed-loop system (2.18) is globally asymptotically stable if there exists a common positive definite matrix P that satisfies the following matrix inequalities:*

$$\begin{aligned} (A_i - B_i K_i)^T P (A_i - B_i K_i) - P &< 0 \quad i = 1, \cdots, r \\ G_{ij}^T P G_{ij} - P &< 0 \quad i < j \leq r \end{aligned} \tag{2.19}$$

where, G_{ij} is the same as in (2.13).

The Proof of this theorem can be found in [22]. ■

In the next chapter, we present an LMI framework for the stability analysis as well as the design of continuous-time and discrete-time T-S fuzzy systems.

Chapter 3

Linear Matrix Inequalities (LMIs)

In this chapter we present an overview of Linear Matrix Inequalities and their applications in control theory. Most of the material in this chapter is adopted from [21]. We will show that the problems described in the previous chapter regarding the stability of T-S fuzzy systems can be easily formulated in terms of LMIs. These LMIs can be solved numerically in an efficient and tractable way. In the next section, we present a brief history of LMIs in control theory.

3.1 History of LMIs In Control Theory

The history of LMIs in the analysis of dynamical systems goes back more than 100 years. In 1890, A. M. Lyapunov published his seminal work introducing what we now call Lyapunov theory. Lyapunov showed that the system of ordinary linear differential equations

$$\dot{x} = Ax(t) \tag{3.1}$$

is globally asymptotically stable, (all trajectories converge to zero) if and only if there exists a positive definite matrix P that would satisfy

$$A^T P + PA < 0 \tag{3.2}$$

the inequality $P > 0$ as well as (3.2) is a special form of an LMI. Lyapunov showed that this LMI can be solved analytically by a set of linear algebraic equations.

In 1940's Lur'e [25], Postnikov [24], and others applied Lyapunov's method to some specific practical problems in control engineering. Specifically, the problem of stability of a control system with a nonlinearity in the actuator was studied. Their stability criteria had the form of LMIs which were reduced to polynomial inequalities and then checked by "hand". This limited their application to first, second, and third-order systems.

In the early 1960's Yakubovich [26], Popov [27], Kalman [28], and others succeeded in reducing the solution of the LMIs that arose in the problem of Lur'e to simple graphical criteria, using what we now call the positive-real (PR) Lemma, or Kalman-Yakubovich-Popov Lemma. This resulted in the celebrated Popov criterion [27], circle criterion [28], Tsypkin criterion [29] and many other variations. Although these criteria worked well for systems with one nonlinearity, they did not usefully extend to systems with more than one nonlinearity.

Perhaps the most important role of LMIs in control theory was recognized in early 1960's by Yakubovich [26, 30]. This is clear simply from title the of some of his papers from 1962-1965, e.g., *The solution of certain matrix inequalities in automatic control theory.*

The PR lemma and its extensions were studied in the latter half of the 1960's and were found to be related to the ideas of passivity, and small-gain introduced by Sandberg [31], and Zames [32], and to quadratic optimal control. By 1971, it was known that the LMIs appearing in the PR lemma could be solved not only by graphical means, but also by solving a certain algebraic Riccati equation (ARE). In a 1971 paper by Willems [33], we find the following: " The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms, for example."

The above suggestion by Willems, foreshadows the next chapter in the LMI history. The next major breakthrough, was the simple observation that the solution set of LMIs arising in system and control theory is *convex*, therefore these LMIs are amenable to a computer solution. This simple observation underscores the fact that although we may not be able to solve many of these LMIs analytically, we can solve them numerically in a reliable way. This observation was made for the first time, by Pyatnitskii, and Skrodinskii [34]. These researchers reduced the LMIs arising in the Lur'e problem to a convex optimization problem, which then they solved using an algorithm known as *ellipsoid algorithm.* They were the first to recast the problem of finding a Lyapunov function to a convex optimization problem. Despite the fact that the ellipsoid algorithm solved the convex optimization problem in polynomial-time (i.e., the complexity of the problem increases in a polynomial fashion when the size of the problem increases), it was not practically efficient.

The final chapter in the history of LMIs is quite recent and important. In 1984, N. N. Karmarkar [35] introduced a new linear programming algorithm that solves linear programs in polynomial-time, and in contrast to the ellipsoid method, is very efficient in practice too. For years, the Simplex method [36] was thought to be the best method for solving linear programs, and although it was an algorithm with combinatoric complexity, it was much more efficient than the ellipsoid algorithm. Karmarkar's algorithm made such a big impact in mathematical programming literature, that it even made its way into the first page of the New York Times. Karmarkar's work spurred an enormous amount of research in the area of interior-point algorithms for linear programming.

In 1988, Nesterov and Nemirovskii [37] developed interior-point methods that apply directly to convex problems involving LMIs. These algorithms have been found to be very efficient. In the next section, we introduce the LMIs formally and discuss their applications afterwards.

3.2 What is an LMI?

A Linear Matrix Inequality (LMI) is an inequality in the following form:

$$F(x) \stackrel{\triangle}{=} F_0 + \sum_{i=1}^{r} x_i F_i > 0 \tag{3.3}$$

where $x \in \mathbb{R}^{\mathrm{r}}$ is a vector variable to be found, and the symmetric matrices $F_i = F_i^T \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, $i = 1, \cdots, r$, are given. The inequality symbol in (3.3) means that the matrix $F(x)$ is positive-definite, i.e., $u^T F(x) u > 0 \;\; \forall u \neq 0 \in \mathrm{I\!R^n}$. Of course, the LMI (3.3) is equivalent to a set of n polynomial inequalities in x, i.e., according to Sylvester's theorem, the leading principal minors of $F(x)$ should be positive. As was mentioned in the previous section, the solution set of the LMI is a convex set. A proof of this fact is provided in the next theorem.

Theorem 5 *The solution set of the LMI in (3.3) is convex.*

proof: The proof is easily obtained by using the definition of a convex set. By definition, the set $\{x|F(x) > 0\}$ is convex if for any two points $x^{(1)}, x^{(2)}$ in the solution set, the convex combination $\lambda x^{(1)} + (1-\lambda)x^{(2)}$ is also a solution, for all $\lambda \in [0, 1]$. This can be shown directly by applying the definition to (3.3). ■

Multiple LMIs

$$F^{(1)}(x), \cdots, F^{(p)}(x) > 0$$

can be expressed as a single LMI

$$diag(F^{(1)}(x), \cdots, F^{(p)}(x)) > 0.$$

Also, LMIs in terms of matrix variables can be written as (3.3). For example, in a 2×2 case, we can write the inequality

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0$$

as :

$$p_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} > 0 \tag{3.4}$$

which is exactly in the form of (3.3). We can also convert nonlinear (convex quadratic) inequalities into LMIs using Schur complements, or the LMI Lemma [38]. The basic idea is as follows: the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 \tag{3.5}$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, and $S(x)$ are LMIs in the vector x in the form of (3.3), is equivalent to

$$R(x) > 0, \quad Q(x) - S(x)R^{-1}(x)S^T(x) > 0 \tag{3.6}$$

In other words, the inequalities (3.6) can be written as the LMI (3.5). Given an LMI $F(x) > 0$, the corresponding LMI problem is to find x^{feas} such that $F(x^{feas}) > 0$ or determine that the LMI is infeasible. A simple example of such LMI problems is the problem of "simultaneous Lyapunov stability problem": We are given r plants A_i, $i = 1, \cdots, r$, and need to find a positive definite Lyapunov matrix P satisfying the following LMIs:

$$P > 0 \qquad A_i^T P + PA_i < 0, \qquad i = 1, \cdots, r$$

or determine that no such P exists. We recall from the previous chapter that this is exactly the same condition for stabilization of open-loop T-S fuzzy systems. In order to present LMI conditions for closed-loop T-S fuzzy systems, we first need to give a brief description of Linear Differential Inclusions, which can be considered a general framework for T-S fuzzy systems, and at the same time, uncertain linear time-varying systems.

3.3 Linear Differential Inclusions

A linear differential inclusion (LDI) is given by

$$\dot{x} \in \Omega x \tag{3.7}$$

where Ω is a subset of $\mathbb{R}^{n \times n}$. The LDI in (3.7) may for example describe a family of linear time-varying systems. In this case, every trajectory of the LDI satisfies

$$\dot{x} = A(t)\, x(t)$$

When Ω is a polytope, the LDI is called polytopic or PLDI, i.e. ,

$$A(t) \in Co\{A_1, A_2, \cdots, A_r\}$$

where Co is the convex hull of A_i, $i = 1, \cdots, n$. This means that we can write $A(t)$ as a convex combination of the vertices of the polytope. Note that this is also true for the case when the coefficients of the convex combination depend on x as well, i.e., we can write the following:

$$A(t, x) = \alpha_1(t, x)A_1 + \alpha_2(t, x)A_2 + \cdots + \alpha_r(t, x)A_r \tag{3.8}$$

where $\{A_1, \cdots, A_r\}$ are known matrices and $\alpha_1, \ldots, \alpha_r$ are positive scalars which satisfy

$$\sum_{i=1}^{r} \alpha_i(t, x) = 1$$

Using a technique known as global linearization [21], we can use PLDIs to study properties of nonlinear time varying systems. In fact, consider the system

$$\dot{x} = f(t, x, u) \tag{3.9}$$

If the Jacobian of the system matrix $A(t,x) = \frac{\partial f}{\partial x}$ lies in the convex hull defined in (3.8), then every trajectory of the nonlinear system is also a trajectory of the LDI defined by Ω.

Much of our motivation for studying PLDIs comes from the fact that we can use them to establish properties of nonlinear, time-varying systems using the global linearization technique. The idea of global linearization, which is basically replacing a nonlinear system by a PLDI, is a rather old one, and can be found in [39, 40]. According to [21], this idea is implicit in the early Soviet literature on the absolute stability problem, e.g., Lur'e and Postnikov [24, 25], and Popov [41]. Of course, we have to bear in mind that approximating the set of trajectories of a nonlinear system via LDIs can be very conservative, i.e., there are many trajectories of the LDI which are not trajectories of the nonlinear system, so we get a rather conservative result. Looking carefully at (3.8), we note that this is exactly in the form of the T-S fuzzy model. In fact, the whole idea of modeling a nonlinear system with T-S fuzzy rules, has the same origin as global linearization. We will revisit the T-S fuzzy systems in the next session, and derive the LMI stability conditions.

3.4 LMI Stability Conditions for T-S Fuzzy Systems

3.4.1 Continuous-Time Case

Sufficient stability conditions for open-loop continuous time T-S systems were derived using Theorem 1. These conditions, as was discussed earlier, are LMIs in the matrix variable P. Note that equation (2.4) is the equation for a PLDI.

On the other hand, the closed-loop case is different. Theorem 2 provides sufficient conditions for the stability of the closed-loop system. The Lyapunov inequalities in (2.12) are not LMIs in P and K_i, since we have the product of P and K_i. However, using a clever change of variables due to Bernussou, Peres, and Geromel [42], we can recast the matrix inequalities in (2.12) as LMIs. The change of variables are:

$$\begin{aligned} P^{-1} &= Y \\ X_i &= K_i Y \end{aligned} \tag{3.10}$$

Pre-multiplying and post-multiplying the inequalities in (2.12), and using the above change of variable, we obtain the following LMIs [22]:

$$\begin{aligned} 0 &< Y \\ 0 &< YA_i^T + A_iY - B_iX_i - X_i^TB_i^T \quad \forall\, i = 1, \cdots, r \\ 0 &< Y(A_i + A_j)^T + (A_i + A_j)Y - (B_iX_j + B_jX_i) - (B_iX_j + B_jX_i)^T \quad j < i \le r \end{aligned} \tag{3.11}$$

If the above LMIs have a solution, stability of the closed loop T-S system is guaranteed. We can then find the T-S controller gains by reversing the variable transformations in (3.10), i.e.

$$K_i = X_i Y^{-1}$$

Again, we point out the fact that the resulting T-S controller is conservative, because we have forced the Lyapunov matrices to be the same for all inequalities. This conservatism is helpful in compensating the approximation errors that appear due to modeling the nonlinear system with T-S fuzzy systems or, to be precise, as PLDIs. In the next chapter, we will extend these results to the case were the states are not available for feedback, and we only have the outputs available for measurement. First, however, we derive the LMI conditions for stability of discrete-time T-S fuzzy systems in the next section.

3.4.2 Discrete-Time Case

The sufficient conditions for stability of discrete-time open-loop T-S systems given in Theorem 3 are exactly LMIs in P, and like the continuous-time case for the open-loop, we do not need any further change of variables. The closed-loop case is however more complicated than the continuous-time counterpart. In addition to the change of variables in (3.10) we need to use the LMI lemma, discussed earlier in this chapter. Closed-loop stability conditions in (2.19) can be recast as the following LMIs [22, 19].

$$\begin{aligned} Y &> 0 \\ \begin{bmatrix} Y & (A_iY - B_iX_i)^T \\ (A_iY - B_iX_i) & Y \end{bmatrix} &> 0 \quad i = 1, \cdots, r \end{aligned}$$

$$\begin{bmatrix} Y & [(A_i + A_j)Y - M_{ij}]^T \\ (A_i + A_j)Y - M_{ij} & Y \end{bmatrix} > 0 \quad j < i \leq r \tag{3.12}$$

where, Y, X_i are defined in (3.10), and M_{ij} are given by

$$M_{ij} = B_i X_j + B_j X_i \tag{3.13}$$

If the LMIs are feasible, the controller gains can be obtained from

$$K_i = X_i Y$$

In the remaining chapters, we extend the results obtained so far, first to the output feedback case, both for continuous-time and discrete-time systems, and then, we present a guaranteed-cost framework to achieve robust performance.

Chapter 4

Fuzzy Observers

4.1 Why Output Feedback

So far, we have developed a systematic framework for the design of Takagi-Sugeno state feedback controllers. An implicit assumption in all previous sections was that the states are available for measurement. However, we know that this is not true in many practical cases. Measuring the states can be physically difficult and costly. Moreover, sensors are often subject to noise and failure. This motivates the question:"How can we design output feedback controllers for T-S fuzzy systems?"

We already know from classical control theory that using an observer, we can estimate the states of an observable LTI system by measuring the output. In fact, we even know how to estimate the states of an LTI system in the presence of additive noise in the system, and measurement noise in the output, using a Kalman filter [43]. Our goal is to generalize the observer methodology to the case of a PLDI instead of a single LTI system, or more specifically, to the case of T-S fuzzy systems. We present a new approach, which is to design an observer based on fuzzy implications, with fuzzy sets in the antecedents, and an asymptotic observer in the consequents. Each fuzzy rule is responsible for observing the states of a locally linear subsystem. The following section will describe the observer design in the continuous-time case [44]

4.2 Continuous-Time T-S Fuzzy Observers

Consider the closed-loop fuzzy system described by r plant rules and r controller rules as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(x)\alpha_j(x)(A_i - B_iK_j)x(t) \\ y(t) &= \sum_{i=1}^{r}\alpha_i(x)C_ix(t) \end{aligned} \tag{4.1}$$

We define a *fuzzy observer* as a set of T-S **If ... Then** rules which estimate the states of the system (4.1). A generic observer rule can be written as

ith Rule: If $Y_1(t)$ is M_{i1} and $\cdots Y_p(t) is \;\; M_{ip}$ THEN: $$\dot{\hat{x}} = A_ix + B_iu + L_i(y - \hat{y})$$

where p is the number of measured outputs, $y = C_ix$ is the output of each T-S plant rule, $\hat{y}$ is the global output estimate, and $L_i \in \mathbb{R}^{n\times p}$ is the local unknown observer gain matrix. The deffuzified global output estimate can be written as:

$$\hat{y}(t) = \sum_{j=1}^{r} \alpha_j C_j \hat{x}(t)$$

where α_is are the normalized membership functions as in (2.3). The aggregation of all fuzzy implications results in the following state equations:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \alpha_i(y)(A_i\hat{x} + B_i u) + \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y) L_i C_j (x - \hat{x}) \tag{4.2}$$

since

$$\sum_{j=1}^{r} \alpha_j(y) = 1$$

we can write equation (4.2) as

$$\dot{\hat{x}} = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y) \left[(A_i - L_i C_j)\hat{x} + B_i u + L_i C_j x\right] \tag{4.3}$$

Note that we wrote the normalized membership functions as a function of y instead of x since, the antecedents are measured output variables and not the states. The controller is also based on the estimate of the state rather than the state itself, i.e., we have

$$u(t) = -\sum_{j=1}^{r} \alpha_j(y) K_j \hat{x}(t) \tag{4.4}$$

replacing (4.4) in (2.9) we get the following equation for the closed-loop system.

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y)(A_i x - B_i K_j \hat{x}) \tag{4.5}$$

Defining the state estimation error as

$$\tilde{x} = x - \hat{x}$$

and subtracting (4.5) from (4.3) we get

$$\dot{\tilde{x}} = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y)(A_i - L_iC_j)\tilde{x} \tag{4.6}$$

To guarantee that the estimation error goes to zero asymptotically, we can use Theorem 2. The observer dynamics is stable if a common positive definite matrix P_2 exists such that the following matrix inequalities are satisfied:

$$\begin{aligned} (A_i - L_iC_i)^T P_2 + P_2(A_i - L_iC_i) &< 0 \quad i = 1, \cdots, r \\ H_{ij}^T P_2 + P_2 H_{ij} &< 0 \quad j < i \leq r \end{aligned} \tag{4.7}$$

where H_{ij} is defined as:

$$H_{ij} = A_i - L_iC_j + A_j - L_jC_i \tag{4.8}$$

Although the inequalities in (4.7) are not LMIs, they can be recast as LMIs by the following change of variables:

$$W_i = P_2L_i \tag{4.9}$$

Using the above variable change and also utilizing the LMI lemma, we obtain the following LMIs in P_2 and W_i:

$$P_2 > 0$$

$$\begin{aligned} A_i^T P_2 + P_2 A_i - W_i C_i - C_i^T W_i^T &< 0 \quad i = 1, \cdots, r \\ (A_i + A_j)^T P_2 + P_2(A_i + A_j) - (W_i C_j + W_j C_i) - (W_i C_j + W_j C_i)^T &< 0 \quad j < i \leq r \end{aligned} \tag{4.10}$$

The observer gains are obtained by the following equation:

$$L_i = P_2^{-1} W_i \tag{4.11}$$

By augmenting the states of the system with the state estimation error, we obtain the following $2n$ dimensional state equations for the closed-loop observer/controller system:

$$\begin{aligned} \left[\begin{array}{c} \dot{x} \\ \hline \dot{\tilde{x}} \end{array}\right] &= \left[\begin{array}{c|c} \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j (A_i - B_i K_j) & \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j B_i K_j \\ \hline 0 & \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j (A_i - L_i C_j) \end{array}\right] \left[\begin{array}{c} x \\ \hline \tilde{x} \end{array}\right] \\ y &= \left[\begin{array}{c|c} \sum_{j=1}^r \alpha_j C_j & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline \tilde{x} \end{array}\right] \end{aligned} \tag{4.12}$$

We have the following theorem for the stability of the closed-loop observer/controller system.

Theorem 6 *The closed-loop observer/controller system (4.12) is globally asymptotically stable, if there exists a common positive definite matrix $\tilde{P}$ such that the following Lyapunov inequalities are satisfied:*

$$\begin{aligned} A_{ii}^T \tilde{P} + \tilde{P} A_{ii} &< 0 \quad i = 1, \cdots, r \\ (A_{ij} + A_{ji})^T \tilde{P} + \tilde{P}(A_{ij} + A_{ji}) &< 0 \quad j < i \leq r \end{aligned} \tag{4.13}$$

where A_{ij} can be defined as

$$A_{ij} = \begin{bmatrix} A_i - B_iK_j & B_iK_j \\ 0 & A_i - L_iC_j \end{bmatrix} \tag{4.14}$$

Proof: The proof directly follows from Theorem 2. ■

Note that the above matrix inequalities are not LMIs in $\tilde{P}$, K_is, and L_is. We would like to know if with the same change of variables as in (3.10) and (4.9), we can rewrite the inequalities in (4.13) as LMIs. In fact, we would like to check if we can extend the separation property of the observer/controller of a single LTI system to the case of (4.12). We will show in the next section, that in the case of (4.12), we indeed have the separation property, and we have two separate sets of LMIs for the observer and the controller [44].

4.3 Separation Property of Observer/Controller

To show that the separation property holds, we have to prove that $\tilde{P}$, the common positive definite solution of the inequalities in (4.13), is a block diagonal matrix with $\lambda P = \lambda Y^{-1}$ and P_2 as diagonal elements, where P is the positive-definite solution of inequalities in (2.12), λ is a positive constant, and P_2 is the solution of (4.7). We can express the separation property in the following theorem:

Theorem 7 *(Separation Theorem for T-S fuzzy systems): The closed-loop system (4.12) is globally asymptotically stable if inequalities in (2.12) and (4.7) are satisfied independently.*

Proof: We choose $\tilde{P}$ as a block diagonal matrix with λP, and P_2 as the block diagonal elements, i.e., we have the following:

$$\tilde{P} = \left[\begin{array}{c|c} \lambda P & 0 \\ \hline 0 & P_2 \end{array}\right] \tag{4.15}$$

We show that there always exists a $\lambda > 0$ such that $\tilde{P}$ satisfies the inequalities in (4.13), provided (2.12) and (4.7) are satisfied. Substituting for $\tilde{P}$ and A_{ij} in (4.13) we obtain the following:

$$\left[\begin{array}{c|c} \lambda\left[(A_i - B_iK_i)^T P + P(A_i - B_iK_i)\right] & \lambda P(B_iK_i) \\ \hline \lambda(B_iK_i)^T P & (A_i - L_iC_i)^T P_2 + P_2(A_i - L_iC_i) \end{array}\right] < 0 \tag{4.16}$$

Using the LMI lemma [38], (4.16) is negative definite if and only if the following conditions are satisfied:

$$\begin{aligned} \lambda\left[(A_i - B_iK_i)^T P + P(A_i - B_iK_i)\right] &< 0 \\ \lambda P(B_iK_i)\left[(A_i - L_iC_i)^T P_2 + P_2(A_i - L_iC_i)\right]^{-1}(B_iK_i)^T P & \\ - \left[(A_i - B_iK_i)^T P + P(A_i - B_iK_i)\right] &> 0 \end{aligned} \tag{4.17}$$

Since (2.12) is satisfied, the first inequality is already true. The second condition is satisfied for any $\lambda > 0$ such that

$$\lambda \min_{1\le i\le r} \mu_i > \max_{1\le i\le r} \nu_i$$

where

$$\mu_i = \lambda_{min}\{P(B_iK_i)\left[(A_i - L_iC_i)^T P_2 + P_2(A_i - L_iC_i)\right]^{-1}(B_iK_i)^T P\}$$

and

$$\nu_i = \lambda_{max}\left[(A_i - B_iK_i)^T P + P(A_i - B_iK_i)\right]$$

where $\lambda_{min}, \lambda_{max}$ are the minimum and maximum eigenvalues. Since (2.12) and (4.7) are already satisfied, such λ always exists. Using the same argument, we can also show that the second set of inequalities in (4.13) is satisfied. Therefore, the two sets of inequalities can be solved independently, and the separation holds. ■

In the next section, we present the discrete-time fuzzy observer.

4.4 Discrete-Time T-S Fuzzy Observers

We can define the T-S fuzzy observer in the same fashion as the continuous-time [45]. A generic rule for the discrete-time T-S fuzzy observer is:

ith Rule: If $y_1(k)$ is M_{i1} and $\cdots y_p(k) is \;\; M_{ip}$ THEN:

$$\hat{x}(k+1) = A_i x(k) + B_i u(k) + L_i(y(k) - \hat{y}(k))$$

where p is the number of measured outputs, and $y(k) = C_i x(k)$ is the output of each T-S plant rule, $\hat{y}$ is the global output estimate, and $L_i \in \mathrm{I\!R}^{\mathrm{n}\times\mathrm{p}}$ is the local observer gain matrix.

The deffuzified output estimate can be written as:

$$\hat{y}(k) = \sum_{j=1}^{r} \alpha_j C_j \hat{x}(k)$$

where α_i are the normalized membership functions as in (2.3). The overall output can also be written in a similar manner,

$$y(k) = \sum_{j=1}^{r} \alpha_j C_j x(k)$$

The aggregation of all fuzzy implications results in the following state equation:

$$\hat{x}(k+1) = \sum_{i=1}^{r} \alpha_i(y)(A_i\hat{x} + B_i u) + \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y) L_i C_j (x - \hat{x}) \tag{4.18}$$

. Since

$$\sum_{j=1}^{r} \alpha_j(y) = 1$$

we can write equation (4.18) as

$$\hat{x}(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y) \left[(A_i - L_i C_j)\hat{x} + B_i u + L_i C_j x\right] \tag{4.19}$$

By defining the estimation error as before, we can write the estimation error $\tilde{x}(k)$ as follows:

$$\tilde{x}(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y)(A_i - L_i C_j)\tilde{x}(k) \tag{4.20}$$

To guarantee that the estimation error goes to zero asymptotically, we can use theorem 4. The observer dynamics is stable if a common positive definite matrix P_2 exists such that the following matrix inequalities are satisfied:

$$\begin{aligned} (A_i - L_iC_i)^T P_2 (A_i - L_iC_i) - P_2 &< 0 \quad i = 1, \cdots, r \\ H_{ij}^T P_2 H_{ij} - P_2 &< 0 \quad j < i \leq r \end{aligned} \tag{4.21}$$

where H_{ij} is defined as in (4.8). Although the inequalities in (4.21) are not LMIs, they can be recast as LMIs using the change of variables of equation (4.9). Using the above variable change and also utilizing the LMI lemma, we obtain the following LMIs in P_2 and W_i:

$$\begin{aligned} \left[\begin{array}{c|c} P_2 & (P_2A_i - W_iC_i)^T \\ \hline P_2A_i - W_iC_i & P_2 \end{array}\right] &> 0 \quad i = 1, \cdots, r \\ \left[\begin{array}{c|c} P_2 & (P_2(A_i + A_j) - W_iC_j + W_jC_i)^T \\ \hline P_2(A_i + A_j) - W_iC_j + W_jC_i & P_2 \end{array}\right] &> 0 \quad j < i \leq r \end{aligned} \tag{4.22}$$

The closed-loop observer/controller system can be written as:

$$\begin{aligned} \left[\begin{array}{c} x(k+1) \\ \hline \tilde{x}(k+1) \end{array}\right] &= \left[\begin{array}{c|c} \sum_{i=1}^r \sum_{j=1}^r \alpha_i\alpha_j (A_i - B_iK_j) & \sum_{i=1}^r \sum_{j=1}^r \alpha_i\alpha_j B_iK_j \\ \hline 0 & \sum_{i=1}^r \sum_{j=1}^r \alpha_i\alpha_j (A_i - L_iC_j) \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline \tilde{x}(k) \end{array}\right] \\ y &= \left[\begin{array}{c|c} \sum_{j=1}^r \alpha_j C_j & 0 \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline \tilde{x}(k) \end{array}\right] \end{aligned} \tag{4.23}$$

Using Theorem 4, the system (4.23) is globally asymptotically stable, if there exists a positive definite matrix $\tilde{P} > 0$ such that :

$$\begin{aligned} A_{ii}^T \tilde{P} A_{ii} - \tilde{P} &< 0 \quad i = 1, \cdots, r \\ (A_{ij} + A_{ji})^T \tilde{P} (A_{ij} + A_{ji}) - \tilde{P} &< 0 \quad j < i \leq r \end{aligned} \tag{4.24}$$

where A_{ij} is the same as in (4.14).

As in the continuous-time case, we can show that the Lyapunov matrix $\tilde{P}$ is indeed block diagonal, i.e., the discrete-time version of the Theorem 4.16 holds, and observer and controller gains can be found via separate LMI feasibility problems. A proof of the separation proven in the discrete-time case is given in [45].

4.5 Numerical Example

We present a numerical example to illustrate the results obtained in this chapter. We use the two-rule T-S fuzzy model which approximates the motion of an inverted pendulum on a cart. This system has been studied in [22, 46]. The T-S fuzzy rules are obtained by approximation of the nonlinear system around 0° and 88°. The T-S rules can be written as:

Plant Rule 1: If y is *around* 0 Then $\dot{x} = A_1 x + B_1 u$

Plant rule 2: If y is *around* $\pm\pi/2$ Then $\dot{x} = A_2 x + B_2 u$

Controller Rule 1: If y is *around* 0 Then $u = -K_1 x$

Controller Rule 2: If y is *around* $\pm\pi/2$ Then $u = -K_2 x$

Observer Rule 1:If y is *around* 0 Then $\dot{\hat{x}} = A_1\hat{x} + B_1 u + L_1 C(x - \hat{x})$

Observer Rule 2: If y is *around* $\pm\pi/2$ Then $\dot{\hat{x}} = A_2\hat{x} + B_2 u + L_2 C(x - \hat{x})$

where A_1, A_2, B_1, B_2, C are given as follows:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 17.3 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -0.177 \end{bmatrix}$$

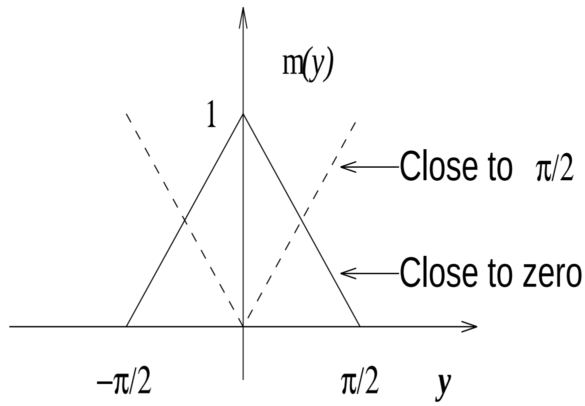

Figure 4.1: Membership functions for the angle.

$$A_2 = \begin{bmatrix} 0 & 1 \\ 9.45 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -0.03 \end{bmatrix} \tag{4.25}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{4.26}$$

The membership functions μ_1, μ_2 for the two fuzzy sets *close to zero*, and *close to* $\pm\pi/2$ are plotted in Figure 4.1. We fix the observer and controller gains by local pole placement, and look for common Lyapunov matrices P, and P_2. We place the closed-loop poles of the system at $-2, -2$, and the poles of the observer dynamics at $-6, -6.5$. The observer and controller gains are:

$$K_1 = \begin{bmatrix} -120.67 & -66.67 \end{bmatrix} \quad K_2 = \begin{bmatrix} -2551.6 & -764.0 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 12.5 & 57.3 \end{bmatrix}^T \quad L_2 = \begin{bmatrix} 12.5 & 50.0 \end{bmatrix}^T \tag{4.27}$$

Fortunately, the LMIs are feasible and we can find positive-definite Lyapunov matrices P and P_2. The simulation results for the states of the system $x(1)$ and $x(2)$ as well as the estimation error $x(3), x(4)$ are depicted in Figures 4.2 through 4.5. Although we were able to solve for

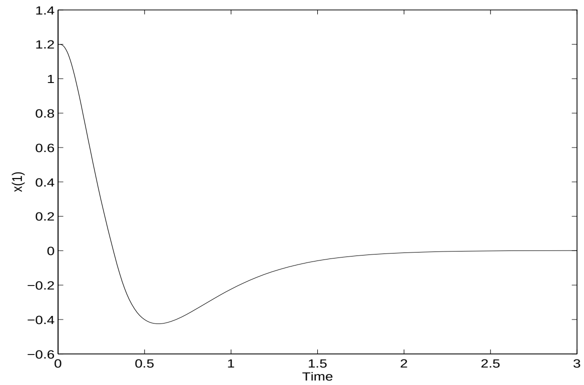

Figure 4.2: Initial condition response of the pendulum angle.

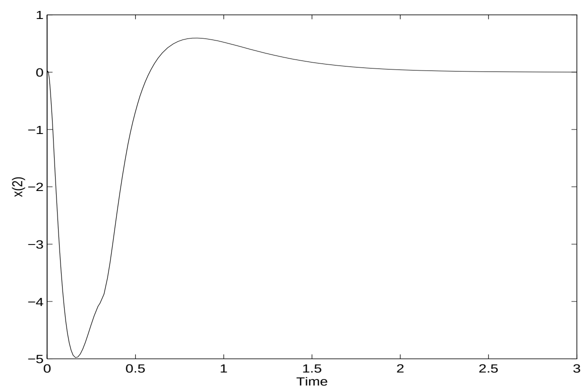

Figure 4.3: Initial Condition response of the angular velocity.

positive-definite Lyapunov matrices P and P_2 using local pole placement, this might not be always possible. This is the reason why we need to obtain some performance in addition to stability. In the next chapter, we will develop a guaranteed-cost approach for minimizing a quadratic cost function [38].

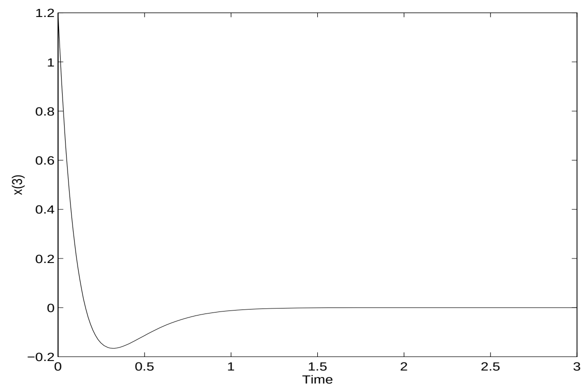

Figure 4.4: Estimation error for angle.

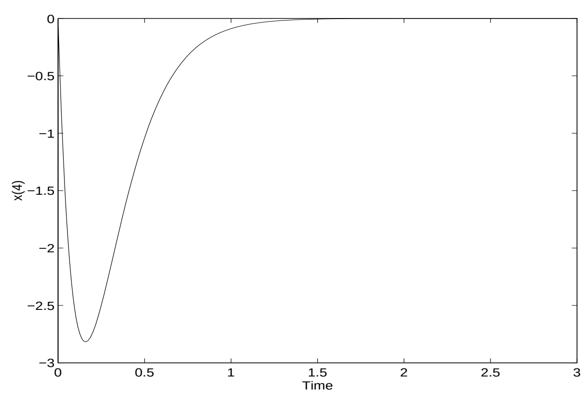

Figure 4.5: Estimation error for angular velocity.

Chapter 5

Guaranteed-Cost Design of T-S Fuzzy Systems

We studied the stability of T-S fuzzy systems in detail in the previous chapters. However, stability is always a primary goal, and we usually need to specify some performance objective in the design procedure as well. There have been few results that have gone beyond stability in order to consider performance for fuzzy systems. The authors in [19] have added a degree-of-stability criterion, and have shown that controller design with a guaranteed degree of stability, can be transformed into a Generalized EigenValue Problem (GEVP) [21]. Recently the authors in [47] and [48] have added an LMI condition that can bound the control action. In this chapter, we generalize these results to the problem of minimizing the expected value a quadratic performance measure with respect to random initial conditions, with zero mean and a covariance equal to the identity. Using the guaranteed-cost approach [38, 49], we minimize an upper bound on an LQ measure representing the control effort and the regulation error. We show that this problem can be transformed into a trace minimization problem, which can then be solved using any of the available convex optimization software packages.

The guaranteed-cost design was first introduced in [49]. Briefly, the idea is to replace the uncertain cost of the system with a certain upper bound, and try to minimize that upper bound. Using this approach, although we may not find the global minimum of the cost functional, we can find the minimum of the upper bound, and by doing so, hope to be close to the actual minimum. The interested reader is referred to [38] for more details on this subject. First, we present a brief review of linear quadratic control (LQR) theory.

5.1 CASE I: Continuous-Time Case

5.1.1 A Brief Review of Continuous-Time LQR Theory

It is a well known result from LQR theory that the problem of minimizing the cost function

$$J = \int_0^\infty (x^T Q x + u^T R u) dt \tag{5.1}$$

where $Q \geq 0$ and $R > 0$, subject to :

$$\dot{x} = Ax + bu, \qquad u = -Kx$$

where $(A, Q^{1/2})$ is detectable, and (A, B) is stabilizable, results in finding a positive solution of the following Algebraic Riccati Equation (ARE)

$$(A - BK)^T P + P(A - BK) + Q + K^T R K = 0$$

and $K = R^{-1}B^TP$. We can write the minimum cost J as [38]:

$$J_m = x(0)^T Px(0)$$

where P is the solution of the above ARE. If we write the ARE as a matrix inequality instead of an equality, the solution of the inequality will be an upper bound on the performance measure J, and we can reach J_m by minimizing that upper bound.

5.1.2 Guaranteed Cost Design of Continuous-Time T-S Systems

While the above result holds for a single LTI system, we can extend it to the case of equation (2.11). To avoid the dependency of the cost function J on initial conditions, we assume the initial conditions are randomized variables with zero mean and a covariance equal to the identity, i.e.,

$$\begin{aligned} \mathbb{E}\{x_0 x_0^T\} &= I \\ \mathbb{E}\{x_0\} &= 0 \end{aligned} \tag{5.2}$$

where $\mathbb{E}$ is the expectation operator. Our objective is to minimize the expected value of the performance index J with respect to all possible initial conditions. Now we can state the following lemma:

Lemma 1 *For random initial conditions with zero mean and covariance equal to the identity, we have*

$$\mathbb{E}_{\mathrm{x}_0}\{x_0^T P x_0\} = tr(P) \tag{5.3}$$

Proof: Note that

$$\mathbb{E}\{x_0^T P x_0\} = \mathbb{E}\{tr(P x_0 x_0^T)\} = tr\{P\mathbb{E}(x_0 x_0^T)\} = tr(P)$$

■

Using the above lemma, we have the following theorem:

Theorem 8 *Consider the closed-loop fuzzy system (2.11). We have the following bound on the performance objective J*

$$J = \mathbb{E}_{x_0} \int_0^{\infty} (x^T Q x + u^T R u) dt < tr(P) \tag{5.4}$$

where P is the solution of the following Ricatti inequality

$$(A_i - B_i K_j)^T P + P(A_i - B_i K_j) + Q + \sum_{i=1}^{r} K_i^T R K_i < 0 \tag{5.5}$$

and u is defined as in equation (2.10).

proof: We already know that $J < tr(\hat{P})$ where $\hat{P}$ satisfies the following Riccati inequality

$$(A_i - B_i K_j)^T \hat{P} + \hat{P}(A_i - B_i K_j) + Q + (\sum_{i=1}^{r} \alpha_i K_i)^T R (\sum_{i=1}^{r} \alpha_i K_i) < 0 \tag{5.6}$$

We just need to show that

$$(\sum_{i=1}^{r} \alpha_i K_i)(\sum_{i=1}^{r} \alpha_i {K_i}^T) < \sum_{i=1}^{r} K_i^T K_i \tag{5.7}$$

For simplicity, we will show that the above inequality is true when we only have two rules for the controller, the extension to more than two can be done using induction. We need to show that

$$(\alpha_1 K_1 + \alpha_2 K_2)^T R(\alpha_1 K_1 + \alpha_2 K_2) < {K_1}^T R K_1 + {K_2}^T R K_2 \tag{5.8}$$

To illustrate this, we rewrite the left hand side of (5.8) as the following quadratic form:

$$\begin{bmatrix} K_1^T R^{1/2} & K_2^T R^{1/2} \end{bmatrix} \begin{bmatrix} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^T \end{bmatrix} \begin{bmatrix} R^{1/2} K_1 & R^{1/2} K_2 \end{bmatrix} \tag{5.9}$$

The right hand side of (5.8) can be written as:

$$\begin{bmatrix} K_1^T R^{1/2} & K_2^T R^{1/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^{1/2} K_1 & R^{1/2} K_2 \end{bmatrix} \tag{5.10}$$

To prove the theorem we have to show that

$$\begin{bmatrix} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^T \end{bmatrix} < \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.11}$$

This is already satisfied since the difference of the two matrices is positive definite i.e., we have the following

$$\begin{bmatrix} 1-\alpha_1^2 & 1-\alpha_1\alpha_2 \\ 1-\alpha_1\alpha_2 & 1-\alpha_2^2 \end{bmatrix} > 0 \tag{5.12}$$

This concludes the proof. ■

Now, using the same change of variables as (3.10), and pre-multiplying and post-multiplying equation (5.5) by P^{-1} and also using Theorem 2, we can write (5.5) as the following inequalities

$$
\begin{aligned}
0 &> YA_i^T + A_iY - B_iXi - X_i^TB_i^T + YQY + \sum_{i=1}^{r} X_i^TRX_i \\
0 &> Y(A_i + Aj)^T + (A_i + A_j)Y - M_{ij} - M_{ij}^T + YQY + \sum_{i=1}^{r} X_i^TRX_i \\
& \quad i = 1, \cdots, r \quad j < i \leq r
\end{aligned}
\tag{5.13}
$$

where M_{ij} is defined in (3.13). Using the LMI Lemma [38], we can write the above inequalities as follows

$$
\begin{aligned}
0 &> \begin{bmatrix} YA_i^T + A_iY - B_iX_i - X_i^TB_i^T & YQ^{1/2} & X_1^TR^{1/2} & \cdots & X_r^TR^{1/2} \\ Q^{1/2}Y & -I_{n\times n} & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & -I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & \cdots & -I_{m\times m} \end{bmatrix} \\
0 &> \begin{bmatrix} Y(A_i + A_j)^T + (A_i + A_j)Y - M_{ij} - M_{ij}^T & YQ^{1/2} & X_1^TR^{1/2} & \cdots & X_r^TR^{1/2} \\ Q^{1/2}Y & -I_{n\times n} & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & -I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & \cdots & -I_{m\times m} \end{bmatrix} \\
& i = 1, \cdots, r \quad j < i \leq r
\end{aligned}
\tag{5.14}
$$

To obtain the least possible upper bound using a quadratic Lyapunov function, we have the following optimization problem

Min $tr(Y^{-1})$
Subject To: LMIs in (5.14)

This is a convex optimization problem which can be solved in polynomial time [37], using any of the available LMI toolboxes. To make it possible to use MATLAB® LMI Toolbox, we introduce an artificial variable Z, which is an upper bound on Y^{-1}, and minimize $tr(Z)$ instead, i.e, we recast the problem in the following form

Min $tr(Z)$
Subject To LMIs in (5.14), and

$$\begin{bmatrix} Z & I_{n\times n} \\ I_{n\times n} & Y \end{bmatrix} > 0 \tag{5.15}$$

If the above LMIs are feasible, we can calculate the controller gains as $K_i = X_i Y^{-1}$.

Next, we present a numerical example, to illustrate these results.

5.1.3 Numerical Example

Consider the problem of balancing an inverted pendulum on a cart. We use the same model as in [22]. The equations for the motion of the pendulum are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g\sin(x_1) - amlx_2^2\sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)} \end{aligned} \tag{5.16}$$

where x_1 denotes the angle of the pendulum (in radians) from the vertical axis, and x_2 is the angular velocity of the pendulum (in radians per second), $g = 9.8m/s^2$ is the gravity constant, m is the mass of the pendulum (in Kilograms), M is the mass of the cart (in Kilograms), $2l$ is the length of the p endulum (in meters), and u is the force applied to the cart (in Newtons). Using the same values as in [22], we have $a = \frac{1}{m+M}$, $m = 2$ kg, $M = 8.0$ kg, and $2l = 1.0$m. We approximate the nonlinear plant by two Takagi-Sugeno fuzzy rules as follows:

Plant Rule (1): If x_1 is *close to zero* **Then** $\dot{x} = A_1 x + B_1 u$

Plant Rule (2): If x_1 is *close to* $\pm\pi/2$ **Then** $\dot{x} = A_2 x + B_2 u$

where *close to zero* and *close to* $\pm\pi/2$ are the input fuzzy sets defined by the membership functions $\mu_1 = 1 - \frac{2}{\pi}|x_1| \quad \mu_2 = \frac{2}{\pi}|x_1|$

respectively, (see figure 4.1), and A_1, A_2, B_1, B_2 are given as follows

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ \frac{g}{4l/3 - aml} & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -\frac{a}{4l/3 - aml} \end{bmatrix} \\ A_2 &= \begin{bmatrix} 0 & 1 \\ \frac{9g}{4\pi(4l/3 - aml\beta^2)} & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -\frac{a\beta}{4l/3 - aml\beta^2} \end{bmatrix} \\ \beta &= \cos(80^\circ) \end{aligned} \tag{5.17}$$

We also choose the following values for Q and R

$$Q = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.18}$$

$$R = 2 \tag{5.19}$$

Solving the LMI optimization problem in the previous section, we obtain the following values for the controller gain

$$\begin{aligned} K_1 &= \begin{bmatrix} -225.6 & -55.8 \end{bmatrix} \\ K_2 &= \begin{bmatrix} -272.1 & -75.9 \end{bmatrix} \end{aligned} \tag{5.20}$$

The resulting global controller is

$$u = -(\mu_1(x_1)K_1 + \mu_2(x_1)K_2)x \tag{5.21}$$

Simulations indicate that the above control law can balance the pendulum for initial conditions between $[-80°, 80°]$. Results are depicted in Figures 5.1 to 5.3. As is evident from the simulation results, the controller gains are much smaller than the ones given in [22]. It is worthwhile to note that we can design nonlinear controllers for the plant (5.16) based on feedback linearization techniques, but these controllers are usually very complicated. One such controller was given in [50]:

$$\begin{aligned} u &= k(x_1, x_2) \\ &= -\frac{g}{a}\tan(x_1) - \frac{4le_1e_2}{3a}\ln[\sec(x_1) + \tan(x_1)] \\ &\quad + e_1e_2ml\sin(x_1) + \frac{(e_1+e_2)x_2}{a}[\frac{4l}{3}\sec(x_1) - aml\cos(x_1)] \end{aligned}$$

where e_1 and e_2 are the desired closed loop eigenvalues. Note that here we do not have any measure for optimality. Instead, By linearizing the dynamics of the system for angles greater

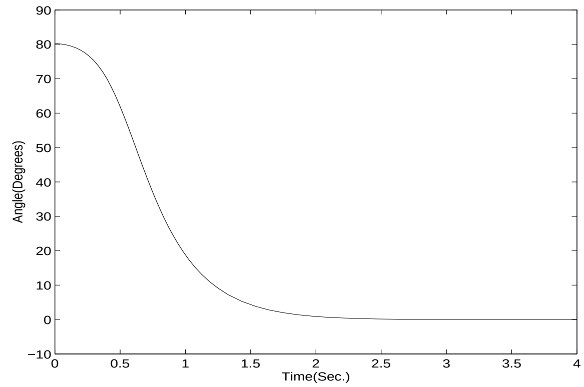

Figure 5.1: Initial condition Response of the Angle.

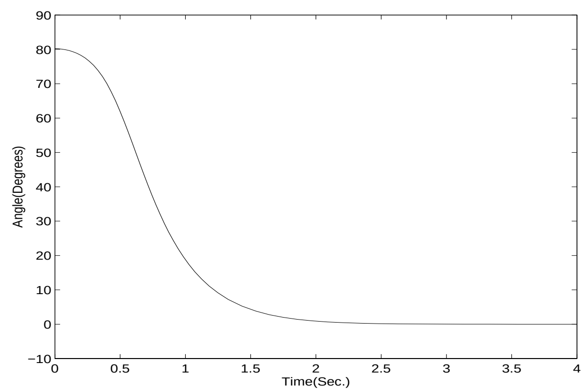

Figure 5.2: Initial condition Response of Angular Velocity.

than $\pi/2$ and those close to π, we can balance the pendulum at any initial condition while feedback linearization works only in the $[-\pi/2, \pi/2]$ interval [22].

5.2 CASE II: Discrete-Time Case

In the discrete-time case, the problem of minimizing the cost function

$$J = \sum_{k=0}^{\infty} x^T(k)Qx(k) + u^T(k)Ru(k) \tag{5.22}$$

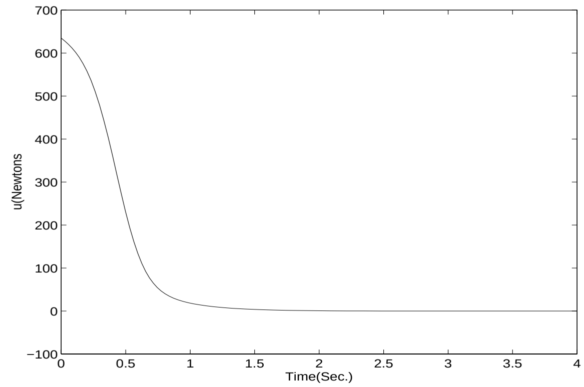

Figure 5.3: Control Action.

reduces to finding a positive definite solution $P > 0$ of the following discrete-time Riccati equation:

$$(A - BK)^T P(A - BK) - P + Q + K^T RK = 0 \tag{5.23}$$

where $Q \geq 0$ and $R > 0$. We can write the minimum cost of J as [51]:

$$min\{J\} = x_0^T P x_0$$

Again, as in the continuous-time case, we can extend these results to the case of discrete-time PLDIs, or discrete-time T-S fuzzy systems using a guaranteed-cost framework. The cost function is exactly the same as in the continuous-time case, and the only difference is that the Riccati equation is in discrete-time. It can be shown [38, 46] that we can write the upper bound on the performance objective as: $J = tr(P)$ where P satisfies the following Lyapunov inequalities:

$$(A_i - B_i K_i)^T P(A_i - B_i K_i) - P + Q + \sum_{i=1}^{r} K_i^T R K_i \quad < \quad 0 \quad i = 1, \ldots, r$$

$$G_{ij}^T P G_{ij} - P + Q + \sum_{i=1}^{r} K_i^T R K_i \quad < \quad 0 \quad j < i \leq r \tag{5.24}$$

and G_{ij} is the same as in (2.13). With the usual change of variables and by using the LMI lemma [38], we can write the Riccati inequalities (5.24) as the following LMI's:

$$\begin{bmatrix} Y & N_i^T & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ N_i & Y & 0 & 0 & \cdots & 0 \\ Q^{1/2}Y & 0 & I_n & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & 0 & \cdots & I_m \end{bmatrix} \quad < \quad 0$$

$$\begin{bmatrix} Y & O_{ij}^T & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ O_{ij} & Y & 0 & 0 & \cdots & 0 \\ Q^{1/2}Y & 0 & I_n & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & 0 & \cdots & I_m \end{bmatrix} \quad < \quad 0$$

$$i = 1, \cdots, r \; j < i \leq r \tag{5.25}$$

where

$$\begin{aligned} N_i &= A_i Y - B_i X_i \\ O_{ij} &= (A_i + A_j)Y - M_{ij} \end{aligned} \tag{5.26}$$

and M_{ij} is defined in (3.13).

5.3 Limitations of Our Approach

Despite the fact that the methods presented so far seem to be very appealing, they do not work for all systems. The main limitation being the implicit assumption in our design procedure that the local subsystems, which are basically linearization of the original system, are quadratically stabilizable. There are many systems that can be stabilized, but are not quadratically stabilizable. To illustrate this point, we present the following example, known as a *benchmark problem for nonlinear control design* [52]. This system was originally proposed as a simplified 4-state model of a dual-spin space craft. The problem involves a cart of mass M whose mass center is constrained to move along a straight horizontal line. Attached to the cart is a "proof body" actuator of mass m and moment of inertia I. Relative to the cart, the proof body rotates about a vertical line passing through the cart mass center. The nonlinearity of the problem co mes from the interaction between the translational motion of the cart and the rotational motion of the eccentric proof mass. (See [52] for more details). The state space representation of the system is as follows:

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{-x_1 + \epsilon x_4^2 \sin x_3}{1 - \epsilon^2 \cos^2 x_3} \\ x_4 \\ \frac{\epsilon \cos x_3 (x_1 - \epsilon x_4^2 \sin x_3)}{1 - \epsilon^2 \cos^2 x_3} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-\epsilon \cos x_3}{1 - \epsilon^2 \cos^2 x_3} \\ 0 \\ \frac{1}{1 - \epsilon^2 \cos^2 x_3} \end{bmatrix} u \tag{5.27}$$

where ϵ is a positive number between zero and one. We obtain the following T-S fuzzy model for the system

Plant Rule (1): If x_3 is *close to zero* **Then** $\dot{x} = A_1 x + B_1 u$

Plant Rule (2): If x_3 is *close to* $\pm\pi/2$ **Then** $\dot{x} = A_2 x + B_2 u$

where *close to zero* and *close to* $\pm\pi/2$ are the input fuzzy sets defined by the membership functions

$$\mu_1 = 1 - \frac{2}{\pi}|x_1| \quad \mu_2 = \frac{2}{\pi}|x_1|$$

respectively, (see Figure 4.1), and A_1, A_2, B_1, B_2 are given as follows

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{1-\epsilon^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon}{1-\epsilon^2} & 0 & 0 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -\frac{\epsilon}{1-\epsilon^2} \\ 0 \\ \frac{1}{1-\epsilon^2} \end{bmatrix} \\
A_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{1-\epsilon^2\beta^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon\beta}{1-\epsilon^2\beta^2} & 0 & 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -\frac{\epsilon\beta}{1-\epsilon^2\beta^2} \\ 0 \\ \frac{1}{1-\epsilon^2\beta^2} \end{bmatrix} \\
\beta &= \cos(80^\circ)
\end{aligned} \tag{5.28}$$

Using the guaranteed-cost approach with the following weighting matrices: $Q = I$, and $r = 0.1$, we solve the LMI optimization algorithm. Unfortunately, we can not achieve a very satisfactory performance, since the LMIs are marginally feasible. The simulation results are depicted in Figures 5.4 through 5.7. Simulation results in [52] indicate that this system can be stabilized with a better performance using a nonlinear controller. In other words, we are limiting ourselves to quadratic Lyapunov functions, and in the case of the benchmark system, while the controller in [52] achieves a better performance.

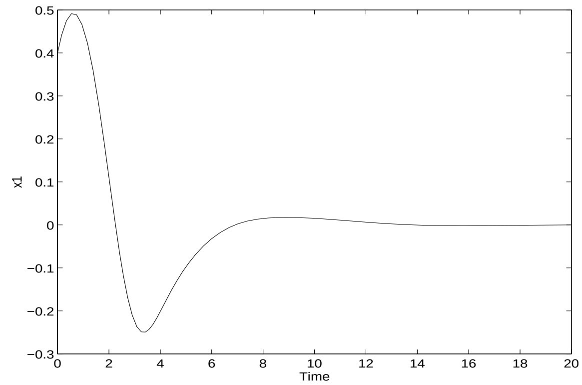

Figure 5.4: (nondimensionalized) position .

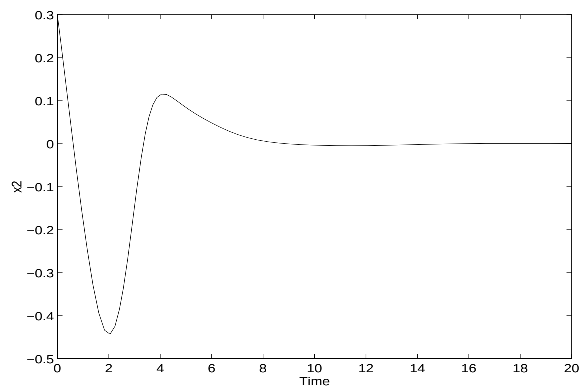

Figure 5.5: (nondimensionalized) velocity .

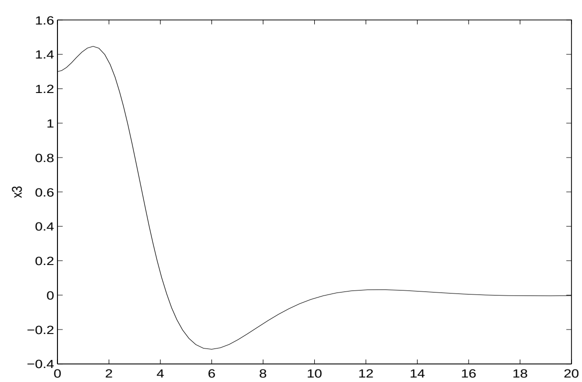

Figure 5.6: (nondimensionalized) angular position .

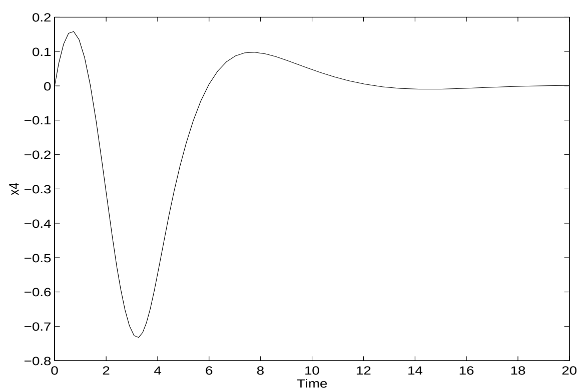

Figure 5.7: (nondimensionalized) angular velocity .

Chapter 6

Non-Fragile Controller Design via LMIs

6.1 Introduction

One of the most active areas of research in linear control systems is robust and optimal controller design. For the past 15 years several researchers have come up with different methods that enable the controller to cope with uncertainties in the plant dynamics. Some of these methods deal with the so-called structured uncertainty, while others deal with unstructured uncertainty. A majority of these methods rely on the Youla-Kučera Q parameterization of all stabilizing controllers. Elegant techniques for minimizing H_2 [53], H_∞ [54, 55] and L_1 [56] norms of different closed-loop transfer functions have been developed using this parameterization. Although these methods cope with uncertainty in the plant dynamics, they all assume that the controller derived is precise, and exactly implemented. Unfortunately, this is not the case in practice. The controller implementation is subject to round-off errors in numerical computations, in addition to the need of providing the practicing engineer with safe-tuning margins. Therefore, the design has to be able to tolerate some uncertainty in the controller as well as the plant dynamics. Recent results in [15] have brought attention to this problem. The authors in [57] have come up with a method to deal with the uncertainty in a fixed-structure dynamical controller, but have not taken into account the uncertainty in the plant dynamics.

The basic premise of this chapter is that one can not achieve "resiliency" if robustness is all that is demanded, and as motivated by [15] and discussed in [57], there exists a trade-off between the system's ability to tolerate both. The numerical examples in [15] suggest that if the only uncertainty is in the plant, all of the available margins will be used, making the closed-loop system extremely fragile with respect to uncertainties in the controller. Since designing a dynamical controller as in [57] for the case where both system and controller are uncertain makes the problem very complicated, we consider in this chapter the design of robust, yet resilient static state feedback controllers using the methodology of T-S fuzzy Systems [58]. We also stress that recent results in [59] suggest that the order of the controller is not the only cause of fragility, i.e, the controller can be of low order, yet still be fragile.

6.2 Polytopic Uncertainty

We discussed in the previous chapters the design of T-S fuzzy controllers for nonlinear systems. Now, we are going to use the same methodology for the design of uncertain linear systems. The main difference between this chapter and the previous ones is that we assume that the α_is in

(3.8) are uncertainties, instead of known functions. As far as stability and robust performance are concerned, we do not need to know the exact value of α_is. In other words, the theory is the same for uncertain systems and systems modeled by T-S fuzzy systems. Modeling uncertain systems in a PLDI format is very common in the robust control literature. We extend the results in [21] regarding state feedback controller design for PLDIs to the case where the controller gains are uncertain as well, i.e., the controller gains also lie in a polytope. The design of non-fragile controllers with affine uncertainty is studied in [62]. To illustrate this method, we write the uncertain system in a polytopic form as follows [60]:

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(t,x)(A_i x + B_i u) \tag{6.1}$$

where, $x(t) \in R^n$, $u(t) \in R^m$, $A_i \in R^{n\times n}$, $B_i \in R^{n\times n}$ $\sum_{i=1}^{r} \alpha_i(t,x) = 1$, and $\alpha_i(t,x) > 0$, $\forall i \in \{1,\ldots,r\}$. For simplicity, we assume that the states are available for measurement and feedback. Using a similar form of polytopic uncertainty for the controller, the control input can be written as

$$u = -\sum_{j=1}^{r} \beta_j(t,x) K_j x \tag{6.2}$$

where $\beta_j(t,x) > 0$ $\forall j \in \{1,\ldots,r\}$, and $\sum_{j=1}^{r} \beta_j(t,x) = 1$. Replacing u in (6.1) with (6.2), and keeping in mind that $\sum_{i=1}^{r} \alpha_i(t,x) = 1$, the closed-loop system can be written as

$$\dot{x} = \sum_{j=1}^{r}\sum_{i=1}^{r} \alpha_i(t,x)\beta_j(t,x)(A_i - B_i K_j)x \tag{6.3}$$

The stability of (6.3) can be checked by the following theorem:

Theorem 9 *: The closed-loop system (6.3) is globally asymptotically stable if there exists a common positive-definite matrix P that satisfies the following Lyapunov inequalities :*

$$(A_i - B_iK_j)^TP + P(A_i - B_iK_j) < 0 \quad i, j = 1, \ldots, r \tag{6.4}$$

Proof:The proof is easily obtained by multiplying inequalities (6.4) by $\alpha_i\beta_j$ and adding them up. ■

Pre-multiplying and post-multiplying the inequalities in (6.4) by $Y = P^{-1}$, and introducing $X_i = K_iY$, we can write inequalities (6.4) as the following LMIs

$$YA_i^T + A_iY - M_{ij} - M_{ij}^T < 0 \quad i, j = 1, \ldots, r \tag{6.5}$$

where $M_{ij} = B_iX_j$. Note that the above conditions are more strict than the ones in theorem 2, since we assume different uncertain parameters for the controller and the plant. Same as before, we can obtain the vertices of the controller polytope from the following equations:

$$\begin{aligned} P &= Y^{-1} \\ K_i &= X_iY^{-1} \quad i = 1, \cdots, r \end{aligned} \tag{6.6}$$

6.3 Robust Performance

As in Chapter 5, we can obtain robust performance using guaranteed-cost bounds for the uncertain system (6.3). We use the same approach as in Chapter 5, and we get the following

LMIs for guaranteed-cost design [60].

$$\begin{bmatrix} N_{ij} & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ Q^{1/2}Y & -I_n & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & -I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & \cdots & -I_m \end{bmatrix} < 0 \qquad i,j = 1,\cdots,r \tag{6.7}$$

where N_{ij} is defined as

$$N_{ij} = YA_i^T + A_iY - M_{ij} - M_{ij}^T \tag{6.8}$$

To obtain the least possible upper bound provable by a quadratic Lyapunov function, we have the following optimization problem

Min $tr(Y^{-1})$
Subject To: LMIs in (6.7)

To make it possible to use MATLAB® LMI Toolbox, we introduce an artificial variable Z as an upper bound on Y^{-1}, and minimize $tr(Z)$ instead, i.e, we recast the problem in the following form:

Min $tr(Z)$
Subject To LMIs in (6.7), and (5.15)

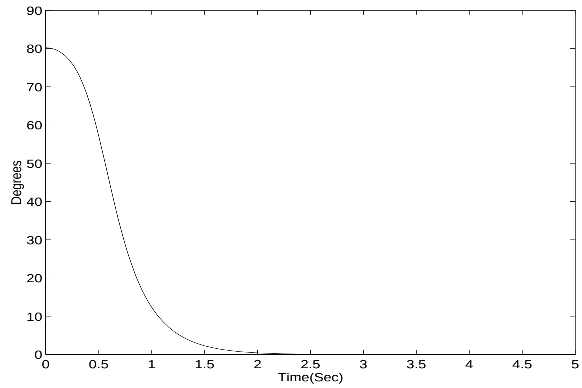

Figure 6.1: Angle Response with 30% change in cart mass and 40% in pole length.

If the above LMIs are feasible, we can find directions in which uncertainty can be tolerated. In other words, any convex combination of the controller gains would guarantee stability. To illustrate our point, we go back to the same problem studied in section 5.1.3, i.e., the problem of balancing the inverted pendulum on a cart. The difference between the current approach and the one in chapter 5 is that we replace the membership functions in the control equation (5.21) with constant numbers between zero and one as long as the closed-loop LDI approximates the closed-loop nonlinear system. One such choice may be

$$u = -(0.4K_1 + .6K_2)x \tag{6.9}$$

To illustrate the robustness of this approach, we gave a 30% increase to the cart mass M and also increased the pole length by 40%. Results are depicted in Figures 6.1 and 6.2. We can also repeat these results in the case of discrete-time systems. Details are discussed in [61].

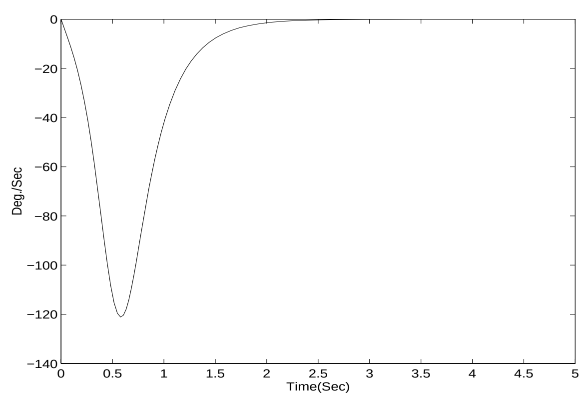

Figure 6.2: Angular velocity with 30% change in cart mass and 40% in pole length.

Chapter 7

Conclusion

7.1 A Brief Summary of The Thesis

The purpose of this thesis was to present a systematic framework for the design of simple controllers for nonlinear systems. The idea was to treat a nonlinear system, as a time-varying uncertain-linear system. The methods described in this thesis present a unified approach to the design of robust controllers and Takagi-Sugeno fuzzy controllers. We showed that T-S fuzzy systems are a special form of Linear Differential Inclusions which can be used for the design of robust as well as *resilient* controllers. The idea of representing a nonlinear system with an uncertain linear system is implicit in the early Soviet literature on absolute stability of control systems. We used these ideas and extended the recent results on stability of Takagi-Sugeno fuzzy systems to a guaranteed-cost method for achieving performance in addition to stability using convex optimization methods. We also discussed the limitations of this approach using numerical examples. Although the results are promising, there is still room for future research in this area which we describe in the next section.

7.2 Future Research Directions

Future research can be done in the area of non-fragile controller design by extending the results obtained here to the case of output feedback controllers. The results can be extended to the problem of non-fragile implementation of such controllers. Our approach was able to provide resiliency with respect to the controller gains. However, in practice one might need to have resilience with respect to variations in the electronic components that the controller is made of. Also the effect of truncation of the parameters can be an important research direction. In the area of T-S fuzzy systems, we need to look for stability results that take into account the properties of fuzzy implications and membership functions to reduce the conservatism in our stability results. In other words, we did not utilize the membership functions in proving our stability results, i.e., we treated the membership functions as unknown uncertainties. However, if the membership functions are available, we should look for stability results that use the information of the membership functions. Another direction for future research is the approximation accuracy of T-S fuzzy systems. At present, this is a very active area, and several researchers have reported some relative success. In years to come, there would perhaps be a closer tie between the so called "classical" control methods and "soft computing" methods. We have tried to reduce the gap between these two important disciplines, and hope that this thesis would be one of the many in this direction.

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