Sampling Theory

In graphics we start off with a continuous function (the scene)

- Triangles 
- Spheres 
- Nurbs 

continuous

$\longrightarrow$ Discrete 2D image

sampling

- screen pixels (spatial sampling)
- frames in time (temporal sampling)
- texels in a texture (texture sampling)

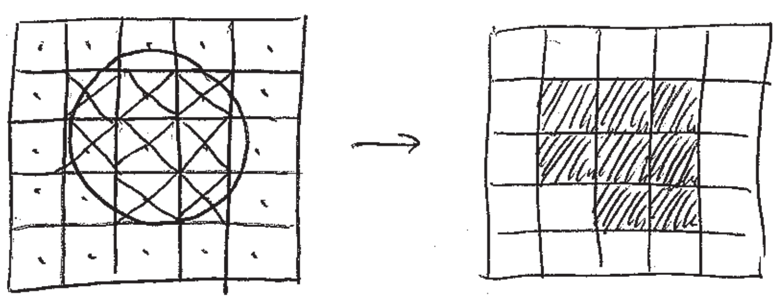
Ex. Scan conversion of a circle:

$$f(x,y) = (x-a)^2 + (y-b)^2 - r^2$$

$f(x,y) > 0$, outside circle

$f(x,y) < 0$, inside circle

$f(x,y) = 0$, on circle boundary



after reconstruction

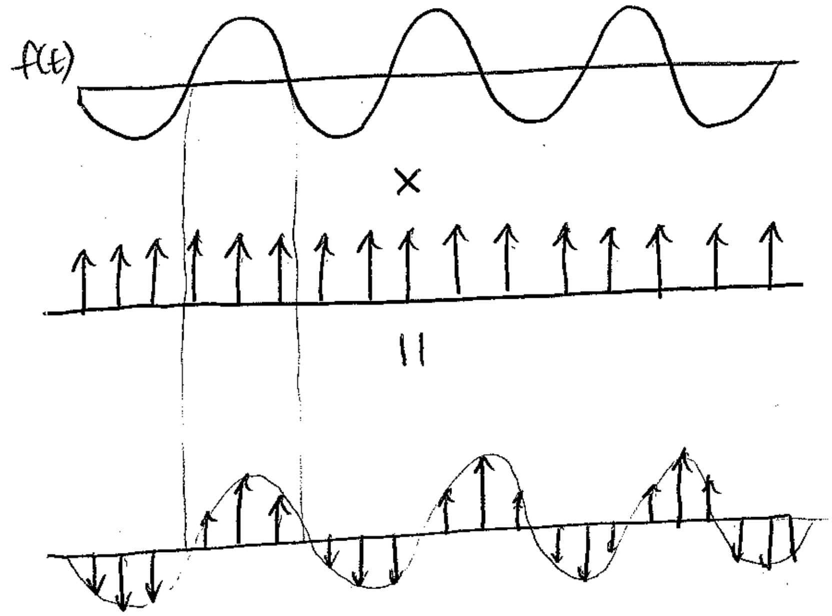
Complete sampling & reconstruction pipeline during rendering

Continuous signal (scene) $\rightarrow$ sampling $\rightarrow$ discrete 2D image $\longrightarrow$ reconstruction $\longrightarrow$ continuous 2D image

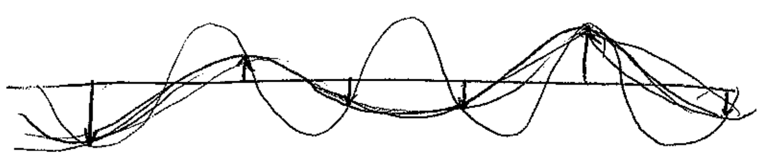
by "pixels" of the display

Sampling

Equivalent to multiplying the signal by a train of delta functions:



Show example of when we undersample...



Aliasing – higher frequencies appearing as lower frequencies.

It turns out that $f_{sample} \geq 2$ max freq of signal $\}$ Nyquist Rate

Two ways to achieve this:

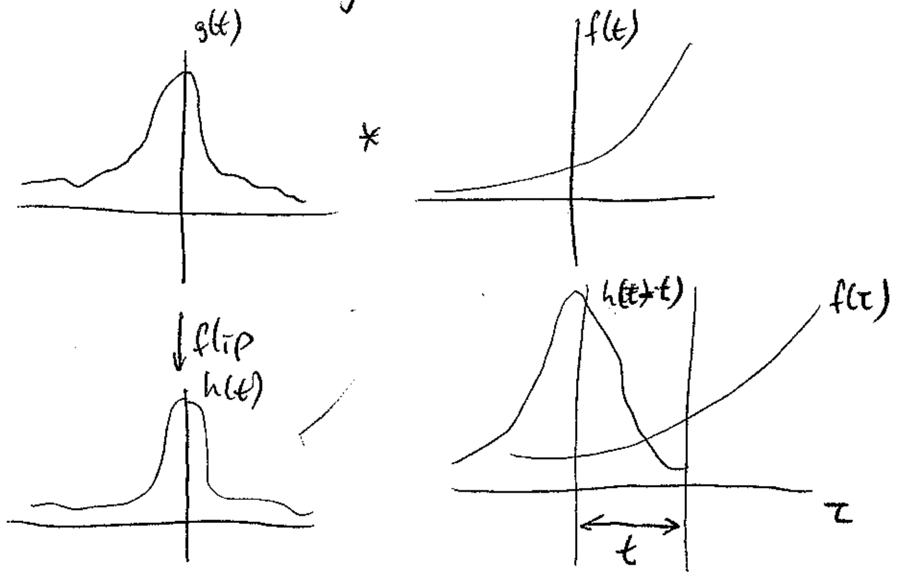
1. increase sampling frequency
2. Blur the signal to reduce high frequency content.

Convolution

$$(f * g)(t) = \int f(\tau)\, g(t-\tau)\, d\tau$$

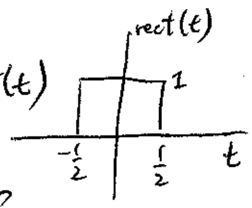
$$h(t) = g(-t)$$

$$= \int f(\tau)\, h(\tau - t)\, d\tau$$



Ex:

$f(t) = \text{rect}(t) = \Pi(t)$



$g(t) = f(t) * f(t) = ?$

$$g(t) = \int \Pi(\tau)\, \Pi(t-\tau)\, d\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Pi(t-\tau)\, d\tau = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \Pi(-\tau)\, d\tau$$

if $t + \frac{1}{2} < -\frac{1}{2} \Rightarrow t < -1$

$$\boxed{g(t) = 0}$$

if $-1 \leq t \leq 0$

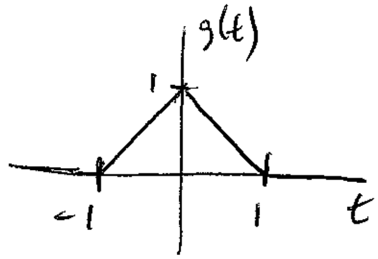
$$\int_{-\frac{1}{2}}^{t+\frac{1}{2}} d\tau = t + \frac{1}{2} + \frac{1}{2} = 1 + t$$

if $\quad 0 \le t \le 1$

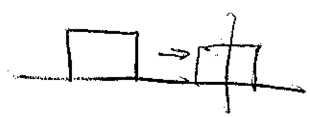
$$g = \int_{t-\frac{1}{2}}^{\frac{1}{2}} d\tau = \frac{1}{2} - t + \frac{1}{2} = 1 - t$$

if $\quad t \geq 1 \quad g(t) = 0$

$$g(t) = \begin{cases} 0, & \text{if } t \le -1,\ t > 1 \\ 1+t, & \text{if } -1 \le t \le 0 \\ 1-t, & \text{if } 0 \le t \le 1 \end{cases}$$



Also know how to do the geometric way ...



Convolution with a delta function:

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x - x_0) = f(x_0)$$

$$g(t) = f(t) * \delta(t)$$

$$= \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\,\delta(\tau - t)\,d\tau$$

$$= f(t)$$

Fourier Transform

FT: $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

IFT: $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, dt$$

Euler's formula

$$e^{jx} = \underbrace{\cos x + j \sin x}_{\text{projecting onto this basis}}$$

$$\overbrace{\int_{-\infty}^{\infty} x(t)\, \underbrace{\sin(\omega t)}_{\text{odd}}\, dt}^{\text{sine transform}}$$

Ex: Fourier transform of a rect

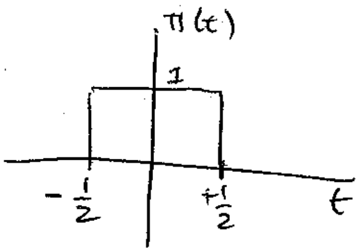
$$\mathcal{F}\{\Pi(t)\} = ?$$

$$= \int_{-\infty}^{\infty} \Pi(t)\, e^{-j\omega t}\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t}\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\cos(\omega t) - \cancel{j\sin(\omega t)}\right) dt$$

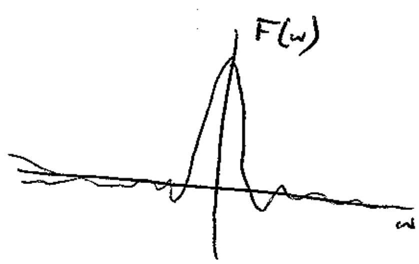
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(\omega t)\, dt = \left.\frac{\sin \omega t}{\omega}\right|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\sin\frac{\omega}{2}}{\omega} + \frac{\sin\frac{\omega}{2}}{\omega} = \frac{2\sin\frac{\omega}{2}}{\omega}$$

$$= \frac{\sin\alpha}{\alpha} = \operatorname{sinc}(\alpha)$$

$$= \operatorname{sinc}\left(\frac{\omega}{2}\right)$$



$\xrightarrow{\mathcal{F}}$



Ex: Fourier transform of impulse train $Ш(t)$

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - Tn) \xrightarrow{\mathcal{F}} F(\omega) = \int_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \delta(t - Tn)\, e^{-j\omega t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} e^{-j\omega Tn}$$

$$f(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j2\pi kt/T}$$

$$= \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{n}{T}\right)$$