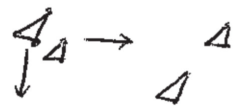
Transformations

We need a way to compute the position of the vertices after we have applied transformation to them.

Transformations: 1) Rotation: 

2) Scaling: 

3) Translation: 

etc

EX: Suppose you have a car modelled by vertices and polygons:

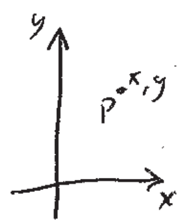
Now you want to do things like: · translate it (move it forward)

· rotate it (make a turn)

etc.

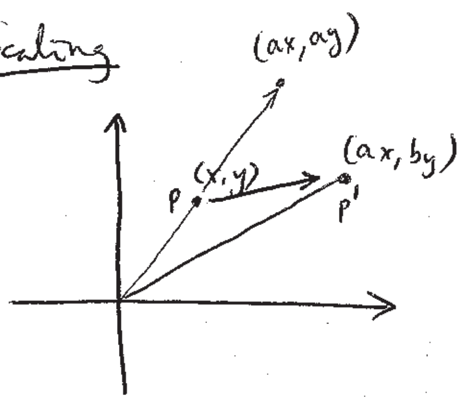
We are going to start with 2D Transformations
(ie. transformations on the plane)

- Vertices will be represented by vectors: e.g.



$$P = \begin{bmatrix} x \\ y \end{bmatrix}$$

Scaling



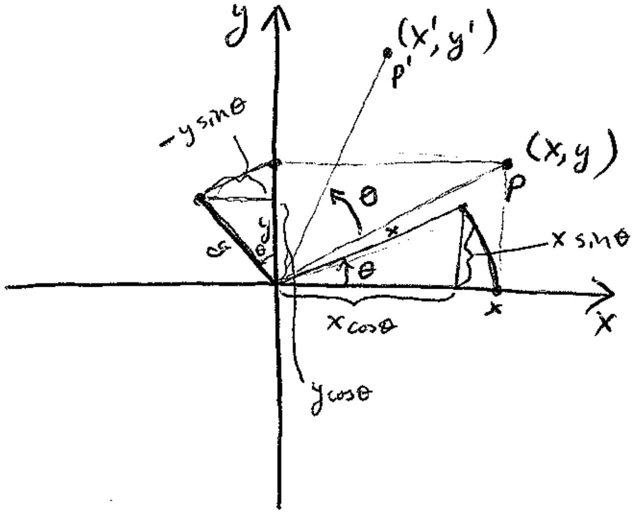
$$\underset{\text{2×1 vector}}{P'} = \underset{\substack{\text{2×2} \\ \text{matrix}}}{M}\, \underset{\text{2×1 vector}}{P}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{aligned} x' &= ax \\ y' &= by \end{aligned}$$

Rotation

(Counterclockwise)
Rotate about z axis

$$P = x\vec{i} + y\vec{j}$$



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

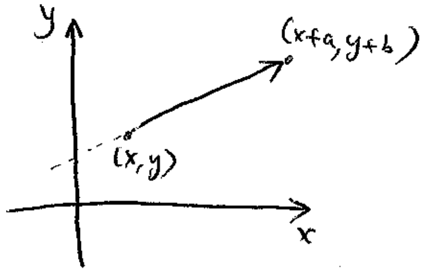
in Matrix notation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R(\theta)} \begin{bmatrix} x \\ y \end{bmatrix}$$

Important! Know how to derive this

Translation

A simple way to do it is with addition:



$$P' = P + \underset{\text{shift}}{\overset{\text{translation}}{}}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$$

Not the cleanest way to describe translation. Before we used matrix multiplication, here we use addition. We'll see a better way in a moment.

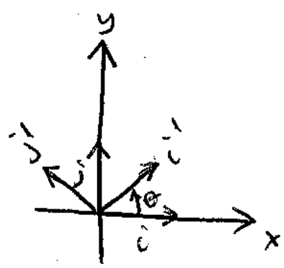
Column space

Apply vectors $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to our $R(\theta)$ matrix:

We're essentially grabbing the columns of $R(\theta)$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



The columns of the $R(\theta)$ are the axes of the new rotated coordinate system.

We can write these vectors as an identity matrix:

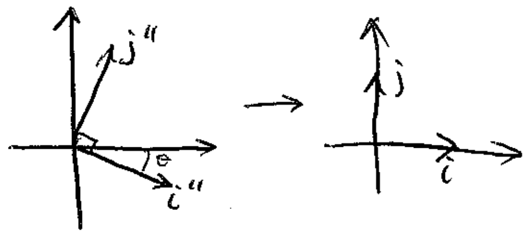
$$I = \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

$$RI = R$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The column space of $R(\theta)$ is the space formed by i' and j'

Row space:



Use the columns of $R(\theta)$

$$R(\theta) = \begin{array}{r c} & i''^T \\ & \begin{bmatrix} [\cos\theta & -\sin\theta] \\ [\sin\theta & \cos\theta] \end{bmatrix} \\ j''^T & \end{array}$$

Rotate these by $R(\theta)$:

$$R(\theta)\, i'' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta \\ \cancel{\sin\theta\cos\theta} - \cancel{\sin\theta\cos\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i$$

$$R(\theta)\, j'' = \begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos\theta \end{bmatrix} \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)\sin\theta - \cos\theta\sin\theta \\ \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = j$$

This means that:

$$R R^T = I$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{array}{c c} i'' & j'' \\ \multicolumn{2}{c}{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}} \end{array} = \begin{array}{c c} i & j \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \end{array} = I$$

This works because R has some interesting properties:

- R is an orthogonal matrix : $RR^T = I = R^T R$

Orthonormal {

- Columns of R are orthogonal : Ex. $-\cos\theta\sin\theta + \cos\theta\sin\theta = 0$
  (and rows) (dot product is zero)
- Columns are normalized Ex: $\sqrt{\cos^2\theta + \sin^2\theta} = 1$
  (and rows) (unit length)

Translation represented by addition is not very good:

1. Does not allow for easy composition of transformations (some add, some multiply)
2. ↳ Cannot work a single matrix to take all the transformations into account.

Answer: Homogeneous coordinates

$$\text{2D Cartesian coords} \longrightarrow \text{3D homogeneous coordinates}$$

$$\underbrace{(x, y)}_{\text{Euclidean space}} \longrightarrow \underbrace{(x, y, \overset{\text{weight}}{w})}_{\text{Projective space}}$$

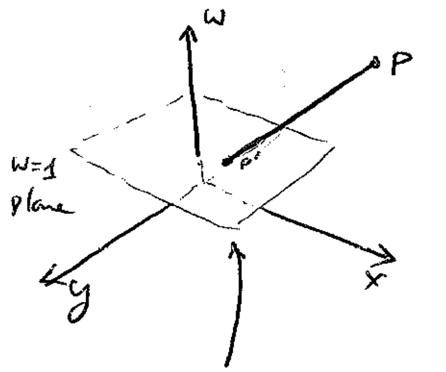
Unlike Cartesian coordinates, in homogeneous coordinates every coordinate component can be multiplied by the same value and it's still the same point:

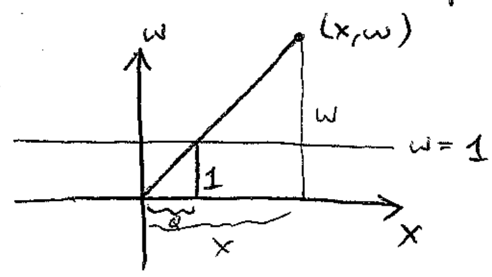
$$(x, y, w) \equiv (ax, ay, aw) \equiv \left(\frac{x}{w}, \frac{y}{w}, 1\right)$$

The process of dividing by w is called homogenization.

By dividing by w:

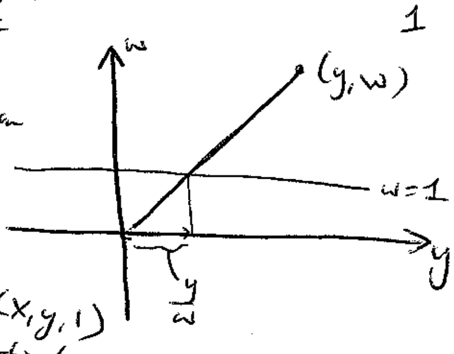
$$(x, y, w) \to \left(\frac{x}{w}, \frac{y}{w}, 1\right)$$ we effectively project the point to the $w=1$ plane.





$$\frac{w}{1} = \frac{x}{q} \Rightarrow \boxed{q = \frac{x}{w}}$$

You can think of this w=1 as being the Euclidean space $\mathbb{R}^2$. So that's why point (x,y) in Cartesian coordinates would become point $(x,y,1)$ in homogeneous coordinates



Points with $w=0$ specify a point at ∞

$$\underbrace{(x, y}_{\infty \text{ number of points at infinity.}}, 0) \leftarrow \text{cannot be homogenized}$$

Back to our transformations:

Rotation:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ 1 \end{bmatrix} \leftarrow \text{correct rotation}$$

Scale:

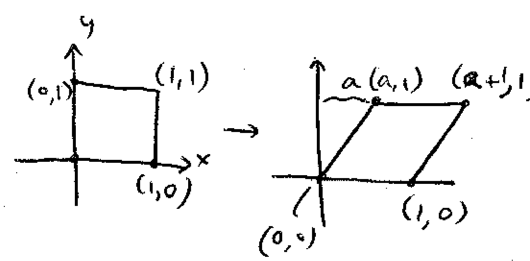
$$S(a,b) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ 1 \end{bmatrix} \leftarrow \text{correct scale}$$

Translation:

$$T(a,b) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix} \leftarrow \text{correct translation}$$

Shears

$$x\text{-shear} \rightarrow Sh_x(a) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+ay \\ y \\ 1 \end{bmatrix}$$



$$y\text{-shear } Sh_y(b) = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y+bx \\ 1 \end{bmatrix}$$

Composition of 2D Transformations

$$T_1 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \text{ followed by } T_2 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \qquad T_2 \circ T_1$$

Matrix Mult is associative!

$$V' = T_2(T_1 V) = (T_2 T_1) V$$

remember $T_1 T_2 \neq T_2 T_1$ for most matrices!

$$T_2 T_1 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix}$$