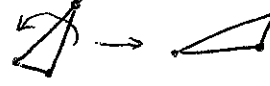
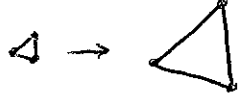
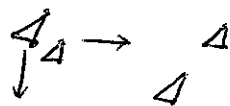


Transformations

9/25/07 ①

We need a way to compute the position of the vertices after we have applied transformations to them.

- Transformations:
- 1) Rotation: 
 - 2) Scaling: 
 - 3) Translation: 
- etc.

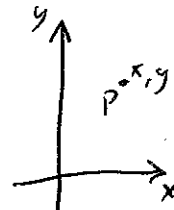
EX: Suppose you have a car modelled by vertices and polygons:



Now you want to do things like: translate it (move it forward)
• rotate it (make a turn)
etc.

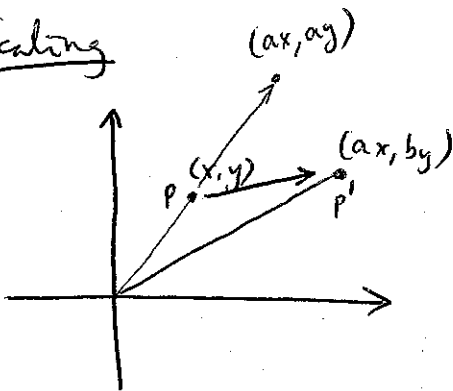
We are going to start with 2D Transformations
(ie. transformations on the plane)

- Vertices will be represented by vectors: e.g.



$$P = \begin{bmatrix} x \\ y \end{bmatrix}$$

Scaling



2x1 vector 2x2 matrix
↓ ↙ ↘
 $P' = M P$

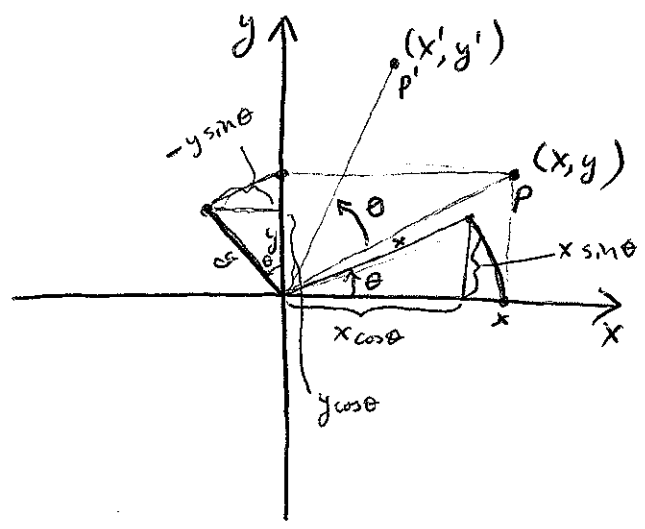
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{aligned} x' &= ax \\ y' &= by \end{aligned}$$

Rotation

(Counterclockwise)

Rotate about z axis

$$P = x\vec{i} + y\vec{j}$$



$$x' = x \cos \theta - y \sin \theta$$

$$y' = x \sin \theta + y \cos \theta$$

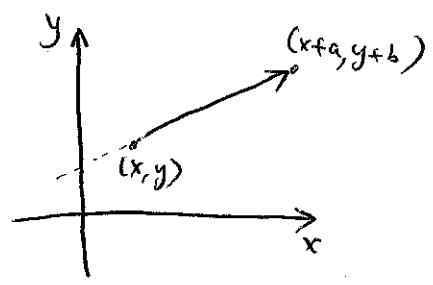
in Matrix notation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{R(\theta)} \begin{bmatrix} x \\ y \end{bmatrix}$$

Important! Know how to derive this

Translation

A simple way to do it is with addition:



$$P' = P + \begin{matrix} \text{translation} \\ \text{shift} \end{matrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$$

Not the clearest way to describe translation. Before we used matrix multiplication, here we use addition. We'll see a better way in a moment.

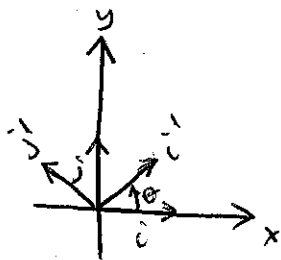
Column space

Apply vectors $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to our $R(\theta)$ matrix:

We're essentially grabbing the columns of $R(\theta)$

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$



The columns of the $R(\theta)$ are the axes of the new rotated coordinate system.

We can write these vectors as an identity matrix:

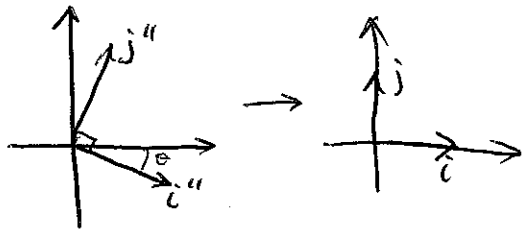
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$RI = R$$

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

The column space of $R(\theta)$ is the space formed by i' and j'

Row space:



Use the columns of $R(\theta)$

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

i''^T (pointing to the top row)

j''^T (pointing to the bottom row)

Rotate these by $R(\theta)$:

$$R(\theta) i'' = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta \\ -\sin \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta \\ \sin \theta \cos \theta - \sin \theta \cos \theta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i$$

$$R(\theta) j'' = \begin{bmatrix} \cos(\theta) & -\sin \theta \\ \sin(\theta) & \cos \theta \end{bmatrix} \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) \sin \theta - \cos \theta \sin \theta \\ \sin^2 \theta + \cos^2 \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = j$$

This means that:

$$R R^T = I$$

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

i'' j'' i j

This works because R has some interesting properties:

• R is an orthogonal matrix: $R R^T = I = R^T R$

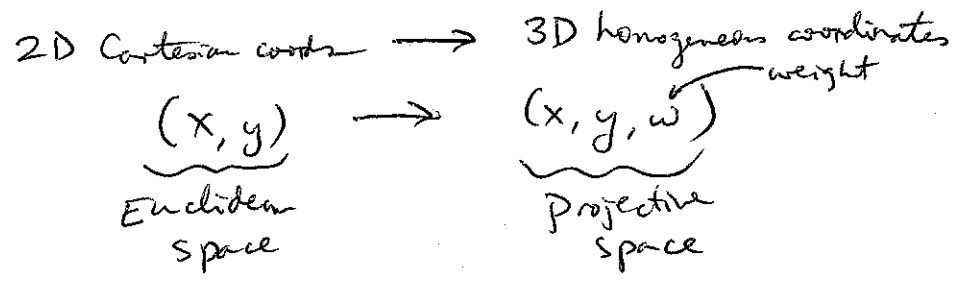
Orthonormal

- Columns of R are orthogonal: (dot product is zero) Ex: $-\cos \theta \sin \theta + \cos \theta \sin \theta = 0$
- Columns are normalized (unit length) Ex: $\sqrt{\cos^2 \theta + \sin^2 \theta} = 1$

Translation represented by addition is not very good:

- 1. Does not allow for easy composition of transformations (some add, some multiply)
- 2. Cannot write a single matrix to take all the transformations into account.

~~Best~~ Answer: Homogeneous coordinates



unlike Cartesian coordinates, in homogeneous coordinates every coordinate component can be multiplied by the same value and it's still the same point:

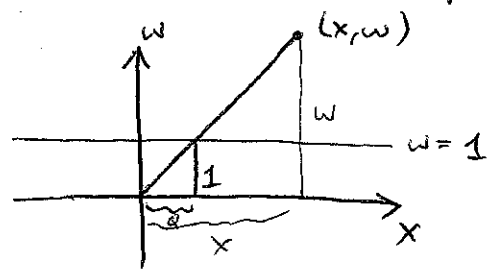
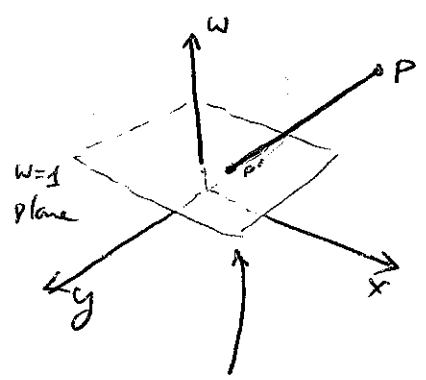
$$(x, y, w) \equiv (ax, ay, aw) \equiv \left(\frac{x}{w}, \frac{y}{w}, 1\right)$$

The process of dividing by w is called homogenization.

By dividing by w :

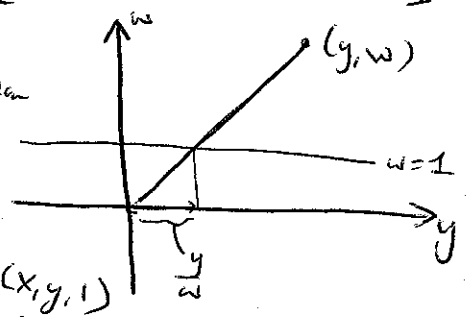
$$(x, y, w) \rightarrow \left(\frac{x}{w}, \frac{y}{w}, 1\right) \text{ we effectively project}$$

the point to the $w=1$ plane.



$$\frac{w}{1} = \frac{x}{q} \Rightarrow \boxed{q = \frac{x}{w}}$$

you can think of this $w=1$ as being the Euclidean space \mathbb{R}^2 . So that's why point (x, y) in Cartesian coordinates would become point $(x, y, 1)$ in homogeneous coordinates



Points with $w=0$ specify a point at ∞

$(x, y, 0)$ ← cannot be homogenized
∞ number of points at infinity.

Back to our transformations:

Rotation:

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \\ 1 \end{bmatrix} \leftarrow \text{correct rotation}$$

Scale:

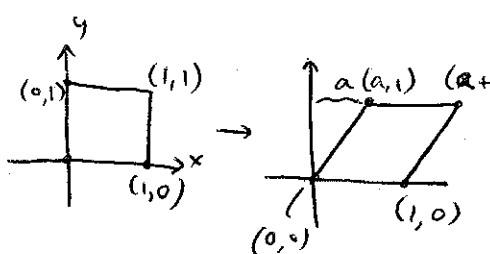
$$S(a, b) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ 1 \end{bmatrix} \leftarrow \text{correct scale}$$

Translation:

$$T(a, b) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix} \leftarrow \text{correct translation}$$

Shears

x-shear $\rightarrow Sh_x(a) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+ay \\ y \\ 1 \end{bmatrix}$



y-shear $Sh_y(b) = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y+bx \\ 1 \end{bmatrix}$

Composition of 2D Transformations

7

$$T_1 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \text{ followed by } T_2 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \quad T_2 \circ T_1$$

$$V' = T_2(T_1 V) = (T_2 T_1) V$$

Matrix Mult is associative!

remember $T_1 T_2 \neq T_2 T_1$
for most matrices!

$$T_2 T_1 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix}$$