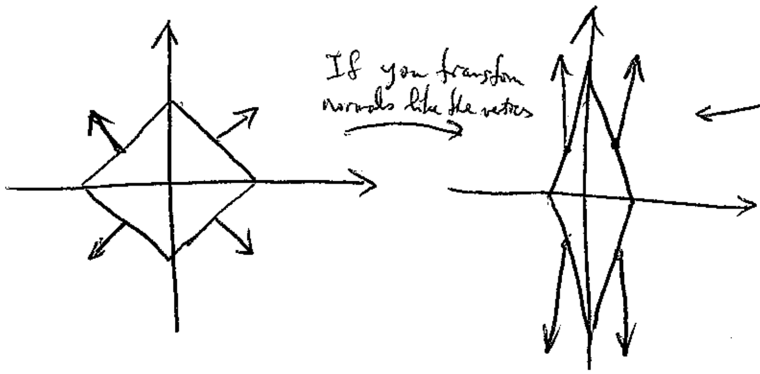
Transforming surface Normals



the normals are no longer $\perp$ to the surface! This is no good!

you want

$$\underbrace{n}_{\text{normal}}^T \overbrace{v}^{\text{some vector on surface}} = 0$$

$$\underbrace{(M_2 n)^T}_{\substack{\text{we want to apply some } M_2 \text{ to } n \text{ s.t. that they}\\ \text{are still } \perp!}} \overbrace{(M_1 v)}^{\text{applying } M_1 \text{ to } v} = 0$$

$$(M_2 n)^T (M_1 v) = 0$$

$$n^T \underbrace{M_2^T M_1}_{\text{we want this to be identity } I \text{ so that } n^T v = 0} v = 0$$

so $\quad M_2^T M_1 = I$

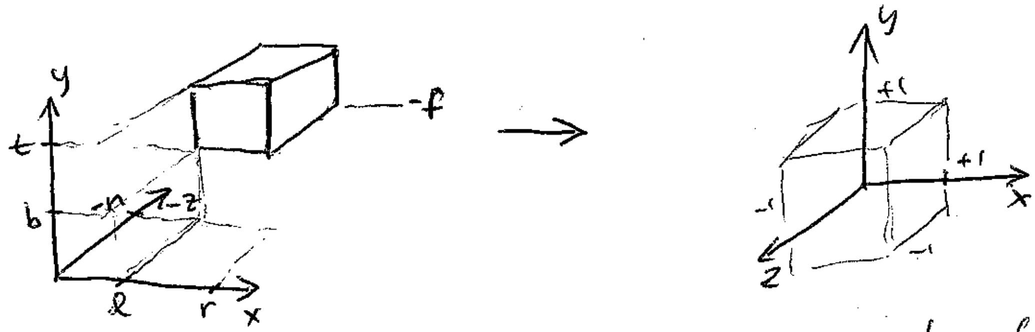
$$M_2^T = M_1^{-1}$$

$$M_2 = (M_1^{-1})^T$$

This is the inverse transpose of the modelview matrix

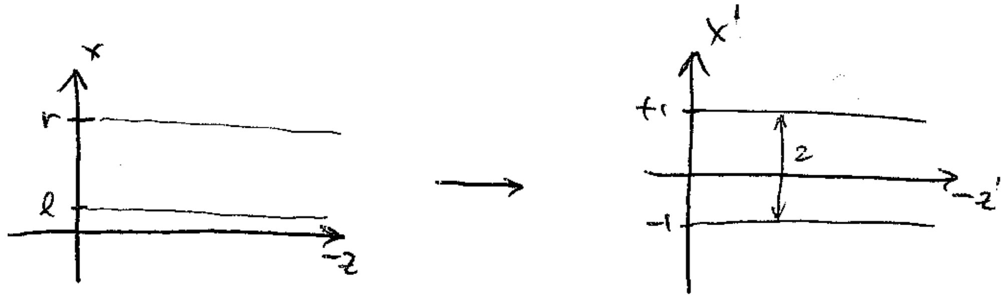
ORTHOGRAPHIC PROJECTION

Task: Take a rectangular volume in eye space defined by (l, r, t, b, n, f) and project it into a cube with sides ± 1.



We need to construct a Matrix M_{ortho} that does this transformation.

First transform the x-coordinates



$$l < x < r \rightarrow -1 < x' < 1$$

distance: $r - l$ $\qquad$ $x' = f(x)$

midpt: $\frac{r+l}{2}$

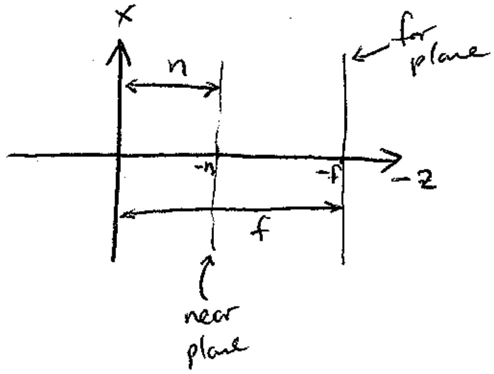
1. first shift x by midpt. This ensures that we are even about 0.
2. Now scale to make length 2

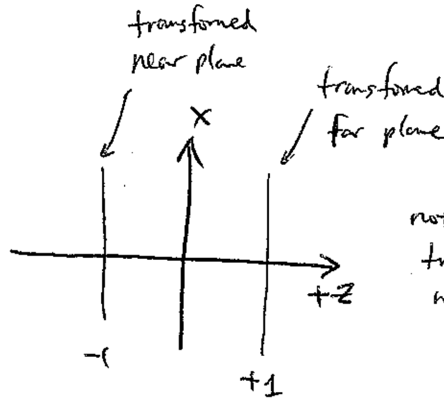
$$x' = \left(\frac{2}{r-l}\right)\left(x - \left(\frac{r+l}{2}\right)\right) = \boxed{\frac{2}{r-l}x - \frac{r+l}{r-l}}$$

Do the same for y:

$$\boxed{y' = \frac{2}{t-b}y - \frac{t+b}{t-b}}$$

Now compute z-coordinate:





note how now the transformed far plane > near plane. So things further away have bigger z.

distance: $f-n$

midpt: $\dfrac{-n-f}{2}$

this "$-$" is to flip the planes so that the far plane is bigger z.

$$z' = \left(\frac{-2}{f-n}\right)\left(z + \frac{n+f}{2}\right)$$

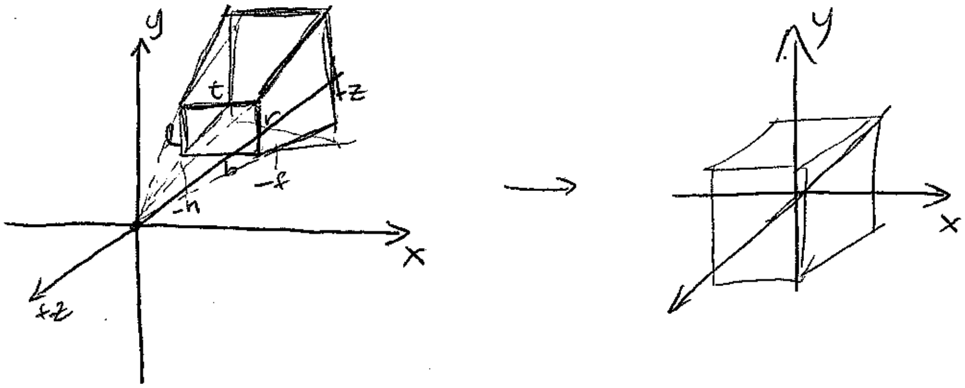
$$\boxed{z' = -\frac{2}{f-n}\,z - \frac{f+n}{f-n}}$$

Now we can write the orthogonal projection matrix

$$M_{ortho} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & -\frac{2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

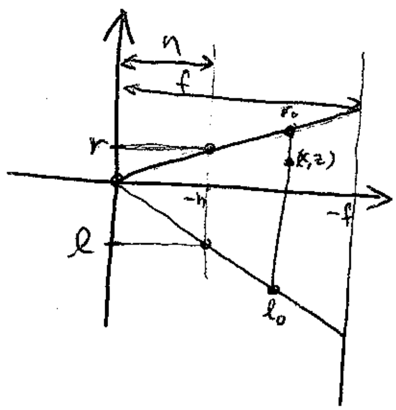
Same as OpenGL spec

PERSPECTIVE PROJECTION



Task: Take a frustum defined by (l, r, t, b, n, f) and project it into a cube with sides ± 1

Again, first do this for the x-coordinate:



we want to transform

$$l_0 \le x \le x_0 \rightarrow -1 \le x' \le 1$$

what are r_0, l_0 in terms of what we know:

$$\frac{r_0}{-z} = \frac{r}{n} \Rightarrow r_0 = -\frac{rz}{n}$$

$$\frac{l_0}{-z} = \frac{l}{n} \Rightarrow l_0 = -\frac{lz}{n}$$

distance between r_0 and l_0 : $-\frac{rz}{n} + \frac{lz}{n} = \frac{(l-r)z}{n}$

midpt between r_0 and l_0 = $\frac{-rz-lz}{2n}$

Now write x':

$$x' = \frac{2n}{(l-r)z}\left(x - \frac{-rz-lz}{2n}\right) = \frac{2n}{(l-r)z}\left(x + \frac{(r+l)z}{2n}\right)$$

$$\boxed{x' = \frac{2nx}{(l-r)z} + \frac{r+l}{l-r}}$$

Hmm... there is $\frac{1}{z}$ here. How do we write this in matrix form? We'll deal with this later!

We can do same for y':

$$\boxed{y' = \frac{2ny}{(b-t)z} + \frac{t+b}{b-t}}$$

To get rid of that weird z in the denominator, let's multiply both sides by $-z$ (later we'll have to remember to divide)

$$\boxed{-zx' = \frac{2n}{r-l}x + \frac{r+l}{r-l}z}$$

$$\boxed{-zy' = \frac{2n}{t-b}y + \frac{t+b}{t-b}z}$$

we can divide out the $-z$ by putting $-z$ in the homogeneous coordinate w!

- Now for the z-component:

$$z' = f(z) = \frac{Az+B}{-z}$$

Now plug in what we know:

$$f(-n) = -1 \Rightarrow \frac{-An+B}{n} = -1 \Rightarrow -An+B = -n$$

$$f(-f) = 1 \Rightarrow \frac{-Af+B}{f} = 1 \Rightarrow -Af+B = f$$

OpenGL perspective projection matrix

$$-An + Af = -n - f$$

$$A(f-n) = -(n+f)$$

$$\boxed{A = -\frac{f+n}{f-n}}$$

plug in to get B

$$\boxed{M_{pers} = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}}$$

to get a $-z$ in the w component!

$$B \Rightarrow \frac{f+n}{f-n} \cdot n + B = -n$$

$$B = -\frac{f+n}{f-n} \cdot n - \frac{nf-n^2}{f-n}$$

$$= \frac{-fn-\cancel{n^2}-nf+\cancel{n^2}}{f-n}$$

$$\boxed{B = -\frac{2fn}{f-n}}$$

VIEWPORT TRANSFORMATION

Map $-1 < x' < 1$ to $0 < i <$ screen width

$$i = \left(\frac{x'}{2} + \frac{1}{2}\right) \cdot \text{screen width}$$

$$i = \frac{w}{2} x + \frac{w}{2}$$

$$j = \frac{h}{2} y + \frac{h}{2}$$

$$M_{\text{Viewport}} = \begin{bmatrix} \frac{w}{2} & 0 & 0 & \frac{w}{2} \\ 0 & \frac{h}{2} & 0 & \frac{h}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$