Fitting Lines

Least Squares

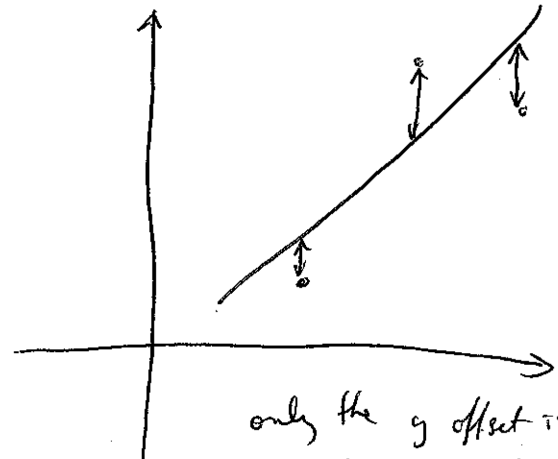
represent line as:

$$y = ax + b$$

Find a, b such that

$$\sum_i (y_i - ax_i - b)^2$$

is reduced



only the y offset is considered in calculating the error

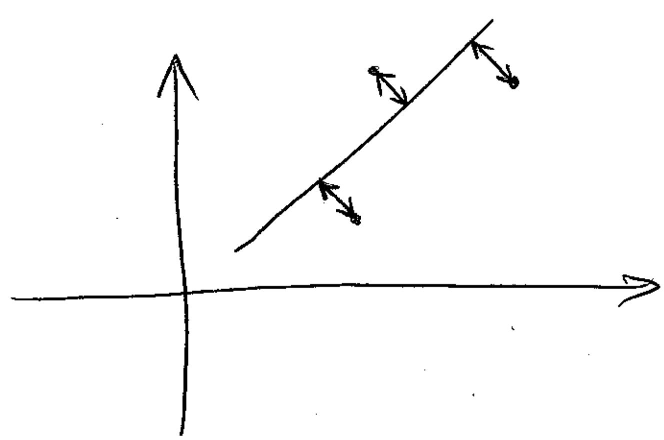
Total least squares

line: $ax + by + c = 0$

(a, b, c)

distance from point (u, v) to line $ax + by + c = 0$
is $\text{abs}(au + bv + c)$ if $a^2 + b^2 = 1$

so minimize: $\sum_i (ax_i + by_i + c)^2$ with constraint $a^2 + b^2 = 1$



Implicit Curves

$\phi(x, y) = 0$

Line: $ax + by + c = 0$

circle: $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$ ← centre (a, b) radius r

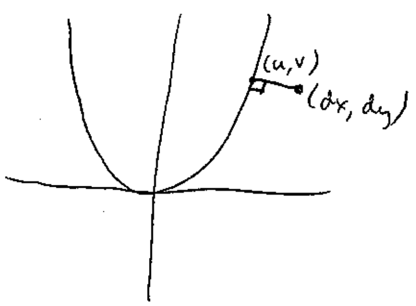
ellipses: $ax^2 + bxy + cy^2 + dx + ey + f = 0$ where $b^2 - 4ac < 0$

Hyperbolae: $ax^2 + bxy + cy^2 + dx + ey + f = 0$ where $b^2 - 4ac > 0$

Parabolae: $ax^2 + bxy + cy^2 + dx\ ey + f = 0$ where $b^2 - 4ac = 0$

General conic: $ax^2 + bxy + cy^2 + dx + ey + f = 0$

Testing distance from point to implicit curve:



1. (u,v) is on the curve

   e.g. $\phi(u,v) = 0$

2. $s = (d_x - d_y) - (u, v)$ is $\perp$ to the curve

Normal of curve $N = \left(\dfrac{\partial \phi}{\partial x}, \dfrac{\partial \phi}{\partial y}\right)$

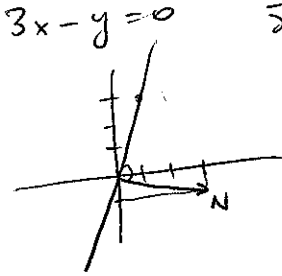
so tangent to curve $T = \left(\dfrac{\partial \phi}{\partial y}, -\dfrac{\partial \phi}{\partial x}\right)$

see example on line:

$3x - y = 0$

$\dfrac{\partial \phi}{\partial x} = 3$

$\dfrac{\partial \phi}{\partial y} = -1$



$T \cdot s = 0$

so

$$\frac{\partial \phi}{\partial y}(u,v)(d_x - u) - \frac{\partial \phi}{\partial x}(u,v)(d_y - v) = 0$$

This, with original equation gives us two equations and two unknowns.

Ex: Conic section:

$$ax^2+bxy+cy^2+dx+ey+f=0$$

plug in (u,v)

$$au^2+buv+cv^2+du+ev+f=0$$

$$\frac{\partial\phi}{\partial x}(u,v) = 2ax+by+d\Big|_{(u,v)} = 2au+bv+d$$

$$\frac{\partial\phi}{\partial y}(u,v) = b\cancel{x}+2cy+e\Big|_{(u,v)} = b\cancel{u}+2cv+d$$

putting them together (for pt (d_x, d_y))

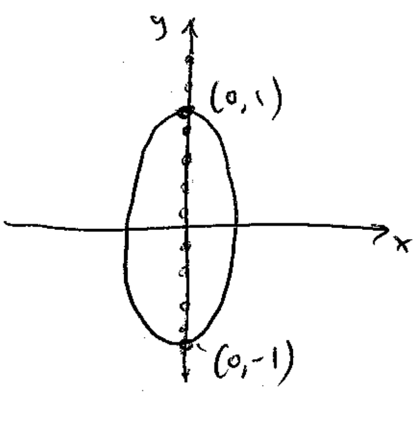
$$2(a-c)uv-(2ad_y+e)u+(2cd_x+d)v+(ed_x-dd_y)=0$$

Let's look at a specific example:

ellipse: $2x^2+y^2-1=0$

1. $2u^2+v^2-1=0$
2. $2uv-4d_y u+2d_x v=0$

Suppose our points are along the line $(d_x, d_y)=(0,\lambda)$



plug in to second equation:

$$2uv-4\lambda u=0$$

$$2u(v-2\lambda)=0$$

either $u=0 \Rightarrow v=\pm 1$ so $(0,1)$ $(0,-1)$ are solutions

or $v=2\lambda$ plug in to 1

$2u^2+4\lambda^2-1=0$ ← has solutions only if $-\frac{1}{2}\le\lambda\le\frac{1}{2}$

Parametric curves

$$f = (x(t), y(t)) \qquad t \in [t_{min}, t_{max}]$$

Tagent at τ: $\left(\frac{dx}{dt}(\tau), \frac{dy}{dt}(\tau)\right)$

so nearest point to (d_x, d_y):

$$\frac{dx}{dt}(\tau)\left(d_x - x(\tau)\right) + \frac{dy}{dt}(\tau)\left(d_y - y(\tau)\right) = 0$$

only one equation

Fitting as a probabilistic inference

$$\underbrace{P(\text{measurements} \mid a, b, c)}_{\text{probability of measurements given } a, b, c} = \prod_i P(\overbrace{x_i, y_i}^{\text{datapts measured}} \mid \overbrace{a, b, c}^{\text{line}})$$

log-likelihood: $-\frac{1}{2\sigma^2} \sum_i (ax_i + by_i + c)^2$ maximize this!

same as minimizing the total least squares!