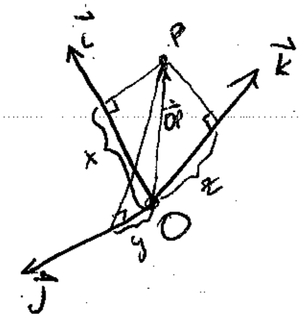
Coordinate system

$F = (O, \vec{i}, \vec{j}, \vec{k})$

orthonormal coordinate frame

origin

basis vectors
- orthogonal
- normalized



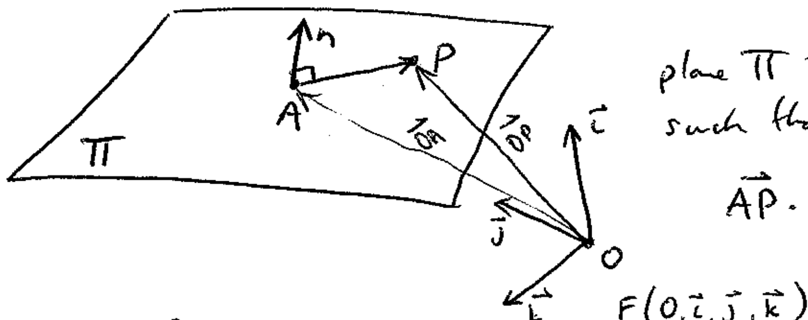
right hand coordinate system

$\vec{i} \times \vec{j} = \vec{k}$

To compute the ${}^F(x, y, z)$ coords of a point in F: (coordinates)

$$\left.\begin{aligned} x &= \vec{OP} \cdot \vec{i} \\ y &= \vec{OP} \cdot \vec{j} \\ z &= \vec{OP} \cdot \vec{k} \end{aligned}\right\} \text{equal to} \quad \vec{OP} = x\vec{i} + y\vec{j} + z\vec{k}$$

Example:



plane Π is defined by all pts P such that:

$$\vec{AP} \cdot \vec{n} = 0$$

${}^F P = (x, y, z)$ & ${}^F n = (a, b, c)^T$

$$\vec{AP} = \vec{OP} - \vec{OA}$$

$$\vec{AP} \cdot n = (\vec{OP} - \vec{OA}) \cdot n = 0$$

$$\Rightarrow \vec{OP} \cdot n - \underbrace{\vec{OA} \cdot n}_{d} = 0$$

$$ax + by + cz - d = 0$$

Homogeneous coordinates

$$(a, b, c, -d)\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

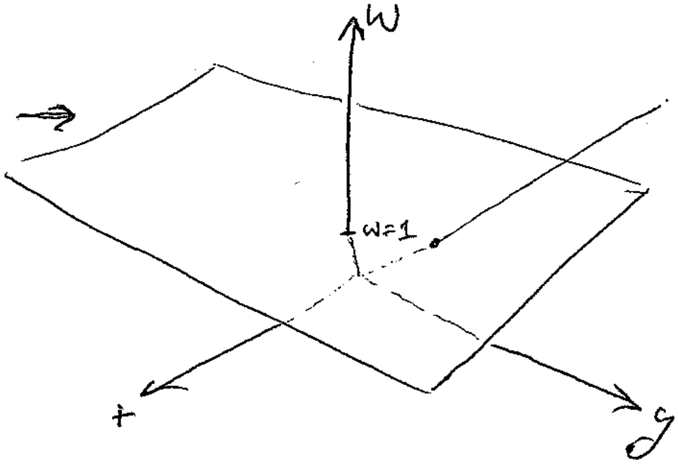
Homogeneous coords are a projective space!

Euclidean space		projective space
2D cartesian coordinates	$\longleftrightarrow$	3D homogeneous coords
(x, y)	$\longleftrightarrow$	(x, y, w)

w: homogeneous coordinate

(x, y, w)

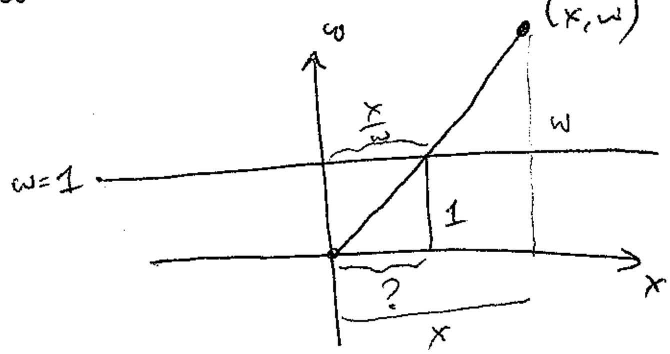
$w=1$ plane



the $w=1$ plane is the Euclidean $\mathbb{R}^2$ space

$$\therefore\ (ax, ay, aw) \equiv (x, y, w) \equiv \underbrace{\left(\frac{x}{w}, \frac{y}{w}, 1\right)}_{\text{homogenization (divide by } w)}$$

look from the side:



so $(x, w) \equiv \left(\frac{x}{w}, 1\right)$

$$\frac{w}{x} = \frac{1}{?}$$

$$? = \frac{x}{w}$$

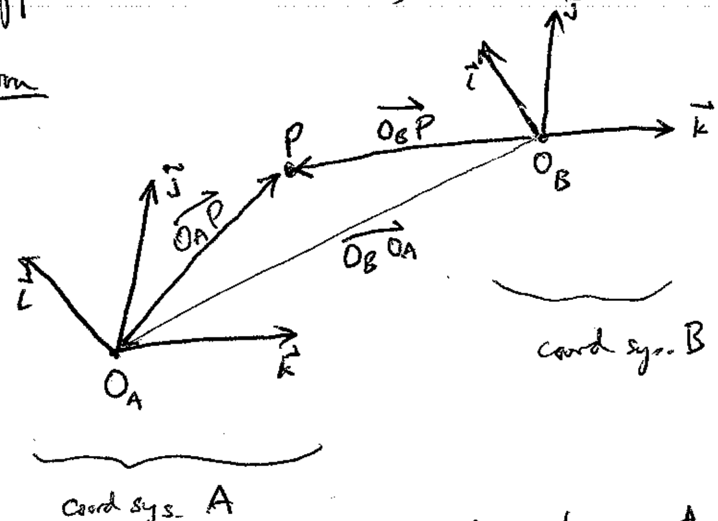
Points with a homogeneous coordinate of 0 cannot be homogenized!

Points at infinity!

Transforming coordinate systems

Suppose the two are only translated from one another

Translation



so $\vec{i}_A = \vec{i}_B$

$\vec{j}_A = \vec{j}_B$

$\vec{k}_A = \vec{k}_B$

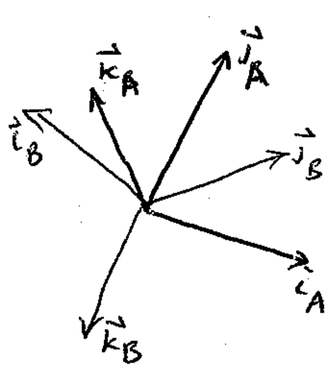
but $O_A \neq O_B$

if we know ${}^{A}P$, what's ${}^{B}P$?

$$\underbrace{\overrightarrow{O_B P}}_{} = \underbrace{\overrightarrow{O_B O_A}}_{} + \underbrace{\overrightarrow{O_A P}}_{}$$

$${}^{B}P = \underbrace{{}^{B}O_A}_{\text{translation between the reference frame}} + {}^{A}P$$

Rotation say $O_A = O_B$



the columns tell us how each basis of A maps into the 3 basis of B

the rows tell us how to project a vector in A ${}^{A}v$ into each of the components in B.

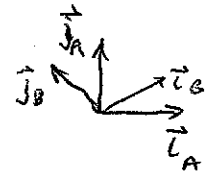
$$\underbrace{{}^{B}_{A}R}_{\text{rotation between reference frame of A to B}} = \begin{bmatrix} \vec{i}_A \cdot \vec{i}_B & \vec{j}_A \cdot \vec{i}_B & \vec{k}_A \cdot \vec{i}_B \\ \vec{i}_A \cdot \vec{j}_B & \vec{j}_A \cdot \vec{j}_B & \vec{k}_A \cdot \vec{j}_B \\ \vec{i}_A \cdot \vec{k}_B & \vec{j}_A \cdot \vec{k}_B & \vec{k}_A \cdot \vec{k}_B \end{bmatrix}$$

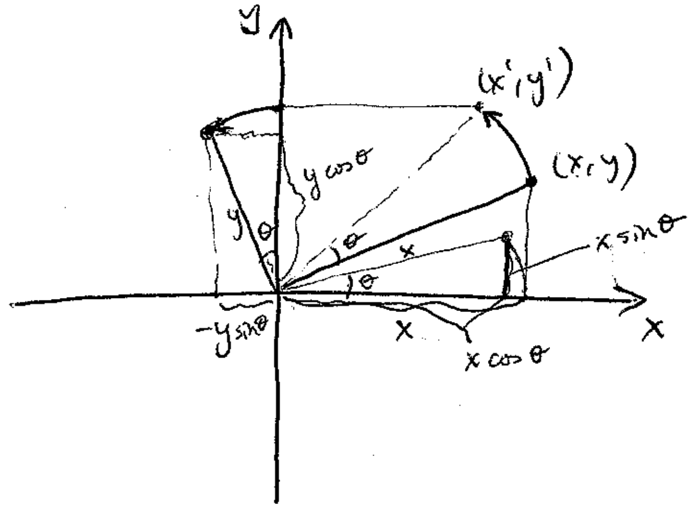
Note: ${}^{A}_{B}R = {}^{B}_{A}R^{T} = {}^{A}_{B}R^{-1}$

orthogonal matrix!

Derivation of Rotation

Assume rotation about z axis: $\vec{k}_A = \vec{k}_B = \vec{k}$





$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

in Matrix notation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R(\theta)} \begin{bmatrix} x \\ y \end{bmatrix}$$

When the origin and basis vectors are different, we now have a translation and a rotation.

$$^{B}P = \underbrace{{}^{B}_{A}R}_{\substack{\text{rot from} \\ A \text{ to } B}} \underbrace{{}^{A}P}_{\substack{\text{point in A} \\ \text{coord sys}}} + \underbrace{{}^{B}O_A}_{\text{translation}}$$

Take advantage of the following property of matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

the coordinates of P in ref B

transformation from A to B

coordinates of P in A

rigid body transformation

$$\begin{pmatrix} {}^{B}P \\ 1 \end{pmatrix} = {}^{B}_{A}T \begin{pmatrix} {}^{A}P \\ 1 \end{pmatrix} \quad \text{where} \quad {}^{B}_{A}T = \begin{pmatrix} {}^{B}_{A}R & {}^{B}O_A \\ \emptyset^T & 1 \end{pmatrix}$$

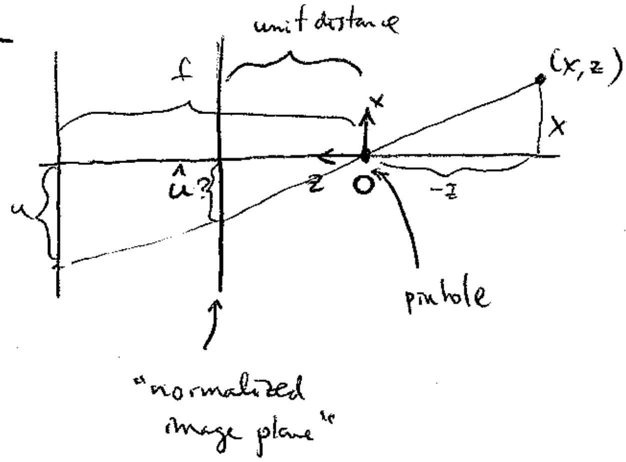
if ${}^{B}_{A}R$ is set to a general 3x3 matrix A:

$$T = \begin{pmatrix} A & t \\ 0^T & 1 \end{pmatrix} \leftarrow \text{affine transformation}$$

if T becomes an arbitrary 4x4 matrix (non-singular) it is a projective transform. Talk about those later.

Camera Parameters

Intrinsic



$$u = \frac{fx}{z}$$

$$v = \frac{fy}{z}$$

$$-\frac{\hat{u}}{1} = \frac{x}{-z} \Rightarrow \hat{u} = \frac{x}{z}$$

$$\hat{v} = \frac{y}{z}$$

However the real sensor is not this ideal!

pixel dimensions $\frac{1}{k} \times \frac{1}{l}$

$$u = kf\frac{x}{z} = \alpha \frac{x}{z}$$

$$v = lf\frac{y}{z} = \beta \frac{y}{z}$$

shift origin of sensor to lower corner:

$$u = \alpha \frac{x}{z} + u_0$$

$$v = \beta \frac{y}{z} + v_0$$