Circular Points

Back to conics:

$$a x_1^2 + b x_1 x_2 + c x_2^2 + d x_1 x_3 + e x_2 x_3 + f x_3^2 = 0$$

for a circle $a = c$, $b = 0$:

$$x_1^2 + x_2^2 + d x_1 x_3 + e x_2 x_3 + f x_3^2 = 0$$

Where does it intersect the line at ∞? i.e. $x_3 = 0$

$$x_1^2 + x_2^2 = 0$$

$$x_1 = 1 \qquad x_2 = \pm i$$

$$\underbrace{I = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad J = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}}_{\text{"circular points"}}$$

fixed under similarity transformation:

$$\overbrace{\begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}}^{\text{similarity transformation}} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \underbrace{s e^{-i\theta}}_{\text{constant}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

<u>Thm.</u> The circular points I, J are fixed points under the projective transformation H iff H is a similarity.

Circular points are orthogonal directions of Euclidean geometry:

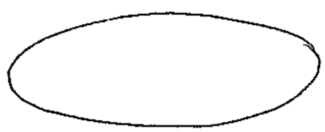
$$I = (1,0,0)^T + i(0,1,0)^T$$

$\therefore$ Once circular points are identified, we can recover orthogonality and other metric properties.

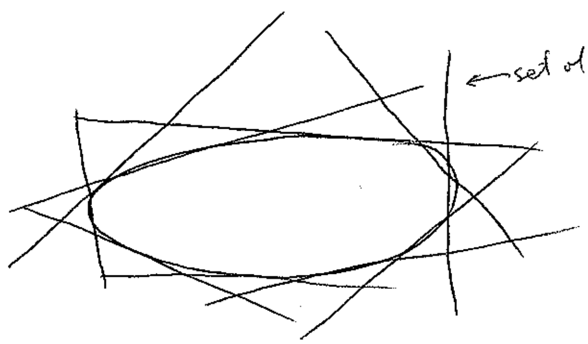
Recovery of metric properties from images

- Suppose the circular points are identified in an image
- Projective transform H can be found that maps imaged circular points to their canonical position $(1, \pm i, 0)^T$ on l_∞.
- The transformation between the world plane and the rectified image will then be a similarity, because it is a projection where the circular points are fixed.

Dual Conic



set of points satisfying $x^T C x = 0$



← set of lines satisfying $l^T C^* l = 0$

dual conic

It turns out that C^* is the adjoint of C (conjugate transpose)

if $x' = Hx$ then $C^* \to C^{*\prime}$

$C^{*\prime} = H C^* H^T$

Define a conic $C^*_\infty = IJ^T + JI^T = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} (1 \ -i \ 0) + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} (1 \ i \ 0)$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

if $x' = \underset{\text{similarity transform}}{H_s} x$

$C^{*\prime}_\infty = H_s C^*_\infty H_s^T = C^*_\infty$

So the dual conic C^*_∞ is fixed under projective transform iff H is a similarity.

In Euclidean geometry:

angle between two lines:

$$\ell = (\ell_1, \ell_2, \ell_3)^T \qquad \text{normals: } (\ell_1, \ell_2)^T$$

$$m = (m_1, m_2, m_3)^T \qquad (m_1, m_2)^T$$

angle:

$$\cos\theta = \frac{\ell_1 m_1 + \ell_2 m_2 \leftarrow \text{dot product}}{\sqrt{(\ell_1^2 + \ell_2^2)(m_1^2 + m_2^2)} \leftarrow \text{normalized}}$$

Equivalent expression invariant under projective transformations:

$$\cos\theta = \frac{\ell^T C_\infty^* m}{\sqrt{(\ell^T C_\infty^* \ell)(m^T C_\infty^* m)}}$$

C_∞^* ← conic dual to circular points

Thus, once the conic C_∞^* is identified on the projective plane, then the Euclidean angles may be measured

Thus, Lines ℓ and m are $\perp$ if $\ell^T C_\infty^* m = 0$

A projective transformation H can be decomposed:

$$H = \underbrace{H_S}_{\substack{\text{similarity}\\ \text{transform}}} \underbrace{H_A}_{\substack{\text{affine}\\ \text{transform}}} \underbrace{H_P}_{\substack{\text{projective}\\ \text{transform}\\ \text{(elation)}}} = \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} \underbrace{\begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix}}_{H_A} \underbrace{\begin{bmatrix} I & 0 \\ v^T & v \end{bmatrix}}_{H_P} = \begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$$

non-singular $A = sRK + tv^T$

K is upper-triangular matrix s.t. $|K| = 1$

$$H = H_P H_A H_S$$

back to conics:

$$C_\infty^{*\prime} = H C^* H^T$$

$$C_\infty^{*\prime} = H C_\infty^* H^T = (H_P H_A H_S)\, C_\infty^* \,(H_P H_A H_S)^T$$

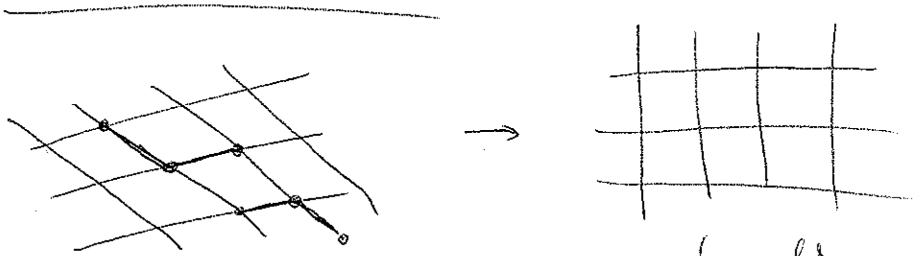
$$= (H_P H_A)\underbrace{(H_S C_\infty^* H_S^T)}(H_A^T H_P^T)$$

Remember C_∞^* is invariant under similarity transform

$$= (H_P H_A)\, C_\infty^* \,(H_A^T H_P^T)$$

$$= \begin{bmatrix} KK^T & KK^T v \\ v^T KK^T & v^T KK^T v \end{bmatrix}$$

Metric Rectification I



from affine image → to world

$$l'^T C_\infty^{*'} m' = 0 \text{, with } v = 0$$

$$C_\infty^{*'} = \begin{bmatrix} KK^T & 0 \\ 0^T & 0 \end{bmatrix}$$

symmetric matrix $S = KK^T$
2 degrees of freedom

$$(l_1' \; l_2' \; l_3') \begin{bmatrix} KK^T & 0 \\ 0^T & 0 \end{bmatrix} \begin{pmatrix} m_1' \\ m_2' \\ m_3' \end{pmatrix} = 0$$

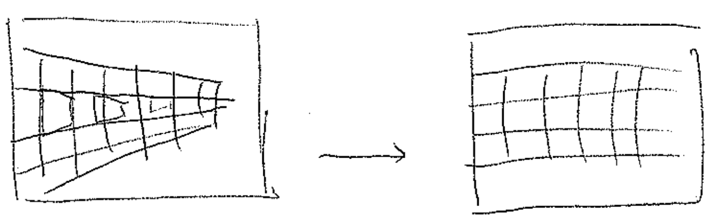
$$(l_1', l_2') \, S (m_1', m_2')^T = 0$$

$$\Downarrow$$

$$(l_1' m_1', \; l_1' m_2' + l_2' m_1', \; l_2' m_2') \, s = 0 \qquad s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

Metric Rectification II:

From



perspective projection → world

$\perp$ lines in the world: $l^T C_\infty^* m = 0$

$$(l_1 m_1, \; (l_1 m_2 + l_2 m_1)/2, \; l_2 m_2, \; (l_1 m_3 + l_3 m_1)/2, \; (l_2 m_3 + l_3 m_2)/2, \; l_3 m_3) c$$

$$C = \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \leftarrow \text{conic matrix of } C_\infty^*$$