2/13/08 (1)

Fourier Transforms

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

2D

$$\mathcal{F}\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, \underbrace{e^{-i2\pi(ux+vy)}}_{\text{sinusoidal basis (complex)}}\, dx\, dy$$

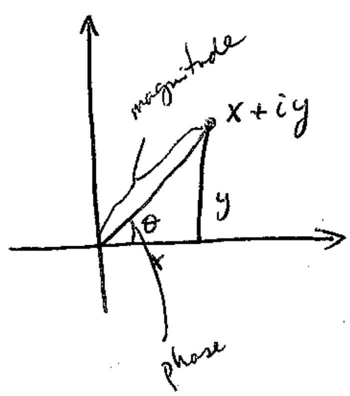
1D Fourier coefficients

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\cos(2\pi(ux+vy))\,dx\,dy \; + \quad \leftarrow \text{real part } \mathcal{F}_R(g)$$

$$i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\sin(2\pi(ux+vy))\,dx\,dy \quad \leftarrow \text{imaginary part } \mathcal{F}_i(g)$$

Magnitude & phase



magnitude: $\sqrt{\mathcal{F}_R^2 + \mathcal{F}_i^2}$

phase: $\operatorname{atan}\left(\frac{\mathcal{F}_i}{\mathcal{F}_R}\right)$

Properties

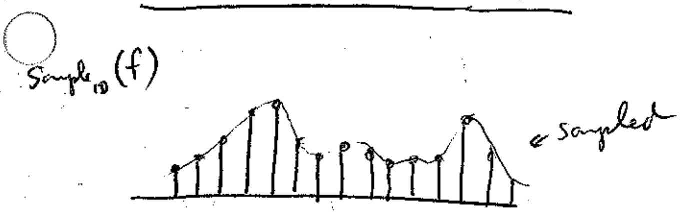
Linear: $\mathcal{F}\{g(x,y) + h(x,y)\} = \mathcal{F}\{g(x,y)\} + \mathcal{F}\{h(x,y)\}$

$$\mathcal{F}\{k\,g(x,y)\} = k\,\mathcal{F}\{g(x,y)\}$$

Convolution Theorem, see next page!

Sampling:

$$\text{sample}_{2D}(f) = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \underbrace{f(i,j)}_{\text{"sample" function value at }(i,j)}\, \underbrace{\delta(x-i,\, y-j)}_{\text{delta function at }(i,j)}$$



$$\text{Sample}_{2D}(f) = \underbrace{f(x,y)}_{\text{original signal}} \times \underbrace{\left\{\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\delta(x-i,\, y-j)\right\}}_{\text{array of spikes}}$$

Convolution Theorem

$$h(t) = \int f(\tau)\, g(t-\tau)\, d\tau$$

$$\mathcal{F}\{h(t)\} = F(u)\,G(u)$$

convolution $\overset{\mathcal{F}}{\Longleftrightarrow}$ multiplication

$$\mathcal{F}\{h(t)\} = \int \overbrace{\int f(\tau) g(t-\tau)\, d\tau}^{\text{convolution}}\; e^{-i2\pi u t}\, dt$$

$$= \int f(\tau) \int g(t-\tau)\, e^{-i2\pi u t}\, dt\;\, d\tau$$

$$x = t - \tau$$
$$dx = dt$$
$$t = x + \tau$$

$$= \int f(\tau) \underbrace{\int g(x)\, e^{-i2\pi u x}\, dx}\; e^{-i2\pi u \tau}\, d\tau$$

$$= G(u) \underbrace{\int f(\tau)\, e^{-i2\pi u \tau}\, d\tau}_{F(u)}$$

$$= F(u)\,G(u)$$

So using it in sampling:

$$\mathcal{F}(\text{Sample}_{2D}(f(x,y))) = \mathcal{F}\Big(f(x,y) \overset{\text{multiplication}}{\cdot} \Big\{ \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \delta(x-i, y-j) \Big\} \Big)$$

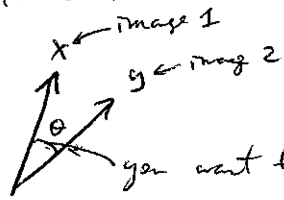
$$= \mathcal{F}(f(x,y)) \overset{\text{2D convolution}}{**} \mathcal{F}\Big\{ \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \delta(x-i, y-j) \Big\}$$

$$= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} F(u-i, v-j)$$

essentially, replicas of F( ) at every i, j

Correlation

Think of it as vectors



you want to find cos θ!

Observation: an n pixel grayscale image is simply a vector in n-dimensional space.

$$\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$$

$x \cdot y$ ← correlation is effectively computing the dot product.

$\|x\|\|y\|$ ← normalization

(see example (next page).

Noise

Gaussian distribution: $\dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$

Applying finite difference filters to noise:

$$\underbrace{\begin{matrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{matrix}}_{\text{1st derivative in the x direction}}$$

To compute the 2nd derivative, we apply the derivative to itself:

$$\begin{matrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{matrix} \qquad 0 \ \boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{matrix}} \ 0 \leftarrow \text{2nd derivative}$$

$$\begin{matrix} 0 & 0 \\ -1 & 1 \\ 0 & 0 \end{matrix} \rightarrow$$

3rd derivative → $\boxed{\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{matrix}}$

$1 \quad -2 \quad 1$

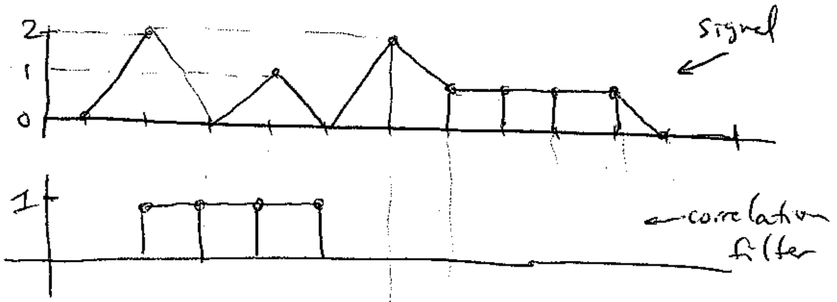
$-1 \quad 1 \rightarrow$

Same as Pascal's triangle:

$$\begin{matrix} 1 \\ 1 \ -1 \\ 1 \ -2 \ \ 1 \\ 1 \ -3 \ \ 3 \ -1 \\ 1 \ -4 \ \ 6 \ -4 \ \ 1 \\ \vdots \end{matrix}$$

for each the mean is zero!

Correlation Example:



First, sliding "dot product"