10/9/07

CHARACTERISTIC EQUATION METHOD

- Similar to that used to solve differential equations
  1) Assume a specific form for the solution
  2) Use initial conditions to resolve coefficients in order to determine actual solution

A k-th order homogeneous linear recurrence with constant coefficients

$$(\text{Eq. } 1) \quad \overset{T(n)}{t_n} + a_1 t_{n-1} + \dots + a_k t_{n-k} = 0$$

$$\left.\begin{array}{l} t_{k-1} = i_1 \\ \vdots \\ t_0 = i_0 \end{array}\right\} \text{initial conditions}$$

- Parameter k is a fixed value that depends on the problem, a_i's constant coefficients

- Equation is homogeneous because RHS is 0
- Equation is linear because LHS is a first degree polynomial in the unknown t_i's.

Principle of superposition

if $t(n) = f(n)$ and $t(n) = g(n)$ are both solutions, then $t(n) = \alpha f(n) + \beta g(n)$ is also a solution.

To see why plug into the equation

$$\alpha f(n) + \beta g(n) + a_1(\alpha f(n-1) + \beta g(n-1)) + \dots \quad a_k(\alpha f(n-k) + \beta g(n-k)) =$$

$$\alpha(f_n + a_1 f_{n-1} + \dots \; a_k f_{n-k}) + \beta(g_n + a_1 g_{n-1} + \dots + a_n g_{n-k}) =$$

$$= \alpha \cdot 0 + \beta \cdot 0 = 0$$

Assume a general solution of the form $t_n = r^n$

Substituting this into equation:

$$r^n + a_1 r^{n-1} + \dots + a_k r^{n-k} = 0$$

divide by r^{n-k}:

$$r^k + a_1 r^{k-1} + \dots + a_k = 0 \leftarrow \text{characteristic equation}$$

polynomial of degree k.

The roots of this equation are called the characteristic roots.

- If the k roots are all distinct $(r_1, \dots, r_k)$ then any linear combination of them

determined by initial conditions.

$$t_n = \sum_{i=1}^{k} c_i r_i^n$$

is a solution to our equation

Ex:

$$t_n - 3t_{n-1} - 4t_{n-2} = 0,\ n \geq 2 \Rightarrow \overbrace{t_n = 3t_{n-1} + 4t_{n-2}}^{\text{recurrence relation}}$$

$t_1 = 1$
$t_0 = 0$

Characteristic equation:

$$r^2 - 3r - 4 = 0 \Rightarrow (r+1)(r-4) = 0$$

roots: $r = -1$ & 4

General solution. $t_n = c_1(-1)^n + c_2(4)^n$

If all we wanted was asymptotic bound, $t_n = \Theta(4^n)$

Let's get exact solutions:

Initial conditions:

$$\begin{aligned} n=0 &\Rightarrow c_1 + c_2 = 0 \\ n=1 &\Rightarrow -c_1 + 4c_2 = 1 \end{aligned} \Rightarrow c_1 = -\frac{1}{5} \quad c_2 = \frac{1}{5}$$