ECE 537 - Foundations of Computing
Prof. Sen
Homework #7
Solutions

The last problem from Homework 6 was re-assigned as Homework 7.

1. Use generating functions to find closed forms for the following sums:

   (a) $\sum_{i=0}^{n} a^i$
   Solution: First we need to put this sum into a form that we can work with. That is, let us rewrite the sum as a recurrence relation:

$$t_n = \sum_{i=0}^{n-1} a^i + a^n = t_{n-1} + a^n.$$

Now let's apply the four step procedure to solve this recurrence using generating functions.
Step 1: This step was already completed above when we rewrote the sum as a recurrence relation.
Step 2: Transform to the z domain, and solve for $G(z)$.

$$G(z) = \sum_{i=0}^{\infty} t_i z^i = \sum_{i=0}^{\infty} t_{i-1} z^i + \sum_{i=0}^{\infty} a^i z^i$$

Notice that the last term in the equation above is in Table 1 (the fifth entry up from the bottom), and it's the generating function for the sequence that has a closed form given by $1/(1-az)$. Thus, we can write

$$G(z) = zG(z) + \frac{1}{1-az}$$

Step 3: Solve for $G(z)$:

$$\begin{aligned} G(z) &= zG(z) + \frac{1}{1-az} \\ &= \frac{1}{(1-z)(1-az)} \end{aligned}$$

Step 4: Determine $[z^n]G(z)$. Using partial fraction expansion we get

$$G(z) = \frac{1}{(1-z)(1-az)} = \frac{A}{(1-z)} + \frac{B}{(1-az)}$$

where

$$\begin{aligned} A &= [(1-z)G(z)]_{z=1} = \frac{1}{1-az}\Big|_{z=1} = \frac{1}{1-a} \\ B &= [(1-az)G(z)]_{z=1/a} = \frac{1}{1-z}\Big|_{z=1/a} = \frac{1}{1-\frac{1}{a}} = \frac{a}{a-1}. \end{aligned}$$

Thus,

$$G(z) = \frac{\left(\frac{1}{1-a}\right)}{(1-z)} + \frac{\left(\frac{a}{a-1}\right)}{(1-az)}.$$

The first term on the LHS corresponds to the third entry in Table 1, while the second term on the LHS corresponds to the fifth entry up from the bottom of Table 1. Therefore,

$$\begin{aligned} t_n &= \frac{1}{1-a} + \frac{a}{a-1} \cdot a^n \\ &= \frac{-1}{a-1} + \frac{a^{n+1}}{a-1} \\ &= \frac{a^{n+1}-1}{a-1} \qquad a \neq 1 \end{aligned}$$

(b) $\sum_{i=0}^{n} ia^i$

Solution (hard way): Rewriting the sum as a recurrence relation we get:

$$t_n = \sum_{i=0}^{n-1} ia^i + na^n = t_{n-1} + na^n.$$

Then

$$\begin{aligned} G(z) &= \sum_{i=0}^{\infty} t_i z^i \\ &= \sum_{i=0}^{\infty} t_{i-1} z^i + \sum_{i=0}^{\infty} ia^i z^i. \end{aligned}$$

The first term on the RHS is no problem, it's the second entry in Table 2 with $m = 1$, so we can write $zG(z)$ as the generating function for that term. The second term, however, requires some generating function manipulations in order to get it into a form that we can use. Specifically, we need to derive a generating function for $t_i = ia^i$. Recall that differentiation is used to bring a factor of i down into the coefficient. That is, the sixth entry in Table 2 has $zG'(z) = \sum_i ig_i z^n$. This matches the second term on the RHS of the above equation if we choose $g_i = a^i$, and the generating function for the sequence $< a, a^1, a^2, \ldots >$ is $\frac{1}{1-az}$. Finally, since

$$\frac{d}{dz}\left[\frac{1}{1-az}\right] = \frac{a}{(1-az)^2}$$

we have that

$$zG'(z) = \frac{az}{(1-az)^2}.$$

So now we can write the generating function for t_n as

$$\begin{aligned} G(z) &= zG(z) + \frac{az}{(1-az)^2} \\ &= \frac{az}{(1-z)(1-az)^2} = \frac{A}{(1-z)} + \frac{Bz+C}{(1-az)^2}. \end{aligned} \tag{1}$$

Using partial fraction expansion, we get

$$A = \frac{az}{(1-az)^2}\Big|_{z=1} = \frac{a}{(1-a)^2}$$

and

$$Bz + C = \frac{az}{(1-z)}\Big|_{z=1/2}$$

which leads to

$$B = -2(a+C).$$

In order to resolve the unknown constants B and C in this equation, we can use Equation (1) to write

$$\begin{aligned} az &= A(a^2z^2 - 2az + 1) + (Bz + C)(1-z) \\ &= (a^2A)z^2 + (-2aA + B - C)z + (A + C) \end{aligned} \tag{2}$$

Evaluating Equation (2) at $z = 0$ leads to $A = -C$, so we have

$$C = \frac{-a}{(1-a)^2}.$$

Now we only need to determine B. We start by evaluating Equation (2) at $z = 1$, but this leads to $a = A(a-1)^2$, or $A = \frac{a}{(1-a)^2}$, which doesn't help us, and is in fact something we already knew, so we need to try something else. Let's try evaluating Equation (2) at $z = -1$. This leads to the equation $-a = A(1+a)^2 - 2B + 2C$, and solving for B we obtain $B = \frac{A(1+a)^2}{2} + C + a$. Finally, substituting the values we have already calculated for A and C into this equation and simplifying leads to:

$$B = \frac{a^3}{(1-a)^2}.$$

Now we use the values we have calculated for A, B, and C in Equation (1) to write

$$\begin{aligned} G(z) &= \frac{\frac{a}{(1-a)^2}}{1-z} + \frac{\frac{a^3z}{(1-a)^2} - \frac{a}{(1-a)^2}}{(1-az)^2} \\ &= \frac{\frac{a}{(1-a)^2}}{1-z} + \frac{\frac{a^3z}{(1-a)^2}}{(1-az)^2} - \frac{\frac{a^3}{(1-a)^2}}{(1-az)^2}. \end{aligned} \tag{3}$$

All that is left to do is to determine the n-th coefficient of $G(z)$, and we can do this term by term. The n-th coefficient of the first term on the RHS of Equation (3) is

$$\frac{a}{(1-a)^2}.$$

The third term on the RHS of Equation (3) can be rewritten as

$$-\frac{\frac{-a}{(1-a)^2}}{(1-x)^2}$$

with $x = az$. This allows us to use the seventh entry in Table 2 to write the n-th term for this generating function as

$$-\frac{a}{(1-a)^2} \cdot (n+1)x^n \;=\; -\frac{a(n+1)}{(1-a)^2} \cdot (az)^n,$$

which means that the coefficient of this term is

$$-\frac{(n+1)a^{n+1}}{(1-a)^2}$$

Finally, the n-th term of the second term on the RHS of Equation (3) can be determined by using the previous result, along with the second entry in Table 1. This yields

$$\frac{a^3(n+1)}{(1-a)^2} \cdot (az)^n \cdot z \;=\; \frac{a^3(n+1)a^n}{(1-a)^2} \cdot z^{n+1}$$

which means that the n-th coefficient for this term must be

$$\frac{a^3(n)a^{n-1}}{(1-a)^2} = \frac{ana^{n+1}}{(1-a)^2}$$

Putting these terms together and simplifying, we get the final solution:

$$\begin{aligned} t_n &= \frac{a}{(1-a)^2} + \frac{ana^{n+1}}{(1-a)^2} - \frac{(n+1)a^{n+1}}{(1-a)^2} \\ &= \frac{(an-n-1)a^{n+1}+a}{(1-a)^2} \qquad a \neq 1. \end{aligned}$$

Alternative Solution (easy way): By taking the derivative with respect to a of the solution from part (a) we get:

$$\sum_{i=0}^{n} ia^{i-1} = \frac{1-(n+1)a^n + na^{n+1}}{(1-a)^2}.$$

Now, multiplying through by a gets us the proper form:

$$\sum_{i=0}^{n} ia^{i} = \frac{a-(n+1)a^{n+1} + na^{n+2}}{(1-a)^2} \qquad a \neq 1.$$