TRANSFORMING NORMALS

$$n^T v = 0$$

$$(M_x n)^T (M v) = 0$$

$$n^T \underbrace{M_x^T M}_{I} v = 0$$

$$n^T v = 0$$

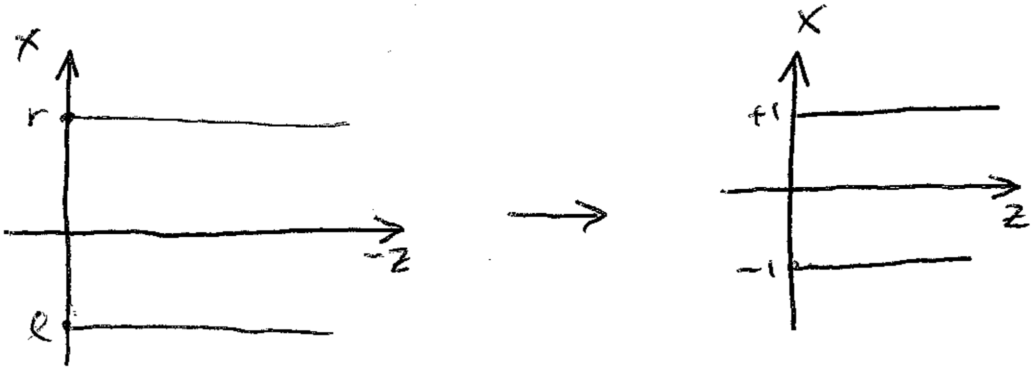
So

$$M_x^T M = I$$

$$M_x^T = M^{-1}$$

$$M_x = (M^{-1})^T$$

COMPUTING X,Y COMPONENTS OF ORTHOGRAPHIC PROJECTION



$l < x < r \longrightarrow -1 < x' < 1$

distance: $r-l$

midpt: $\frac{r+l}{2}$

$$x' = \frac{2}{r-l}\left(x - \frac{r+l}{2}\right)$$

$$\boxed{x' = \left(\frac{2}{r-l}\right)x - \frac{r+l}{r-l}}$$

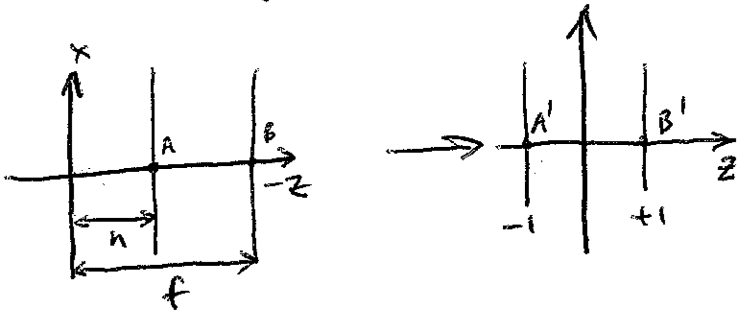
can do the same for y

$$\boxed{y' = \frac{2}{t-b}y - \frac{t+b}{t-b}}$$

Hence the orthogonal projection matrix

$$M_o = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & -\frac{2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

z is a little tricky:



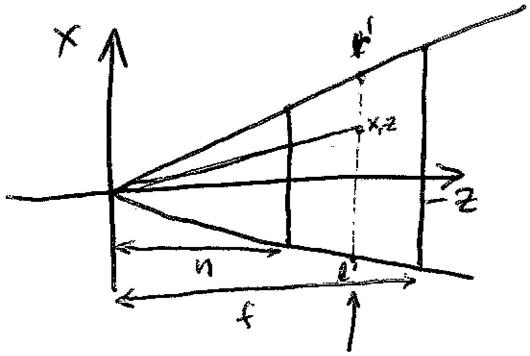
distance: $f-n$

midpt: $\frac{-n-f}{2}$

$$z' = \frac{-2}{f-n}\left(z + \frac{n+f}{2}\right) = \frac{2}{n-f}z + \frac{n+f}{n-f}$$

$$\boxed{z' = -\frac{2}{f-n}z - \frac{f+n}{f-n}}$$

COMPUTING X, Y COMPONENTS OF PERSPECTIVE MATRIX



$l' < x < r' \longleftrightarrow -1 < x' < 1$

$$\frac{r'}{-z} = \frac{r}{n} \Rightarrow r' = \frac{-rz}{n}$$

$$\frac{l'}{-z} = \frac{l}{n} \Rightarrow l' = \frac{-lz}{n}$$

distance $r' \to l' = \frac{-rz}{n} + \frac{lz}{n}$

$$= \frac{(l-r)z}{n}$$

midpt: $\frac{-rz-lz}{2n}$

$$x' = \frac{2n}{(l-r)z}\left(x - \frac{-rz-lz}{2n}\right)$$

$$= \frac{2nx}{(l-r)z} + \frac{\cancel{2n}(l+r)\cancel{z}}{\cancel{2n}(l-r)\cancel{z}}$$

$$x' = \frac{2nx}{(l-r)z} + \frac{(l+r)}{(l-r)}$$ ← let's get this in matrix form

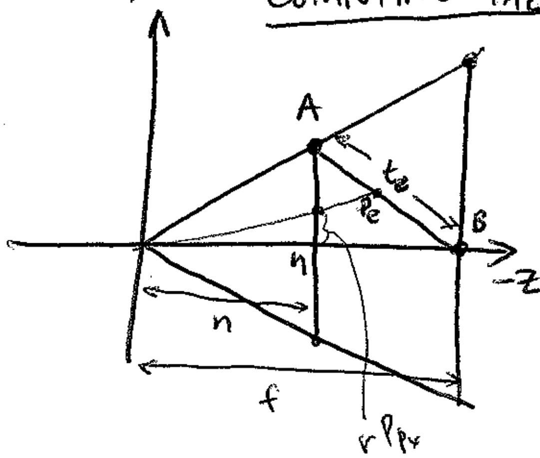
We will divide by $-z$, so multiply by $-z$

$$x' = -\frac{2n}{(l-r)}x - \frac{(l+r)}{(l-r)}z$$

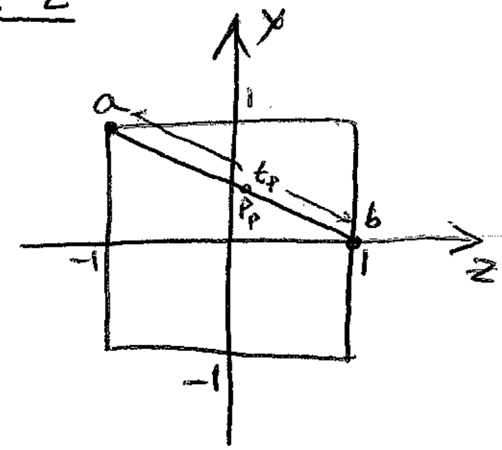
$$\boxed{x' = \frac{2n}{(r-l)}x + \frac{r+l}{r-l}z}$$

Can do the same for y: $\boxed{y' = \frac{2n}{t-b}y + \frac{t+b}{t-b}z}$

COMPUTING THE PERSPECTIVE Z



$\longrightarrow$



$$P_e = (1-t_e)A + t_e B$$

$$P_e = A + (B-A)t_e$$

$$P_p = (1-t_p)a + t_p b$$

$$P_p = a + (b-a)t_p$$

What is the relationship between P_{e_x} ? P_{p_x} ?

$$\frac{P_{ex}}{-P_{ez}} = \frac{rP_{px}}{n} \Rightarrow \boxed{P_{p_x} = \frac{nP_{ex}}{-P_{e_z}r}}$$

$$\overset{+1}{a_x} + (\overset{0}{b_x} - \overset{1}{a_x})t_p = \frac{n(\overset{r}{A_x} + (\overset{0}{B_x} - \overset{r}{A_x})t_e)}{(-\underset{-n}{A_z} - (\underset{-f}{B_z} - \underset{-n}{A_z})t_e)r}$$

$$1 - t_p = \frac{n(r - rte)}{(n - (-f+n)te)r}$$

$$1 - t_p = \frac{n(r - rte)}{(n + (f-n)te)r}$$

$$\boxed{t_p = 1 - \frac{n(1-t_e)}{n+(f-n)t_e}} \quad \text{eq.1}$$

$$t_p = \frac{\cancel{n} + (f-n)t_e - \cancel{n} + nt_e}{n + (f-n)t_e}$$

$$= \frac{ft_e - \cancel{nt_e} + \cancel{nt_e}}{n + (f-n)t_e}$$

$$\boxed{t_p = \frac{ft_e}{n + (f-n)t_e}} \quad \text{eq. 2}$$

What we want $\quad P_{e_z} \Longleftrightarrow P_{p_z}$

$$P_{p_z} = \overset{-1}{a_z} + (\overset{1}{b_z} - \overset{-1}{a_z})t_p = -1 + 2t_p$$

$$\boxed{P_{p_z} = 2t_p - 1}$$

plug in eq. 2

$$P_{p_z} = \frac{2ft_e}{n + (f-n)t_e} - 1$$

$$t_e \overset{?}{\Longleftrightarrow} P_{e_z}$$

$$\frac{P_{e_z} - \overset{-n}{A_z}}{\underset{-f}{B_z} - \underset{-n}{A_z}} = t_e$$

Plug in

$$t_e = \frac{P_{e_z} + n}{n - f}$$

$$P_{p_z} = \frac{2f\left(\frac{P_{e_z} + n}{n-f}\right)}{n + (f-n)\left(\frac{P_{e_z} + n}{n-f}\right)} - 1$$

$$= \frac{2f\left(\frac{P_{e_z} + n}{n-f}\right)}{\cancel{n} - P_{e_z} - \cancel{n}} - 1$$

$$= \frac{2f(P_{e_z} + n)}{P_{e_z}(f-n)} - 1 = \frac{2fP_{e_z} + 2fn - P_{e_z}f + P_{e_z}n}{P_{e_z}(f-n)} = \boxed{\frac{P_{e_z}(f+n) + 2fn}{P_{e_z}(f-n)}}$$

$$P_{pz} = \frac{P_{ez}(f+n) + 2fn}{P_{ez}(f-n)}$$

To get it into matrix notation, note I will divide by $-P_{ez}$.
So multiply by $-P_{ez}$.

$$P_{pz} = \frac{-P_{ez}(f+n) - 2fn}{f-n}$$

$$\boxed{P_{pz} = -\frac{f+n}{f-n} P_{ez} - \frac{2fn}{f-n}}$$

Hence the Perspective Projection matrix is:

$$M_p = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

VIEWPORT TRANSFORMATION

Map $-1 < x < 1$ to $0 < x' <$ screen width

$$x' = \left(\frac{x}{2} + 0.5 \right) \text{screen width}$$

$$x' = \left(\tfrac{1}{2} \text{width} \right) x + \tfrac{1}{2} \text{width}$$

$$y' = \left(\tfrac{1}{2} \text{height} \right) y + \tfrac{1}{2} \text{height}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} \frac{w}{2} & 0 & 0 & \frac{w}{2} \\ 0 & \frac{h}{2} & 0 & \frac{h}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$