Random Process: Example

Outcome	Sample function
$\omega = 1$	$x_1(t) = 1$
$\omega = 2$	$x_2(t) = -1$
$\omega = 3$	$x_3(t) = t$
$\omega = 4$	$x_4(t) = -t$
$\omega = 5$	$x_5(t) = t^2$
$\omega = 6$	$x_6(t) = -t^2$

Figure 1: Random process example: mapping from sample Ω to the collection of member functions $x_i(t), i = 1, 2, 3, 4, 5, 6$. Note the continuity of the sample functions for the example.

Consider a random process $X(t, \omega)$ that is defined via the random experiment of the throw of a fair dice. The sample functions of the process are given by the expressions described in Table 1. For this example, we have that the sample functions of the process are uniformly continuous functions. If we now sample the random process at an instant of time $t = t_o \in \mathcal{T} \subseteq \mathbf{R}$, then the values of the different member functions at this instant, denoted, $X_{t_o}(\omega)$ constitutes a discrete random variable that adopts the probability law of the underlying dice throw experiment, i.e.,

$$X_{t_o}(\omega) \in \{1, -1, t_o, -t_o, t_o^2, -t_o^2\}.$$

The PDF associated with this random variable is therefore:

$$\begin{aligned} f_{X_{t_o}(\omega)}(x; t_o) &= \frac{1}{6}\left\{\delta(x+1) + \delta(x-1) + \delta(x - t_o)\right\} \\ &+ \frac{1}{6}\left\{\delta(x + t_o) + \delta(x - t_o^2) + \delta(x + t_o^2)\right\}. \end{aligned}$$

Note the inherent dependence of this PDF on the sampling time instant t. The ensemble mean of this random variable is given by:

$$\mu_x(t) = E\{X_t(\omega)\} = \frac{1}{6}(1 - 1 + t - t + t^2 - t^2) = 0$$

The mean of the random variable $X_t(\omega)$, i.e., $\mu_x(t)$ constitutes the ensemble mean of the random process. In this example the random process $X(t)$ is a zero mean process. The variance of this zero mean process is therefore:

$$\sigma_x^2(t) = E\{X_t^2(\omega)\} = \frac{1}{6}\left(1 + 1 + t^2 + t^2 + t^4 + t^4\right) = \frac{1}{3}(1 + t^2 + t^4).$$

Note here that the variance of this process is inherently dependent on the sampling time t. This type of a random process where the mean and/or variance are dependent on the sampling instant will be refered to as a *non stationary random* process. Let us now sample the random process at two different instants of time t_1, t_2, where $t_1 < t_2 \subset \mathcal{T}$, then denote resulting random variables obtained as $X_1 = X_{t_1}(\omega)$ and $X_2 = X_{t_2}(\omega)$. Since the sample function of the underlying process are continuous, the joint PDF of the variables X_1, X_2 is given by:

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2;t_1,t_2) &= \frac{1}{6}\{\delta(x_1-1,x_2-1)+\delta(x_1+1,x_2+1)\} \\ &+ \frac{1}{6}\{\delta(x_1-t_1,x_2-t_2)+\delta(x_1+t_1,x_2+t_2)\} \\ &+ \frac{1}{6}\left\{\delta(x_1-t_1^2,x_2-t_2^2)+\delta(x_1+t_1^2,x_2+t_2^2)\right\} \end{aligned}$$

This joint PDF reflects the fact that only certain transitions of the values of the two random variables are possible from the instant t_1 to the instant t_2 due to the continuity of the sample functions. If we integrate out the dependence on one of the variables, the corresponding marginal PDFS of the random variable X_1 is given by

$$\begin{aligned} f_{X_1}(x_1;t_1) &= \frac{1}{6}\{\delta(x+1)+\delta(x-1)+\delta(x-t_1)\} \\ &+ \frac{1}{6}\left\{\delta(x+t_1)+\delta(x-t_1^2)+\delta(x+t_1^2)\right\}. \end{aligned}$$

An identical expression holds good for the random variable X_2. From the information in this particular example, we can conclude that the random variables X_1 and X_2 are not statistically independent because the joint PDF of these random variables is not a separable expression:

$$f_{X_1,X_2}(x_1,x_2;t_1,t_2) \neq f_{X_1}(x_1;t_1) f_{X_2}(x_2;t_2)$$

The autocorrelation function for this example is given by:

$$\begin{aligned} R_{xx}(t_1,t_2) &= E\{X_1X_2^*\} = \frac{1}{6}\left\{(1)(1)+(-1)(-1)+2(t_1)(t_2)+2(t_1^2t_2^2)\right\} \\ &= \frac{1}{3}\left(1+t_1t_2+t_1^2t_2^2\right) \end{aligned}$$

Since the process is zero mean, for this particular example the autocovariance function $C_{xx}(t_1,t_2)$ is the same as $R_{xx}(t_1,t_2)$. The temporal coherence function that measures the normalized statistical correlation between the random variables X_1 and X_2 is given by:

$$\rho_{xx}(t_1,t_2) = \frac{1+t_1t_2+t_1^2t_2^2}{\sqrt{1+t_1^2+t_1^4}\sqrt{1+t_2^2+t_2^4}}.$$

Note clearly the dependence of even the second-order statistics of the random process on the sampling instants and consequently the origin of the measurement system.