Overview of Optimal Wiener Filtering

For the optimal Wiener filter, the estimate of the *signal of interest* (SOI) takes the form:

$$\hat{d}[n] = \sum_{k=-\infty}^{\infty} w[k]x[n-k].$$

The corresponding estimation error is given via:

$$e[n] = d[n] - \sum_{k=-\infty}^{\infty} w[k]x[n-k].$$

The *Wiener–Hopf* equations for the optimal Wiener filter are given by:

$$\sum_{k=-\infty}^{\infty} w_{\text{opt}}[k]R_{xx}[l-k] = R_{dx}[l], \quad l \in \mathbf{Z}.$$

The optimal Wiener filter for the estimation of the SOI is given by:

$$H_{\text{opt}}(z) = \frac{P_{dx}(z)}{P_{xx}(z)}.$$

This optimal filter can be reformulated in terms of the power spectral factorization of $P_{xx}(z)$ as:

$$H_{\text{opt}}(z) = \left(\frac{1}{\sigma_o^2 H(z)}\right)\left(\frac{P_{dx}(z)}{H^*\left(\frac{1}{z^*}\right)}\right).$$

The corresponding *minimum mean-squared error* (MMSE) for the optimal IIR filter is given via:

$$\epsilon^2_{\text{min}} = R_{dd}[0] - \sum_{k=-\infty}^{\infty} w_{\text{opt}}[k]R_{dx}[k]$$

In the frequency–domain this MMSE expression can be reformulated as:

$$\epsilon^2_{\text{min}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{dd}(e^{j\omega})\left(1 - \frac{|P_{dx}(e^{j\omega})|^2}{P_{dd}(e^{j\omega})P_{xx}(e^{j\omega})}\right) d\omega.$$

In terms of the spectral–coherence parameter $\rho_{xy}(e^{j\omega})$ the MMSE is given by:

$$\epsilon^2_{\text{min}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{dd}(e^{j\omega})\left(1 - |\rho_{xy}(e^{j\omega})|^2\right) d\omega.$$

When the spectral-coherence is zero, i.e., there is no statistical correlation between the spectral components of the SOI and $x[n]$ we have:

$$|\rho(e^{j\omega})| = 0 \Longleftrightarrow \hat{d}[n] = 0 \Longleftrightarrow \epsilon^2_{\text{min}} = R_{dd}[0].$$

When the spectral-coherence is unity, i.e., then the spectral components of the SOI and the observations $x[n]$ are perfectly correlated then:

$$|\rho(e^{j\omega})| = 1 \Longleftrightarrow P_{dx}(e^{j\omega}) = P_{xx}(e^{j\omega})P_{dd}(e^{j\omega}) \Longleftrightarrow \epsilon^2_{\text{min}} = 0.$$