Ergodicity in the Mean

A WSS random process is said to be *ergodic* in the mean if the time-average estimate of the mean obtained from a single sample realization of the process converges in both the mean and in the mean-square sense to the ensemble mean, i.e.,

$$\begin{aligned} \lim_{T\to\infty} E\{\langle \mu_x \rangle_T - \mu_x\} &= 0 \\ \lim_{T\to\infty} \mathrm{Var}(\langle \mu_x \rangle_T) &= 0 \end{aligned} \tag{1}$$

Consider a WSS random signal with mean μ_x and autocovariance function $C_{xx}(\tau)$. The goal of this exercise is to develop criteria to assess whether or not a random signal is ergodic from just its autocovariance function. Let us first look at the sample mean random variable is given by:

$$\langle \mu_x \rangle_T = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)dt.$$

The mean of this random variable is given by:

$$E\{\langle \mu_x \rangle_T\} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} E(X(t))dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \mu_x dt = \mu_x.$$

This statement implies that for a WSS random signal the time-average mean estimate will always converge to the ensemble mean. The variance of the sample mean random variable is given by:

$$\mathrm{Var}(\langle \mu_x \rangle_T) = E\{(\langle \mu_x \rangle_T - \mu_x)^2\}$$

Substituting the expression for the sample mean RV into the variance we have that:

$$\mathrm{Var}(\langle \mu_x \rangle_T) = E\left\{\left(\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t_1)dt_1 - \mu_x\right)\left(\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t_2)dt_2 - \mu_x\right)\right\}$$

Upon simplification the variance of the sample mean estimator can be written as:

$$\mathrm{Var}(\langle \mu_x \rangle_T) = \frac{1}{T^2}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} C_{xx}(t_1, t_2)dt_1 dt_2$$

Invoking the WSS property of the random signal the autocovariance function becomes a function of the delay argument, i.e,

$$C_{xx}(t_1, t_2) = C_{xx}(|t_2 - t_1|) = C_{xx}(\tau).$$

Therefore for a WSS random signal to be ergodic in the mean we require that:

$$\lim_{T\to\infty} \frac{1}{T^2}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} C_{xx}(|t_2 - t_1|)dt_1 dt_2 = 0$$

Instead of integrating over both the variables t_1 and t_2 if we integrate over the delay variable $\tau = t_2 - t_1$ we can express the condition given above as:

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} C_{xx}(\tau)\left(1 - \frac{|\tau|}{T}\right)d\tau = 0.$$

It can be verified that decaying functions such as a Gaussian pulse or a Laplacian pulse satisfy this criteria. Loosely speaking a covariance function that asymptotically decays to 0 will satisfy this condition.