Example: Optimal Smoothing

Consider a smoothing example, where the observations are just a noise contaminated version of the SOI: $y(t) = x(t) + v(t)$, where $v(t)$ is zero-mean, unit variance spectrally white observation noise. Specifically the power spectral density of the SOI, $x(t)$, is of the form:

$$P_{xx}(\Omega) = \frac{2\alpha}{\alpha^2 + \Omega^2}, \quad \Omega \in \mathbf{R}.$$

This power-spectrum can be factorized using analytic continuation as:

$$P_{xx}(s) = \frac{2}{\alpha}\left(\frac{\alpha}{\alpha + s}\right)\left(\frac{\alpha}{\alpha - s}\right), \quad |\sigma| < \alpha.$$

The power-spectrum of the observation process is given by:

$$P_{yy}(s) = 1 + \frac{2\alpha}{\alpha^2 - s^2} = \frac{2\alpha + \alpha^2 - s^2}{\alpha^2 - s^2}.$$

This power-spectrum can be factorized via:

$$P_{yy}(s) = \left(\frac{s - \beta}{s - \alpha}\right)\left(\frac{s + \beta}{s + \alpha}\right),$$

where $\beta = \sqrt{2\alpha + \alpha^2}$. The cross power-spectrum $P_{xy}(s)$ is given by:

$$P_{xy}(s) = P_{xx}(s) = \frac{2}{\alpha}\left(\frac{\alpha}{\alpha + s}\right)\left(\frac{\alpha}{\alpha - s}\right).$$

The optimal Wiener smoother is then given by:

$$H_{\text{opt}}(s) = \frac{P_{xy}(s)}{P_{yy}(s)} = \frac{P_{xx}(s)}{P_{yy}(s)} = \frac{2\alpha}{2\alpha + \alpha^2 - s^2}.$$

The optimal causal Wiener smoother is given by:

$$\begin{aligned} H_{\text{cau}}(s) &= \frac{1}{P_{yy}^{+}(s)}\left(\frac{P_{xx}(s)}{P_{yy}^{-}(s)}\right)_{+} = \left(\frac{s + \alpha}{s + \beta}\right)\left[\left(\frac{2\alpha}{\alpha^2 - s^2}\right) / \left(\frac{s - \beta}{s - \alpha}\right)\right]_{+} \\ &= \left(\frac{2\alpha}{\alpha + \beta}\right)\frac{1}{s + \beta} \end{aligned}$$

We may then obtain the MMSE in each case as:

$$\begin{aligned} \text{MMSE}_1 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{vv}(\Omega) H_{\text{opt}}(\Omega) d\Omega = \frac{\alpha}{\beta} \\ \text{MMSE}_2 &= R_{xx}(0) - \int_{0}^{\infty} h_{\text{opt}}(\tau) R_{xx}(\tau) d\tau = \frac{2\alpha}{\alpha + \beta}. \end{aligned}$$