On the Multiangle Centered Discrete Fractional Fourier Transform

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Abstract—Existing versions of the discrete fractional Fourier transform (DFRFT) are based on the discrete Fourier transform (DFT). These approaches need a full basis of DFT eigenvectors that serve as discrete versions of Hermite–Gauss functions. In this letter, we define a DFRFT based on a centered version of the DFT (CDFRFT) using eigenvectors derived from the Grünbaum tridiagonal commutor that serve as excellent discrete approximations to the Hermite–Gauss functions. We develop a fast and efficient way to compute the multiangle version of the CDFRFT for a discrete set of angles using the FFT algorithm. We then show that the associated chirp-frequency representation is a useful analysis tool for multicomponent chirp signals.

Index Terms—Chirp rate estimation, discrete Fourier transform (DFT), discrete fractional Fourier transform (DFRFT), eigenvalues, eigenvectors, fast Fourier transform (FFT), fractional matrix power, Hermite–Gauss functions, multicomponent chirp signals.

I. INTRODUCTION

THE INTEREST for a *discrete fractional Fourier transform* (DFRFT) has increased in recent years due to its potential applications in digital signal processing and optics. In most of the existing approaches, a DFRFT is obtained via the fractional power of the discrete Fourier transform (DFT) matrix $\mathbf{W}$ using the expansion in [1] as

$$\mathbf{A}_\alpha = \mathbf{W}^{\frac{2\alpha}{\pi}} = \mathbf{V}\mathbf{\Lambda}^{\frac{2\alpha}{\pi}}\mathbf{V}^T \tag{1}$$

where $\mathbf{V}$ is a matrix of eigenvectors of $\mathbf{W}$, and $\mathbf{\Lambda}^{(2\alpha/\pi)}$ is a diagonal matrix with the fractional powers of the eigenvalues of $\mathbf{W}$. With this definition, we have the required identities for boundary conditions

$$\mathbf{A}_{\frac{\pi}{2}} = \mathbf{W} \quad \text{and} \quad \mathbf{A}_{2\pi} = \mathbf{W}^4 = \mathbf{I}. \tag{2}$$

An early effort to compute a fractional transform using (1) resulted in the transform [1]

$$\mathbf{A}_\alpha = a_0(\alpha)\mathbf{I} + a_1(\alpha)\mathbf{W} + a_2(\alpha)\mathbf{W}^2 + a_3(\alpha)\mathbf{W}^3$$

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i.e., a linear combination of the powers of $\mathbf{W}$. This transform, however, does not behave similarly to the continuous transform due to the multiplicities expounded in [2].

Following this work, several different discrete fractional transforms have been proposed with the objective of replicating the behavior of the continuous counterpart. Some approaches toward discretization have been to directly sample the continuous kernel. Since the basis functions of the continuous FRFT are not bandlimited, this process produces aliasing, and oversampling translates in a nonorthogonal basis [3], [4]. Earlier work in [5] produced an expression for a set of orthogonal DFT eigenvectors based on sampling and aliasing of the Hermite–Gauss functions but does not yield a computable version, while recent work in [6] uses the centered version of the DFT but is based on approximating the infinite sum in [5].

Recent efforts toward finding a discrete FRFT have specifically focused on the problem of finding an orthogonal set of eigenvectors for the DFT that resemble discrete versions of the Hermite–Gauss functions. Some approaches [7], [8] have obtained a complete set based on a development of a commuting matrix for the DFT [9]. Another approach that uses Kravchuk functions as eigenvectors has been explored in [10]. A different set of eigenvectors can be obtained for a centered version of a the DFT via a tridiagonal commuting matrix [11] using earlier work by Grünbaum [12]. We adopt the Grünbaum eigenvectors for discussions in this letter for the reasons that a) the commutor has been shown to converge to the Hermite–Gauss differential operator asymptotically [12], and b) they furnish a full set of eigenvectors for any matrix size [12], [13]. A DFRFT based on a centered DFT matrix has been proposed in [13], and some of its properties have been explored in [14].

The approach pursued here uses the centered DFT and has further advantages that arise from the symmetric tridiagonal nature of the commuting matrix and the uniform multiplicities of the eigenvalues when the size of the transform is a multiple of four, as will be explained later. There are two different aspects to be considered when computing a DFRFT: a) computation of the eigenvectors, which is a one time task for a given matrix size, and b) computation of the transform itself. For the first part, we invoke symmetries of the Grünbaum eigenvectors for implementing a reduction in the number of multiplies and storage needed, and for the second part, we reformulate the centered DFRFT with a discrete angular parameter (MA-CDFRFT) in a form where the computationally efficient radix-2 FFT algorithms can be applied. Finally, we look at a chirp rate versus frequency representation for linear chirp signals that results from this multiangle transform.

TABLE I
MULTIPLICITIES OF THE EIGENVALUES OF THE DFT AND THE CDFT

N	DFT				CDFT			
	1	-j	-1	j	1	-j	-1	j
4m	m+1	m	m	m-1	m	m	m	m
4m+1	m+1	m	m	m	m+1	m	m	m
4m+2	m+1	m	m+1	m	m+1	m+1	m	m
4m+3	m+1	m+1	m+1	m	m+1	m+1	m+1	m

II. EIGENVALUES OF THE DFT AND THE CENTERED DFT

We define the *centered DFT* (CDFT) as the unitary matrix with elements

$$\{\mathbf{A}\}_{kn} = \frac{1}{\sqrt{N}} e^{(-j\frac{2\pi}{N})(k-\frac{N-1}{2})(n-\frac{N-1}{2})}$$
$$k, n = 0, \ldots, N-1. \quad (3)$$

This definition corresponds to the offsets $a = b = (N-1/2)$, as defined in [6], for the *offset DFT*. One of the main reasons for choosing the CDFT to define a fractional transform is the fundamental difference in the multiplicities of its eigenvalues with respect to the regular DFT that can be exploited to simplify the computations. From the results obtained in [15] for the multiplicities of the eigenvalues, we can observe that when N is a multiple of four, the DFT has different multiplicities for the four eigenvalues, while the CDFT has a multiplicity of $(N/4)$ for each eigenvalue, as it can be observed in Table I.

An immediate consequence of the different multiplicities of the DFT eigenvalues when using (1) is that $\mathbf{\Lambda}^{(2\alpha/\pi)}$ requires a different implementation for the cases of N being even and odd. When using the CDFT to define a fractional transform, we have a single definition for all values of N given by

$$\mathbf{\Lambda}^{\frac{2\alpha}{\pi}} = \text{diag}(e^{-jp\alpha}), \quad 0 \leq p \leq N-1. \quad (4)$$

This permits us to define the CDFRFT matrix as

$$\mathbf{A}_\alpha = \sum_{p=0}^{N-1} e^{-jp\alpha} \mathbf{v}_p \mathbf{v}_p^H \quad (5)$$

for all values of N, where v_p is the pth eigenvector of the commuting matrix defined in [11].

III. FAST COMPUTATION OF THE MULTIANGLE CDFRFT

Using the definition given in the previous section, a method for fast and efficient computation of this fractional transform can be developed as follows. The elements of the CDFRFT matrix in (5) can be expressed as

$$\{\mathbf{A}_\alpha\}_{kn} = \sum_{p=0}^{N-1} v_{kp} v_{np} e^{-jp\alpha} \quad (6)$$

where v_{kp} is the kth element of eigenvector p. Multiplying $\mathbf{A}_\alpha$ by the signal $x[n]$, we obtain the corresponding transform given by

$$X_\alpha[k] = \sum_{n=0}^{N-1} x[n] \sum_{p=0}^{N-1} v_{kp} v_{np} e^{-jp\alpha}. \quad (7)$$

Rearranging the two sums, we obtain

$$X_\alpha[k] = \sum_{p=0}^{N-1} v_{kp} \sum_{n=0}^{N-1} x[n] v_{np} e^{-jp\alpha}. \quad (8)$$

Now, we observe that for the particular set of equally spaced values of α given by

$$\alpha_r = \frac{2\pi r}{N}, \quad r = 0, 1, \ldots, N-1 \quad (9)$$

we can rewrite the transform as

$$X_r[k] = \sum_{p=0}^{N-1} v_{kp} \sum_{n=0}^{N-1} x[n] v_{np} e^{-j\frac{2\pi}{N}pr}. \quad (10)$$

If we now define $z_k[p]$ as

$$z_k[p] = v_{kp} \sum_{n=0}^{N-1} x[n] v_{np} \quad (11)$$

we observe that the transform can be interpreted as the DFT of $z_k[p]$ for each index k, that is

$$X_k[r] = \sum_{p=0}^{N-1} z_k[p] e^{-j\frac{2\pi}{N}pr} \quad (12)$$

where $0 \leq r \leq N-1$ and $0 \leq k \leq N-1$. Note that the indices of the transform $X_k[r]$ have been interchanged [with respect to (10)] to emphasize the fact that the result of each DFT computation corresponds to the kth element of the CDFRFT for the whole set of angles α_r. Expressing the computation of the transform as a DFT allows the use of the FFT algorithm for efficient computation. This reduces[1] the number of computations of the fractional transform for the entire set of uniformly spaced angles from $O(N^3)$, i.e., the case for direct computation using N matrix multiplications to $O(N^2 \log_2 N)$ when using the radix-2 FFT algorithms with $N = 2^\nu$.

IV. EXPLOITING SYMMETRIES OF GRÜNBAUM EIGENVECTORS

In many of the approaches to define fractional transforms, the eigenvectors have even and odd symmetries that can be exploited to further reduce the number of computations. For the particular case of the CDFRFT, the eigenvectors that result from the Grünbaum type of commuting matrix [11] have been shown to have even and odd symmetries [13] similar to the eigenvectors used for the DFRFT. The eigenvectors have even symmetry for an even number of zero crossings (as defined in [8]) and odd symmetry for an odd number of zero crossings. In this section, we present a way to use only the first half of each of the eigenvectors to compute $z_k[p]$, as defined in (11). We are interested only in the case where N is even and, in particular, when $N = 2^\nu$ because this is the case that can take the most advantage of the decimation in time/frequency FFT algorithms.

When N is even, we can express the elements of the eigenvectors as

$$v_{np} = v_{(N-1-n)p}, \quad 0 \leq n \leq \frac{N}{2} - 1, \quad \text{and } p \text{ even} \quad (13)$$

[1]We specifically require the computation of N monodimensional transforms each with a complexity of $O(N \log_2 N)$.
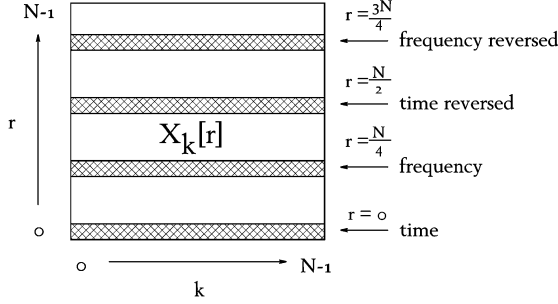
Fig. 1. Graphical representation of $X_k[r]$ that shows how the interpretation of the index k changes depending on the value of index r.

and

$$v_{np} = -v_{(N-1-n)p}, \quad 0 \le n \le \frac{N}{2} - 1, \quad \text{and } p \text{ odd.} \quad (14)$$

Equations (13) and (14) can be put in a single expression as

$$v_{np} = (-1)^p v_{(N-1-n)p}, \quad 0 \le n \le \frac{N}{2} - 1 \quad (15)$$

and we can rewrite (11) as

$$z_k[p] = v_{kp} \sum_{n=0}^{\frac{N}{2}-1} v_{np}(x[n] + (-1)^p x[N-1-n]) \quad (16)$$

for $0 \le k \le N - 1$, reducing the multiplications needed to one half. As a final step, we apply (15) to v_{kp}, and we can rewrite $z_k[p]$ as

$$z_k[p] = (-1)^p (z_{(N-1-k)}[p]), \quad 0 \le k \le \frac{N}{2} - 1. \quad (17)$$

From (17), we observe that we only need to compute half the values of $z_k[p]$, since the others can be obtained by a multiplying by $(-1)^p$. Combined with the reduction in computations to half the terms in the sum of (16), we get an important overall reduction in the number of computations needed to obtain $z_k[p]^2$.

V. CHIRP RATE VERSUS FREQUENCY REPRESENTATION

The resulting transform $X_k[r]$ of (12) is an array that contains the set of CDFRFTs corresponding to angles $\alpha_r = (2\pi r/N), r = 0, 1, \ldots, N - 1$. We call $X_k[r]$ the *multiangle* CDFRFT (MA-CDFRFT). It is important to observe that $X_k[0]$ corresponds to the original signal $x[n]$ with $k = n$, and $X_k[(N/4)]$ corresponds to the CDFT of $x[n]$ when N is a multiple of four, since in this case, $\alpha_{(N/4)} = (\pi/2)$. Fig. 1 shows a graphical representation of the array $X_k[r]$ to illustrate how index k has different interpretations depending on the value of r [for example, when $r = 0, k$ is interpreted as time and when $r = (N/4), k$ corresponds to the frequency]. This representation also shows that the upper half of $X_k[r]$ is a reversed version of the lower half.[3]

Another observation is that the angles of the MA-CDFRFT correspond exactly to the *special angles* of the DFRFTs used

[2]This reduction in complexity is applicable to any set of eigenvectors that have even and odd symmetries [9], [12].

[3]A similar multiangled signal representation in the continuous case, where the fractional Fourier layers are positioned radially, is described in [16].

in [17] for the computation of a DFRFT of an arbitrary angle using a weighted summation of DFRFTs. This is important because the method of weighted summation described in [17], for N odd, can be used to compute the transform for arbitrary angles of the CDFRFT without modification. The CDFRFT has exactly the same definition and eigenvalues as the DFRFT of odd length, with the only difference being the eigenvector set used. The two methods can be combined to compute the CD-FRFT of an arbitrary angle: First, compute the MA-CDFRFT, and then, use the method in [17] to compute the transform for any arbitrary angle.

The computation of the the MA-CDFRFT can be applied to the estimation of the chirp rate of a single signal or a mixture of signals with similar average frequencies but different chirp rates. Defining the chirp rate as the coefficient of the quadratic term of the phase, an example of the application of the MA-CDFRFT to the signal $x[n] = e^{j(-0.003(n-127/2)^2)} + e^{j(0.005(n-127/2)^2)} + e^{j(-0.011(n-127/2)^2)}, n = 0, \ldots, 127$ is presented in Fig. 2. The MA-CDFRFT of the signal has three maxima in the interval $0 \le \alpha \le \pi$ at $r = 24, r = 30$, and $r = 36$ that correspond to the values of α 1.1781, 1.4726, and 1.7671, respectively. Using the formula that relates α with the chirp rate given in [18], the chirp rates result in $-0.0108, -0.0026$, and 0.0053. This is a rough estimation of the actual chirp rates of $-0.011, -0.003$, and 0.005, but it is done using the index where the maximum occurs. A more precise estimation can be obtained by interpolation or by computing the CDFRFT for more angles.

An alternative for the method of weighted summation in [17] when we need to compute the transform at sets of equally spaced angles is to add a constant angle to the expression of α_r, that is

$$\alpha_r = \frac{2\pi r}{N} + \beta, \quad r = 0, 1, \ldots, N - 1. \quad (18)$$

The expression for the transform with this set of angles is

$$X_r[k] = \sum_{p=0}^{N-1} v_{kp} \sum_{n=0}^{N-1} x[n] v_{np} e^{-j\frac{2\pi}{N}pr} e^{-j\beta p} \quad (19)$$

and the last factor in the sum can be taken out to have

$$X_r[k] = \sum_{p=0}^{N-1} (e^{-j\beta p}) v_{kp} \sum_{n=0}^{N-1} x[n] v_{np} e^{-j\frac{2\pi}{N}pr}. \quad (20)$$

The final step is to write it in terms of a DFT of $z_k[p]$ multiplied by a complex exponential, that is

$$X_k[r] = \sum_{p=0}^{N-1} [(e^{-j\beta p}) z_k[p]] e^{-j\frac{2\pi}{N}pr} \quad (21)$$

and we can use the same $z_k[p]$ for different sets of angles. Setting $\beta = \pi/N$, we can compute a second set of angles that falls exactly in the middle of the original angles ($\beta = 0$), doubling the resolution in the angle with two MA-CDFRFT computations. For the signal presented as an example, computing four MA-CDFRFTs corresponding to $\beta = 0, \pi/2N, \pi/N$, and $3\pi/2N$, we find that the three maxima in the interval $0 \le \alpha \le \pi$ occur at the equivalent of $r = 23.75, r = 29.75$, and $r = 36$,
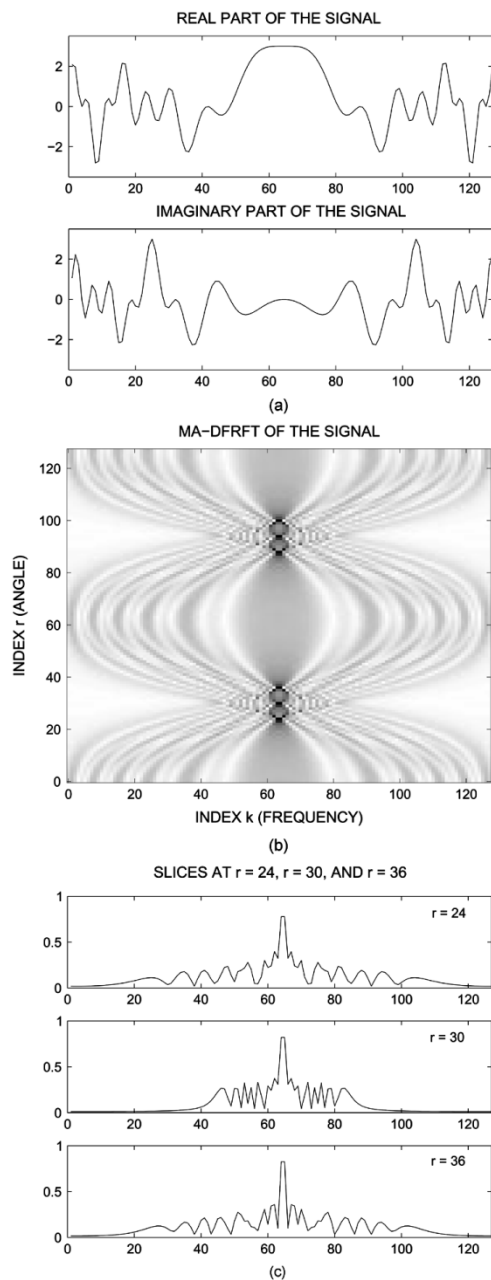
Fig. 2. Complex signal $x[n] = e^{j(-0.003(n-127/2)^2)} + e^{j(0.005(n-127/2)^2)} + e^{j(-0.011(n-127/2)^2)}$, $n = 0, \ldots, 127$ shown in (a). The image in (b) is the magnitude of the MA-DFRFT applied to the signal, and (c) shows the slices of the MA-DFRFT at $r = 24, r = 30$, and $r = 36$ that are the indices at which we have the maxima.

which correspond to the chirp rates of $-0.0112, -0.0029$, and 0.0053, respectively. As a final comment, we would like to emphasize the reduction in the order of computations needed for the MA-CDFRFT. The use of the DFT in real-time applications has been possible because of the efficiency of the FFT algorithm. In the case of the DFRFT, the computation proposed using the FFT results in a similar reduction of computations that we believe will be the key to the eventual use of DFRFT in applications that require analysis of signals in real time.

VI. CONCLUSION

We have presented a algorithm that can be applied to the computation of any fractional transform based on a centered DFT using symmetric eigenvectors. It provides a reduction in the order of the computations from $O(N^3)$ to $O(N^2 \log_2 N)$ when a set of N equally spaced angles is computed by formulating the transform as a DFT and using the FFT algorithm. Further reduction in the computations is achieved by exploiting the symmetries of the eigenvectors. We then presented an application of the MA-DFRFT to the problem of chirp-rate estimation of multicomponent chirp signals and showed that the resultant chirp rate versus frequency representation is a useful tool for the analysis of these signals.

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