An Invariant-Manifold-Based Method for Chaos Control

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Abstract—In this paper, we extend the OGY chaos-control method to be one based on the invariant manifold theory and the sliding mode control concept. This extended-control method not only can deal with higher order chaotic systems in the same spirit of the OGY method, but also can remove the reliance of the control on eigenvalues and eigenvectors of the system Jacobians, resulting in an even simpler but more effective controller. The novelty of the new design lies in the construction of suitable invariant manifolds according to the desired dynamic properties. The controller is then forcing the system state to lie on the intersection of the selected invariant manifolds, so that once the invariant manifolds are reached, the chaotic system will be guided toward a desired fixed point that corresponds to an originally targeted unstable periodic orbit of the given system. Such an idea is directly relevant to the sliding mode control approach. This new method is particularly useful for controlling higher order chaotic systems, especially in the case where some of the eigenvalues of the system Jacobian are complex conjugates. The effectiveness of the proposed method is tested by numerical examples of the third-order continuous-time Lorenz system and the fourth-order discrete-time double rotor map.

Index Terms—Chaos, invariant manifold, stabilization, tracking control.

I. INTRODUCTION

CHAOS control is of significant importance for solving many nontraditional real-world problems by means of nontraditional techniques [1]. Among the existing chaos control methodologies, the model-free chaos control approach has attracted a great deal of attention due to the difficulty in obtaining a faithful modeling for a chaotic system in many real applications. The first model-free chaos control method was suggested by Ott, Grebogi, and Yorke [7], known as the OGY method. It has lately been extended, analyzed, and applied [2]–[8]. For detailed surveys about OGY based controls, readers are referred to [3], [4] and references therein. Essentially, this kind of control techniques require an identification of the stable and unstable manifolds from an available time series and, based on that, a suitable control action is developed

to bring the system orbit to the stable manifold. This type of control methods, although with basic features of classical control, exploits some particular properties of chaos such as the ergodicity and structural stability of chaos.

It is now known that the OGY type of chaos-control methods are effective only for controlling lower dimensional chaotic systems because it utilizes the prominent feature of saddle type of fixed points which have both stable and unstable manifolds. It is common experience that controlling higher dimensional chaotic systems by this methodology is still quite difficult [2]. From an algebraic point of view, its difficulty lies in the situation where the system Jacobian at a fixed point has complex eigenvalues or multiple unstable eigenvalues. Even with distinct real eigenvalues, the construction of stable and unstable manifolds for higher dimensional chaotic systems is a technical challenge [2].

In this paper, we extend the OGY chaos-control method, which not only preserves the spirit of the original method to direct the system orbit to designated stable manifold(s), but also can deal with higher order chaotic systems fairly easily. The novelty lies in the construction of suitable manifolds, independent of the system Jacobian eigenvalues and eigenvectors, which are selected to present a desired dynamics. Once these manifolds are made invariant, the desired dynamics, usually containing the desired fixed point corresponding to an ideal inherent unstable periodic orbit to be stabilized, will take any state in the manifolds to the equilibrium asymptotically. The chaos control task is then to force the system state to reach and lie on these selected invariant manifolds, so that whenever the system state lies on it, the system orbit will be guided to the designated fixed point which corresponds to the originally targeted unstable periodic orbit. This extension, closely related to the sliding mode control design, is analyzed in details in this paper. Simulation results of controlling some typical higher order chaotic systems are presented, to show the effectiveness of the proposed control method.

This paper is organized as follows. The OGY chaos control methodology is first outlined in Section II, which is needed for the development of the paper. Then, the main idea of the new method is presented in Section III, where the design methodology is described and some dynamical and geometrical properties will be examined based on the discrete sliding mode control concept. The proposed chaos control method is then tested in controlling the third-order continuous-time Lorenz system and the fourth-order discrete-time double rotor map. Simulation results for these two representative examples are summarized in Section IV, and conclusions are given in Section V with some comments.

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II. THE ORIGINAL OGY CHAOS CONTROL METHOD

In this approach, while trying to bring the system orbit to a target, one first locally linearizes the system and then uses the controller to slightly push the system orbit toward the target. At this step, the control method is classical (which may be interpreted as a pole placement approach), but the pushing has to be so small that it does not destroy the structural stability of chaos thereby preserving the chaotic property of the system. Then, the controller is turned off, letting the orbit free to travel until it moves back to nearby the target (by the ergodicity of chaos, it will). This step is not typical in classical control, and cannot be applied to nonchaotic systems. To this end, another local linearization of the system is carried out but at the new system parameter values, and then a new and small control push applies, so the next cycle of the process begins. Each time, the system orbit moves closer to the target, while the chaotic dynamics is preserved until the orbit eventually arrives at the target (and chaos is thus eventually suppressed). This control method is based on a classical approach, but it differs from the classical approach in that it utilizes some very nature of chaos, such as the ergodicity and the structural stability of chaos.

More precisely, consider a chaotic dynamic system

$$\dot{x} = f(x, p), \quad x \in R^n. \quad (1)$$

Suppose that the system mathematical model is unknown, but its output time series is available. Using the time series, the phase flow of (1) can be constructed via the Poincaré section method, as

$$x(k+1) = F(x(k), p), \quad x(k) \in R^{n-1}, \quad p \in R^m. \quad (2)$$

As is well known, a fixed point of this map represents a periodic orbit [3]. Only saddle type of unstable periodic orbits (UPO's) is of interest in this discussion.

Furthermore, consider a fixed point, denoted x^* , satisfying

$$x^* = F(x^*, p). \quad (3)$$

Assume that the desired orbit to be stabilized is corresponding to a nominal parameter value, p_0 . This control method suggests to first derive the reduced-order model from the time series about the desired fixed point x^* and the nominal parameter p_0 , so as to obtain a locally linear model of the form

$$\Delta x(k+1) = A\Delta x(k) + B\Delta p(k) \quad (4)$$

where

$$A = \frac{\partial F}{\partial x} \in R^{(n-1) \times (n-1)} \quad \text{and} \quad B = \frac{\partial F}{\partial p} \in R^{(n-1) \times m}$$

with A and B being $(n-1) \times (n-1)$ and $(n-1) \times m$ matrices, respectively, $\Delta x(k) = x(k) - x^* \in R^n$ and $\Delta p = p(k) - p_0 \in R^m$, which are used as control input for stabilization.

Typically, the OGY control method can be illustrated in a two-dimensional (2-D) phase plane [7], [2]. Assume, as mentioned above, the fixed point of

$$\Delta x(k+1) = A\Delta x(k) \quad (5)$$

is a saddle. Then, there exist two eigenvalues, λ_u and λ_s , which satisfy $|\lambda_s| < 1 < |\lambda_u|$. Accordingly, there exist two associated left eigenvectors, v_u and v_s , satisfying $Av_u = \lambda_u v_u$, $Av_s = \lambda_s v_s$. On the other hand, the right eigenvectors, w_u and w_s , are defined as $w_u^T A = \lambda_u w_u^T$, $w_s^T A = \lambda_s w_s^T$. One can easily verify that $[v_u \ v_s]^{-1} = [w_u \ w_s]^T$ in which $v_u^T w_u = 1$, $v_s^T w_s = 1$, $v_u^T w_s = 0$, $v_s^T w_u = 0$. The stable manifold is used as a vehicle to bring the system orbit to the fixed point: any drifting will be taken back to the stable manifold.

The OGY method suggests that we find a displacement of Δp , such that

$$w_u^T \Delta x(k+1) = 0$$

which leads to a simple adjusting law for Δp as follows:

$$\Delta p = -\lambda_u \frac{w_u^T \Delta x(k)}{w_u^T B}. \quad (6)$$

This is a simplified version of the original OGY method, which first performs the shifting and then the mapping. There are some other variants of the method; nevertheless, the essence of this control method is to shift the orbit in such a way that the movement along the direction perpendicular to the tangent of the stable manifold is zero, thereby keeping the orbit on the stable manifold. Thus, at the end, it will bring the orbit to the target.

For those 2-D Poincaré sections used for a third-order chaotic system, if their corresponding A matrix contains complex eigenvalues and/or multiple unstable eigenvalues, the OGY method cannot be applied directly. Partly, this is because (6) cannot be directly applied. In addition, for higher order chaotic dynamic systems, it is technically difficult to construct stable and unstable manifolds from time series, because there most likely exist some conjugate and/or multiple eigenvalues in the system Jacobian. Furthermore, unlike the 2-D Poincaré section where a saddle type of fixed point corresponds to one stable manifold and one unstable manifold, in higher dimensional Poincaré sections, such stable and unstable manifolds are not easily identifiable, due to multiinputs and complex eigenvalues, and the system may even have multiple attractors. In these cases, the original OGY method does not work in general.

In the following, we extend the OGY chaos control method for higher order chaos control, which is based on the invariant manifold theory and the sliding mode control concept. They can together overcome the aforementioned technical problems.

III. THE PROPOSED CHAOS CONTROL METHOD

As reviewed above, the essence of the OGY control method is to restrict the orbital movement along the direction perpendicular to the tangent of the stable manifold, so that sooner or later the system orbit is confined to the stable manifold on which the subdynamics of the manifold drives the orbit to the target fixed point. Such a stable manifold is *invariant* because in a neighborhood of the manifold, the orbit will always be attracted to it under the control, which will stay within the manifold thereafter.

As mentioned, in the 2-D Poincaré section case, the stable and unstable manifolds are constructed by using the system Jacobian eigenvalues and eigenvectors in the OGY method. For higher

order chaotic systems, this construction becomes very difficult, if not impossible, because the system Jacobian may have complex eigenvalues, perhaps with multiplicity, so it is very difficult to determine the stable and unstable manifolds.

Here, we make use of the sliding mode control concept [15] to extend the OGY method to deal with the higher-dimensional case. First, we propose to construct a set of invariant manifolds for the higher order chaotic system, *independent* of the system Jacobian eigenvalues and eigenvectors. This implies that the manifolds, on which the system orbit will eventually stay, can be prescribed independently of the system Jacobians. We then design an OGY-type of control so that in the neighborhoods of the invariant manifolds, the system orbit will reach and then stay within a stable manifold. Note that the parameters of the switching manifolds can be chosen beforehand so that, if needed, the eigenvalues of the resulting dynamics can be assigned to desirable values.

To demonstrate how this idea works, we first choose a set of manifolds which can be independent of the system Jacobian, represented by

$$h(k) = h(\Delta x(k)) = C\Delta x(k) = 0 \in R^m \quad (7)$$

where $\Delta x \in R^{(n-1)}$ and C is a $m \times (n-1)$ matrix. Assuming that the $m \times m$ matrix CB is nonsingular, and we want to find a control action, $\Delta p(k)$, such that

$$\begin{aligned} h(k+1) &= C\Delta x(k+1) \\ &= CA\Delta x(k) + CB\Delta p(k) = 0 \quad \forall k \geq k_0 \end{aligned}$$

so that $h(k) = 0$ becomes the invariant manifolds, i.e., the system orbit will lie on the intersection of these manifolds.

Note that this intersection represents a set described by

$$\Gamma = \left\{ \Delta x \mid \bigcap_{i=1}^m \{h_i(\Delta x) = 0\} \right\}$$

which is of $(n-1-m)$ -dimensional. Note also that the idea here is similar to the discrete sliding mode control [14], but the conventional discrete sliding mode control would require $\Delta h(k) = h(k+1) - h(k) = 0$ instead of $h(k+1) = 0$.

When this is achieved, we have

$$h(k) = C\Delta x(k) = [C_1 \quad C_2] \begin{bmatrix} \Delta x_1(k) \\ \Delta x_2(k) \end{bmatrix} \quad (8)$$

where

$$\begin{aligned} \Delta x_1 &\in R^{n-1-m}; \\ \Delta x_2 &\in R^m; \\ C_1 &\quad m \times (n-1-m) \text{ matrix}; \\ C_2 &\quad \text{nonsingular } m \times m \text{ matrix.} \end{aligned}$$

From (8), and $h(k) = 0$, we have

$$\Delta x_2(k) = -C_2^{-1}C_1\Delta x_1(k). \quad (9)$$

This, although in the discrete-time form, appears to be similar to the continuous-time sliding mode setting.

The question now is how to guarantee that the dynamical system (4) subject to the constraint $h(k) = 0$ is asymptotically stable toward the desired fixed point, i.e., the resulting

Stable manifold

Fig. 1. Illustration of design in the phase plane.

Fig. 2. Illustration of design in 3-D case.

dynamics is asymptotically stable. This can be ensured by choosing suitable matrices C_1 and C_2 in the above formula. Thus, once the system orbit reaches any of these stable manifolds, it will converge to the intersection point. Subsequently, the controlled system orbit converges to the desired fixed point asymptotically, which corresponds to the desired periodic orbit of the given system.

This basic principle of the new control methodology is depicted in Figs. 1 and 2. Fig. 1 shows that in the 2-D phase plane, $\Delta x(k+1) = x(k+1) - x^*$ is forced to be on the chosen stable manifold, which may be different from the system's stable manifold (asymptote) determined by its stable Jacobian eigenvalue. Fig. 2 illustrates the same situation in the three-dimensional (3-D) case.

As pointed out in [2], the one-step-ahead recursive method for the orbit to reach a stable manifold is effective in a small vicinity of the desired fixed point, because the control magnitude is limited by the constraint $|\Delta p(k)| \leq \bar{p}$, where $\bar{p} > 0$, which is small in order not to completely destroy the chaotic nature of the system. The use of the sliding mode concept for chaos control can ensure a local attraction region around the desired fixed point of the Poincaré section, i.e., once the system orbit enters this local attraction region around the intersection of the invariant manifolds, it will stay therein. The asymptotical

stable dynamics of the intersection of the invariant manifolds will then lead the orbit toward the desired fixed point.

The constraint of $|\Delta p(k)| \leq \bar{p}$ with a small $\bar{p} > 0$ may lead to a very narrow attraction region toward the desired fixed point. Outside the region, it would be appropriate to keep the orientation of the system motion toward the intersection of the manifolds, so that the convergence toward the target manifolds becomes faster. An incremental control method may be used here to realize this, in which the control strength is restricted, in a way that the control force is always kept within the allowable range for processing chaos preservation.

Instead of reaching the desired fixed point within one step when the orbit is far away from it, which actually is impossible due the limitation of the control magnitude, we design the small control force such that the system orbit tends to (but may not immediately reach) the invariant manifolds. This idea leads to the following condition:

$$h_i(k)(h_i(k+1) - h_i(k)) < 0, \quad i = 1, \dots, m. \quad (10)$$

This condition indicates that, if $h_i(k) > 0$, then the controller should satisfy $h_i(k+1) < h_i(k)$ (meaning that the orbit approaches $h_i(k) = 0$ from one side, where $h_i(k) > 0$). If $h_i(k) < 0$, then the controller should be such that $h_i(k) > h_i(k+1)$ (the orbit approaches $h_i(k) = 0$ from the other side).

It would be interesting to see the inherent dynamical and geometrical properties of the system when the controlled orbit stays in the invariant manifolds $h(k) = 0$. It should be noted that, the type of “discrete sliding mode control” here is different from the conventional discrete sliding mode control but inherits some interesting dynamical properties from similar structures to both the continuous-time and discrete sliding mode controls.

On the invariant manifolds, an “equivalent” control, denoted $\Delta p_{av}(k)$, can be considered as [15]

$$\Delta p_{av}(k) = -(CB)^{-1}CA\Delta x(k) \quad (11)$$

which is obtained by solving the manifold equation $h(k+1) = 0$. The resulting dynamics on the manifolds is described by

$$\begin{aligned} \Delta x(k+1) &= A\Delta x(k) + B\Delta p(k) \\ &= (I - B(CB)^{-1}C)A\Delta x(k). \end{aligned} \quad (12)$$

It can be proved by using the techniques for continuous-time sliding mode control [10] that such dynamics is invariant with respect to C . Indeed, the linear projector, $P = I - B(CB)^{-1}C$, as discussed in [10], decomposes the space $R^{(n-1)}$ into two subspaces, the null space $N(C)$ and the range $R(C)$ of C , respectively. It can be shown [10] that P is a projector projecting $R^{(n-1)}$ onto $N(C)$ along $R(B)$. Since $\text{rank}(C) = m$ and $\text{rank}(B) = m$, we have $\text{rank}(B(CB)^{-1}C) = m$. Also, $\text{rank}(N(C)) = n - 1 - m$. Therefore, the projector P maps $R^{(n-1)}$ to $N(C)$; hence, P is at most of rank $n - 1 - m$.

The geometrical interpretation is that the effect of the projector P in the order of the given chaotic system is reduced, because the state vector is constrained to lie on $N(C)$, which is an $(n - 1 - m)$ -dimensional subspace. Consequently, we conclude that the chaotic system, when confined on the invariant

manifold so constructed, will have m zero eigenvalues along with $n - 1 - m$ eigenvalues being the transmission zeros of the “equivalent systems” [15], [10].

These $n - 1 - m$ eigenvalues can be arbitrarily assigned by properly chosen C , for example, by choosing all stable eigenvalues for the purpose of constructing the desired invariant manifolds. For a single-stable manifold, one can simply use the pole-assignment technique. For multiple stable manifolds, one can use some well-known algorithms [11], [16] (assigning eigenvalues within a unit disk rather than to the left hand side of the complex plane).

The strength of this new control method lies in that the stable-invariant manifolds are constructed independently of the system, stable and unstable Jacobian eigenvalues, and their associated eigenvectors. As such, it relaxes the restriction that the unstable eigenvalues have to be real in order to construct the desired invariant manifolds for chaos control. Furthermore, it presents a unified framework for designing an OGY-type of control strategy for chaotic systems of any order. It is also worth noting that the OGY method is a special case of our method. For example, for a third-order chaotic system, if the invariant manifold is chosen as one that uses a stable eigenvalue as its exponent, then our method is identical to the OGY method.

We have also noticed that this class of control methods, based on invariant manifolds, are robust against certain system variations. The most influential variation is the modeling error due to the linear regression based modeling, denoted as $\rho(k)$, such that

$$\Delta x(k+1) = A\Delta x(k) + B\Delta p(k) + \rho(k).$$

If $\rho(k) \in R(C)$ (i.e., the matching condition holds[15]), then the invariant manifold $h(k) = 0$ is invariant with respect to $\rho(k)$ [15]. However, since $\rho(k)$ is the modeling error which may not belong to $R(C)$, we will have

$$\begin{aligned} h(k+1) &= CA\Delta x(k) + CB\Delta p(k) + C\rho(k) \\ &= C\rho(k) \\ &\neq 0 \end{aligned}$$

meaning that the invariant manifold cannot be exactly maintained. To ensure that the orbit stay as close as possible to the intersection of the invariant manifolds, one can choose a control consisting of a time delayed term so as to compensate the mismatch

$$\Delta p(k) = -(CB)^{-1}CA\Delta x(k) - (CB)^{-1}C\rho(k-1). \quad (13)$$

This leads to

$$\begin{aligned} \|h(k+1)\| &= \|C\Delta x(k+1)\| \\ &= \|CA\Delta x(k) + CB\Delta p(k) + C\rho(k)\| \\ &= \|C\rho(k) - C\rho(k-1)\|. \end{aligned}$$

The time-delayed chaos control law (13) would be effective if $\rho(k)$ is a slowly varying variation and satisfies the condition

$$\|\rho(k+1) - \rho(k)\| \leq \gamma\|\rho(k)\|, \quad 0 < \gamma < 1. \quad (14)$$

Fig. 3. (a) Lorenz chaos in 3-D space. (b) Time responses of Lorenz chaos.

In fact, the modeling error can be expressed as

$$\rho(k) = \frac{\partial^2 F}{\partial x^2} \Delta^2 x \Big|_{x=x^*} + \frac{\partial^2 F}{\partial p^2} \Delta^2 p \Big|_{x=x^*} + \text{higher order terms.}$$

Since both the terms $(\partial^2 F / \partial x^2) \Delta^2 x|_{x=x^*}$ and $(\partial^2 F / \partial p^2) \Delta^2 p|_{x=x^*}$ have a fixed sign, the variation $\rho(k)$ is slowly varying, hence, the condition (14) can be easily satisfied.

The eigenvalues of the system dynamics on the Poincaré section under the invariant manifold based control can also be arbitrarily assigned by properly chosen C , using for example the algorithm [16] for higher order chaotic systems.

IV. SIMULATION STUDIES

In this section, we report two simulation results for two representative examples: the third-order continuous-time Lorenz system and the fourth-order discrete-time double rotor map. These are used to verify and demonstrate the effectiveness of the proposed chaos control method.

A. The Lorenz System

The Lorenz system is described by [21]

$$\dot{x}_1 = \sigma(x_2 - x_1) \quad (15)$$

$$\dot{x}_2 = rx_1 - x_2 - x_1x_3 \quad (16)$$

$$\dot{x}_3 = x_1x_2 - bx_3 \quad (17)$$

where the constants $\sigma, b, r > 1$. This system has three equilibria, $(0, 0, 0)^T$

$$p = (p_1, p_2, p_3)^T = \begin{bmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r-1 \end{bmatrix}$$

$$q = (q_1, q_2, q_3)^T = \begin{bmatrix} -\sqrt{b(r-1)} \\ -\sqrt{b(r-1)} \\ r-1 \end{bmatrix}. \quad (18)$$

Denote

$$r^* = \frac{\sigma(\sigma + b + 3)}{(\sigma - b - 1)}.$$

It is known that if $r > r^*$ then two eigenvalues of the linearized model have positive real parts, so that p and q are unstable.

The interest here is the case of $r > r^*$ when chaos occurs. A typical chaotic attractor is shown in Figs. 3(a) and (b). The parameter setting for the Lorenz system to display chaos is $\sigma = 10, r = 28, b = 8/3$. The interest of chaos control here is to channel the control input through (16). In [9], a period-one orbit was stabilized by an OGY-based method, the same as in [7]. Here, we choose the Poincaré section to be one parallel to the x - y plane at $x_3 = 29.921$. On the section, the unstable fixed point to be stabilized is located at $(x_1^*, x_2^*) = (13.740, 19.581)$.

By using least-squares' fitting on the sampled data around the unstable fixed point, a linear map was obtained as

$$\begin{bmatrix} \Delta x_1(k+1) \\ \Delta x_2(k+1) \end{bmatrix} = A \begin{bmatrix} \Delta x_1(k) \\ \Delta x_2(k) \end{bmatrix} + B \Delta p(k) \quad (19)$$

where

$$A = \begin{bmatrix} -0.2116 & 1.4727 \\ -0.6363 & 3.4385 \end{bmatrix}, \quad B = \begin{bmatrix} -0.3691 \\ -0.7773 \end{bmatrix}.$$

The eigenvalues of the second-order map (19) were found to be 3.1606 and 0.0664. We first transformed the second-order system (19) into the controllable canonical form, and then assigned the eigenvalue of the invariant manifold, which was of first order, to be 0.4. Transforming back to its original coordinates yields the controller using (13) as

$$\Delta p(k) = [-0.5107 \quad 3.8793] \begin{bmatrix} \Delta x_1(k) \\ \Delta x_2(k) \end{bmatrix} - C \rho(k-1)$$

where $\rho(k-1) = x(k) - Ax(k-1) - Bu(k-1)$ and $C = [0.0206 \quad -0.6297]$. The activating region of the OGY control is limited to $\{\Delta x(k): \Delta^2 x_1(k) + \Delta^2 x_2(k) < 1\}$.

The simulation was carried out with system initial state at (1.5, 5.4). The simulation results are shown in Fig. 4. Fig. 4(a) depicts the orbit of the controlled Lorenz chaos in the 3-D space. From Fig. 4(b), one can see that the control action gradually

Fig. 4. (a) Controlled Lorenz chaos in 3-D space. (b) Time responses of controlled Lorenz chaos.

decreases in time, and eventually the system orbit converges to the desired unstable periodic orbit. We also tested different initial conditions and our chaos control strategy was found to be robust. In fact, without the time delayed term, the control is effective as well.

B. The Fourth-Order Double Rotor Map

The fourth-order discrete kicked double rotor map is described by [18], [19]

$$\begin{bmatrix} Y(k+1) \\ Z(k+1) \end{bmatrix} = \begin{bmatrix} MZ(k) + Y(k) \\ LZ(k) + G(Y(k+1)) \end{bmatrix} \quad (20)$$

where $Y = (y_1, y_2)^\top \in S^1 \times S^1$ is the angular position of the rods at the instant of the k -th kick, $Z = (z_1, z_2)^\top$ is the angular velocity of the rods immediately after the k -th kick, $G(Y(k+1)) = (q_1 \sin x_1(k+1), q_2 \sin x_2(k+1))^\top$, S^1 is the circle (mod 2π), $q_i = f_0 l_i / I$, f_0 is the constant strength of the period impulse kicks, and l_i is the length of the rods ($i = 1, 2$).

Using the same values given in [19], the matrices are obtained as

$$M = \begin{bmatrix} 0.4860 & 0.2134 \\ 0.2134 & 0.6993 \end{bmatrix} \quad L = \begin{bmatrix} 0.2414 & 0.2726 \\ 0.2726 & 0.5140 \end{bmatrix}.$$

Fig. 5. (a) Time responses of x_1 and x_2 of the fourth-order double rotor. (b) Time responses of x_3 and x_4 of the fourth-order double rotor.

To stabilize its unstable periodic orbit of interest, we perturbed the strength of the period impulse kicks f , such that $f = f_0 + \Delta f$ with $f_0 = 9$. The map (20) was linearized about the unstable periodic orbit x^* , as

$$\Delta x(k+1) = A\Delta x(k) + B\Delta f(k) \quad (21)$$

where $\Delta x(k) = (X(k) - X^*Y(k) - Y^*)^\top$, $B = (0 \ 0 \ (l_1/I) \sin(x_1^*) \ (l_2/I) \sin(x_2^*))^\top$ and

$$A = \begin{bmatrix} I_2 & M \\ H(x^*) & L + H(x^*)M \end{bmatrix} \\ H = \begin{bmatrix} q_1 \cos x_1^* & 0 \\ 0 & q_2 \cos x_2^* \end{bmatrix}.$$

We selected two different unstable period-one fixed points from [19] as the control targets, which are

$$x_a^* = (1.4113, 3.9144, 4.5547, -10.3743)^\top$$

and

$$x_b^* = (4.8719, 2.3688, -4.5547, 10.3743)^\top$$

The control task was to force the system orbit to settle at these two fixed points alternatively. Using the algorithm in [16], the following matrices were obtained

$$C_a = (1.1171, -2.5878, 0.2284, -1.2040) \\ C_b = (-1.1171, 2.5878, -0.2284, 1.2040)$$

where C_a is for stabilizing x_a^* and C_b for x_b^* . These two matrices give the four desired eigenvalues of $(I - B(CB)^{-1}C)A$, as -0.15 , 0 , 0.11 , and 0.08 , respectively.

A controller capable of accomplishing the task was then constructed using (11), as

$$\Delta f_a = (-1.3480, -5.1702, -1.4851, -3.3466) \Delta x \\ \Delta f_b = (1.3480, 5.1702, 1.4851, 3.3466) \Delta x. \quad (22)$$

Note that no modeling error is involved in this simulation, so the time delayed term discussed earlier is not needed here.

The eigenvalues of the manifold $h(k)$ are then 0.11 , -0.15 , and 0.08 . The simulation result is shown in Fig. 5, where as one can see the angular positions and velocities of two rods are settled at the two fixed points alternatively, indicating the success of the control.

V. CONCLUSION

We have extended the OGY chaos control methodology based on the invariant manifold theory and the sliding mode control theory for model-free chaos control. Advantages of the proposed control method include: first, it preserves the spirit of the OGY control method in the sense that the controller drives the chaotic orbit to a stable manifold using small perturbation and only when control action is needed; second, the construction of the desired invariant manifolds are independent of the system Jacobian eigenvalues and eigenvectors, which is advantageous in the situation where the system Jacobian possesses multiple and/or complex conjugate eigenvalues for which the original OGY method is generally not applicable. This method is expected to be useful particularly for the control of hyperchaos pertaining to higher order chaotic systems with multiple positive Lyapunov exponents, for such applications as secure image communication and fluid mixing.

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