Homework # 1 Solutions

Problem 1 *Fourier Transforms for Real Signals.*
In what follows, assume the Fourier Transform definition

$$X(\Omega) = \int_R x(t) \exp[-j\Omega t] dt. \tag{1}$$

Assume that $x(t)$ is real. Show that:
1(a) The real and imaginary parts of the spectrum are

$$X_R(\Omega) = \int_R x(t) \cos(\Omega t) dt, \tag{2}$$

where $X_R(\Omega)$ is even and

$$X_I(\Omega) = -\int_R x(t) \sin(\Omega t) dt, \tag{3}$$

where $X_I(\Omega)$ is odd. **Please correct your notes by adding the minus sign to the definition of $X_I(\Omega)$.** Furthermore, show that:

$$X(-\Omega) = X^*(\Omega). \tag{4}$$

Proof: Start with

$$X(\Omega) = \int_R x(t) \exp[-j\Omega t] dt.$$

and note that

$$\exp[-j\Omega t] = \cos(\Omega t) - j \sin(\Omega t).$$

Then, it is clear that

$$X_R(\Omega) = \int_R x(t) \cos(\Omega t) dt,$$

and

$$X_I(\Omega) = -\int_R x(t) \sin(\Omega t) dt.$$

To show that $X_R(\Omega)$ is even, we need to show that $X_R(\Omega) = X_R(-\Omega)$. We have that:

$$X_R(-\Omega) = \int_R x(t) \cos(-\Omega t) dt,$$

which is the same as $X_R(\Omega)$ since $\cos(-\Omega t) = \cos(\Omega t)$.

To show that $X_I(\Omega)$ is odd, we need to show that $X_I(\Omega) = -X_I(-\Omega)$. We have that:

$$-X_I(-\Omega) = \int_R x(t) \sin(-\Omega t) dt.$$

which is the same as $X_I(\Omega)$ since $\sin(-\Omega t) = -\sin(\Omega t)$.

To show that $X(-\Omega) = X^*(\Omega)$, we need to note that:

$$X(-\Omega) = X_R(\Omega) - jX_I(\Omega) = X^*(\Omega).$$

by using the facts that X_R is even and that X_I is odd. **QED**.

1(b) Assume that $x(t)$ is also even: $x(t) = x(-t)$. Prove the expressions for $X_R(\Omega)$, $X_I(\Omega)$, $x(t)$ as given on the top of page 2.8. **Note that the expressions for the odd signals are actually correct!**

Proof: Since $x(t)$ is even, we have

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} x(t)\exp[-j\Omega t]dt \\ &= \int_{-\infty}^{0} x(t)\exp[-j\Omega t]dt + \int_{0}^{\infty} x(t)\exp[-j\Omega t]dt. \end{aligned}$$

Now, apply the change of variables: $\tau = -t$ to the first integral, and use $x(-\tau) = x(\tau)$ in the RHS to get:

$$\begin{aligned} X(\Omega) &= \int_{\infty}^{0} x(\tau)\exp[j\Omega\tau](-1)d\tau + \int_{0}^{\infty} x(t)\exp[-j\Omega t]dt \\ &= \int_{0}^{\infty} x(\tau)\cos(\Omega\tau)d\tau + \int_{0}^{\infty} x(\tau)\sin(\Omega\tau)d\tau + \int_{0}^{\infty} x(t)\cos(\Omega t)dt - \int_{0}^{\infty} x(t)\sin(\Omega t)dt \\ &= 2\int_{0}^{\infty} x(t)\cos(\Omega t)dt. \end{aligned}$$

Thus, it is clear that (as we had to show):

$$X_R = 2\int_{0}^{\infty} x(t)\cos(\Omega t)dt, \quad X_I = 0.$$

Now, we can reconstruct the signal by taking the inverse Fourier Transform:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{R} X(\Omega)\exp[j\Omega t]d\Omega \\ &= \frac{1}{2\pi}\int_{R} (X_R(\Omega) + jX_I(\Omega))\exp[j\Omega t]d\Omega \\ &= \frac{1}{2\pi}\int_{R} X_R(\Omega)\exp[j\Omega t]d\Omega \quad \text{since} \quad X_I(\Omega) = 0 \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} X_R(\Omega)\cos(\Omega t)d\Omega + j\frac{1}{2\pi}\int_{-\infty}^{0} X_R(\Omega)\sin(\Omega t)d\Omega \\ &\quad + \frac{1}{2\pi}\int_{0}^{\infty} X_R(\Omega)\cos(\Omega t)d\Omega + j\frac{1}{2\pi}\int_{0}^{\infty} X_R(\Omega)\sin(\Omega t)d\Omega. \end{aligned}$$

To continue, note that X_R is even as we have shown in $1(a)$. Then, apply the transformation $U = -\Omega$ to all the integrals that have limits from $-\infty$ to 0. This gives:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{\infty}^{0} X_R(U)\cos(Ut)(-1)dU + j\frac{1}{2\pi}\int_{\infty}^{0} X_R(U)\sin(-Ut)(-1)dU \\ &\quad + \frac{1}{2\pi}\int_{0}^{\infty} X_R(\Omega)\cos(\Omega t)d\Omega + j\frac{1}{2\pi}\int_{0}^{\infty} X_R(\Omega)\sin(\Omega t)d\Omega \\ &= \frac{1}{\pi}\int_{0}^{\infty} X_R(\Omega)\cos(\Omega t)dt. \end{aligned}$$

QED

Problem 2. Prove the time-scaling property of page 2.10.

Proof: For $y(t) = x(at)$, we have that

$$Y(\Omega) = \int_R x(at)\exp[-j\Omega t]dt.$$

Now, consider the transformation $\tau = at$, for $a > 0$. We will come back and consider the case when $a < 0$ later. We have that

$$\begin{aligned} Y(\Omega) &= \int_R x(\tau)\exp[-j\Omega\tau/a](1/a)d\tau \\ &= \frac{1}{a}\int_R x(\tau)\exp[-j(\Omega/a)\tau]d\tau \\ &= \frac{1}{a}X\left(\frac{\Omega}{a}\right) \end{aligned}$$

which agrees with the time-scaling property for $a > 0$. For $a < 0$, we similarly have

$$\begin{aligned} Y(\Omega) &= \int_{\infty}^{-\infty} x(\tau)\exp[-j\Omega\tau/a](1/a)d\tau \\ &= -\frac{1}{|a|}\int_{\infty}^{-\infty} x(\tau)\exp[-j(\Omega/a)\tau]d\tau \\ &= \frac{1}{|a|}\int_R x(\tau)\exp[-j(\Omega/a)\tau]d\tau \\ &= \frac{1}{|a|}X\left(\frac{\Omega}{a}\right) \end{aligned}$$

QED

Problem 3. Prove the time shifting property of page 2.11.

Proof: For $y(t) = x(t - t_0)$, we have

$$Y(\Omega) = \int_R x(t - t_0) \exp[-j\Omega t] dt.$$

Now, consider the transformation $\tau = t - t_0$. We have

$$\begin{aligned} Y(\Omega) &= \int_R x(\tau) \exp[-j\Omega(\tau + t_0)] d\tau \\ &= \int_R x(\tau) \exp[-j\Omega\tau] \exp[-j\Omega t_0] d\tau \\ &= \exp[-j\Omega t_0] X(\Omega). \end{aligned}$$

QED

Problem 4. Derive the Fourier Transform of $2p_1(t) + 3p_1(t - 1) - 4p_1(t - 2)$ (page 2.27).
Ans: Since we have the Fourier Transform

$$P(\Omega) = \frac{2 \sin \Omega}{\Omega},$$

we apply linearity and the time shifting property to get

$$\mathcal{F}\{2p_1(t) + 3p(t - 1) - 4p_1(t - 2)\} = 2P(\Omega) + 3P(\Omega) \exp(-j\Omega) - 4P(\Omega) \exp(-2j\Omega).$$

Problem 5. Derive the Fourier Transform of $\exp(-a|t|)$ (page 2.29).
Ans: Let $x(t) = exp(-a|t|)$. Note that $|t| = t$ for $t > 0$ and $|t| = -t$ for $t < 0$. Also note that we can use the Fourier transform of $\exp(-at)u(t)$. We have

$$\begin{aligned} X(\Omega) &= \int_R \exp(-a|t|) \exp[-j\Omega t] dt \\ &= \int_{-\infty}^{0} \exp(-a|t|) \exp[-j\Omega t] dt + \int_{0}^{\infty} \exp(-a|t|) \exp[-j\Omega t] dt \\ &= \int_{-\infty}^{0} \exp(at) \exp[-j\Omega t] dt + \int_{0}^{\infty} \exp(-at) \exp[-j\Omega t] dt. \end{aligned}$$

Now, note that the second integral is the Fourier transform of $\exp(-at)u(t)$. For the first integral, we apply the transformation $\tau = -t$ to get

$$\begin{aligned} X(\Omega) &= -\int_{\infty}^{0} \exp(-a\tau)\exp[-j\Omega(-\tau)]d\tau + \frac{1}{a+j\Omega} \\ &= \int_{0}^{\infty} \exp(-a\tau)\exp[-j(-\Omega)\tau]d\tau + \frac{1}{a+j\Omega} \end{aligned}$$

Now, recognize the first integral as the Fourier transform of $\exp(-a\tau)u(\tau)$ where the frequency Ω has been replaced by $-\Omega$. This gives

$$\begin{aligned} X(\Omega) &= \frac{1}{a-j\Omega} + \frac{1}{a+j\Omega} \\ &= \frac{a+j\Omega+a-j\Omega}{a^2+\Omega^2} \\ &= \frac{2a}{a^2+\Omega^2}. \end{aligned}$$