Homework # 2 (short) Solutions

Problem 1 Derive equivalent time-domain expressions for

$$\frac{d}{dt}u(t-5)$$

and

$$\frac{d^2}{dt^2}u(t)$$

in the sense of generalized functions. Please study the proof of $du(t)/dt = \delta(t)$ prior to answering this question.

Ans: We begin with $\frac{d}{dt}u(t-5)$. Let $g(t)$ be an admissible function and consider

$$\int_{-\infty}^{\infty} \frac{d}{dt}u(t-5)g(t)dt. \tag{1}$$

To continue, simply recall the formula of integration by parts

$$\int xdy = xy - \int ydx. \tag{2}$$

We apply (2) to (1) by moving the differentiation operator from $\frac{d}{dt}u(t-5)$ to the admissible function $g(t)$ using:

$$\begin{aligned} x(t) = g(t) &\Rightarrow \frac{d}{dt}x(t) = \frac{d}{dt}g(t), \\ \frac{d}{dt}y(t) = \frac{d}{dt}u(t-5) &\Rightarrow y(t) = u(t-5). \end{aligned}$$

We thus have that (1) becomes

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d}{dt}u(t-5)g(t)dt &= u(t-5)g(t)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t-5)\frac{d}{dt}g(t)dt \\ &= 0 - \int_{5}^{\infty} \frac{d}{dt}g(t)dt \\ &\quad \text{since} \quad g(\infty) = g(-\infty) = 0 \quad \text{and} \quad u(t-5) = 0 \quad \text{for} \quad t < 5. \\ &= -(g(\infty) - g(5)) \\ &= g(5). \end{aligned}$$

Thus, it appears that $\frac{d}{dt}u(t-5)$ samples the admissible function at $t = 5$. Similarly, for $\delta(t-5)$ we have

$$\int_{-\infty}^{\infty} \delta(t-5)g(t)dt = g(5).$$

Thus, it is clear that $\frac{d}{dt}u(t-5)$ and $\delta(t-5)$ operate in the same way. We thus have that

$$\frac{d}{dt}u(t-5) = \delta(t-5).$$

Similarly for $\frac{d^2}{dt^2}u(t)$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{d^2}{dt^2}u(t)g(t)dt &= \frac{d}{dt}u(t)g(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dt}u(t)\frac{d}{dt}g(t)dt \\
&= 0 - \int_{-\infty}^{\infty} \frac{d}{dt}u(t)\frac{d}{dt}g(t)dt \\
&= -\, u(t)\frac{d}{dt}g(t)\Big|_{-\infty}^{\infty} + \int_{0}^{\infty} \frac{d^2}{dt^2}g(t)dt \\
&= -\frac{d}{dt}g(0).
\end{aligned}$$

Thus, it looks like $\frac{d^2}{dt^2}u(t)$ returns a sample of the derivative at the origin. Similarly, consider the action of $\frac{d}{dt}\delta(t)$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{d}{dt}\delta(t)g(t)dt &= \delta(t)g(t)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t)\frac{d}{dt}g(t)dt \\
&= -\frac{d}{dt}g(0).
\end{aligned}$$

Thus, it is clear that

$$\frac{d^2}{dt^2}u(t) = \delta(t).$$

Problem 2 Derive the Fourier Transforms of

$$\frac{d}{dt}u(t-5),\ \frac{d^2}{dt^2}u(t),\ \cos(\Omega_0 t - 3)u(t),\ \sin(\Omega_0 t - 5)u(t)$$

using properties of the Fourier Transform.

Notes: For full-credit for the second problem, you need to do the derivations by hand and verify your answers using Mathematica. Also, note: (i) the unit step function is represented by `UnitStep`, (ii) the impulse function is represented by `DiracDelta` and (iii) the Fourier Transform is evaluated using (for example): `FourierTransform[UnitStep[t], t, Ω, FourierParameters -> {1, -1}]` .

2(a) For $\frac{d}{dt}u(t-5)$ note that we have a delay $t \to t-5$ followed by differentiation. To see the properties that we need here, simply break these operations one at a time

$$\begin{aligned} g(t) &= \frac{d}{dt}u(t-5) \\ &= \frac{d}{dt}h(t) \quad \text{where} \quad h(t) = u(t-5) \end{aligned}$$

Using the Fourier Transform properties we get

$$\begin{aligned} G(\Omega) &= \mathcal{F}\left\{\frac{d}{dt}u(t-5)\right\} \\ &= j\Omega H(\Omega) \\ &= j\Omega \exp(-j5\Omega)U(\Omega) \\ &= j\Omega \exp(-j5\Omega)\left(\frac{1}{j\Omega} + \pi\delta(\Omega)\right) \\ &= \exp(-j5\Omega) + j\Omega \exp(-j5\Omega)\pi\delta(\Omega). \end{aligned}$$

Now, from the properties of δ we have that

$$f(t)\delta(t) = f(0)\delta(t) \quad \text{provided that } f(t) \text{ is continuous at zero.}$$

We apply this property here to get that

$$\begin{aligned} j\Omega \exp(-j5\Omega)\pi\delta(\Omega) &= \left. j\Omega \exp(-j5\Omega)\pi \right|_{\Omega=0} \delta(\Omega) \\ &= 0. \end{aligned}$$

We thus have that

$$G(\Omega) = \exp(-j5\Omega).$$

Validation using Mathematica

In `Mathematica` we verify this using

```
In[1]:= FourierTransform[D[UnitStep[t-5], {t,1}],
                        t, Omega, FourierParameters->{1,-1}]
Out[1]= Exponential(-5 I Omega)
```

2(b) For $\frac{d^2}{dt^2}u(t)$ we have

$$\begin{aligned} \mathcal{F}\left\{\frac{d^2}{dt^2}u(t)\right\} &= (j\Omega)^2 U(\Omega) \\ &= -\Omega^2\left(\frac{1}{j\Omega} + \pi\delta(\Omega)\right) \\ &= j\Omega. \end{aligned}$$

Validation using Mathematica
Several problems came up when trying to verify this Fourier Transform with Mathematica. My guess is that there are bugs associated with how Mathematica handles Fourier parameters. Fortunately, all of these problems can be easily fixed (for this problem). To see the problems, note that the following evaluations:

```
In[1]:= ExpandAll[FourierTransform[D[UnitStep[t], {t,2}], t, Omega]]
Out[1]= -I Omega / Sqrt [2 Pi]

In[2]:= ExpandAll[FourierTransform[D[UnitStep[t], {t,2}], t, Omega,
                    FourierParameters -> {1,-1}]]
Out[2]= I Omega / Sqrt [2 Pi]
```

```
In[3]:= FourierTransform[DiracDelta[t],t,Omega]
Out[3]= 1 / Sqrt [2 Pi]

In[4]:= FourierTransform[(D[UnitStep[t], {t,2}]), t, Omega]
Out[4]= -I Omega FourierTransform[DiracDelta[t],t,Omega]

In[5]:= FourierTransform[DiracDelta[t],t,Omega, FourierParameters->{1,-1}]
Out[5]= 1
```

give contradictory results. I think the problem is with the way that Mathematica propagates `FourierParameters` through its evaluations.

To fix this, we can simply note that Mathematica's `InverseFourierTransform` is

$$x(t) = \frac{1}{\sqrt{2\pi}} \int X(\Omega) \exp[-j\Omega t] d\Omega. \tag{3}$$

Thus, an equivalent way to evaluate the Fourier Transform that we need is by using the `InverseFourierTransform`:

```
FT [x_] := InverseFourierTransform[x, t, Omega] Sqrt[2 Pi]
```

We can now use `FT` to take the Fourier transforms using the DSP definition

```
In[1]:= FT[DiracDelta[t]]
Out[1]= 1
```

Notice that the new definition assumes that functions are a function of `t` and `Omega` denotes the frequency variable.

Similarly, we want to have `IFT` for taking the inverse Fourier Transform. Similar to what we did before, we define:

```
IFT[x_] := FourierTransform[x, Omega, t] /Sqrt[2 Pi]
```

We can now use our definition to verify some transforms;

```
In[1]:= IFT[1]
Out[1]= DiracDelta[t]
```

We also verify the first problem here

```
In[1]:= FT[D[UnitStep[t-5], {t,1}]]
Out[1]= Exponential (-5 I Omega)
```

For the second derivative of the unit step, we now have the right answer:

```
In[1]:= ExpandAll[FT[UnitStep''[t]]]
Out[1]= I Omega
```

2(c) To take the Fourier transform of $\cos(\Omega_0 t - 3)u(t)$, we note that it is a product of $\cos(\Omega_0 t - 3)$ and $u(t)$. We will tackle each separately and then convolve their Fourier transforms.

We let $g(t) = \cos(\Omega_0 t - 3)$ and $h(t) = \cos(\Omega_0 t)$. Then:

$$\begin{aligned} g(t) &= \cos(\Omega_0 t - 3) \\ &= h(t - a). \end{aligned}$$

To get a note

$$h(t - a) = \cos(\Omega_0 t - \Omega_0 a) = \cos(\Omega_0 t - 3)$$

which gives that $a = 3/\Omega_0$. We thus have

$$\begin{aligned} G(\Omega) &= \exp(-j3\Omega/\Omega_0)H(\Omega) \\ &= \pi \exp(-j3\Omega/\Omega_0) \left(\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)\right). \end{aligned}$$

Recall the Fourier transform of $u(t)$

$$U(j\Omega) = \frac{1}{j\Omega} + \pi\delta(\Omega)$$

Recall the multiplication property

$$\mathcal{F}(g(t)u(t)) = \frac{1}{2\pi} G(\Omega) * U(\Omega).$$

We thus have

$$
\begin{aligned}
\frac{1}{2\pi}G(\Omega) * U(\Omega) &= \frac{1}{2\pi}\pi \exp(-j3\Omega/\Omega_0)\left[\delta(\Omega+\Omega_0)+\delta(\Omega-\Omega_0)\right] * \left(\frac{1}{j\Omega}+\pi\delta(\Omega)\right) \\
&= 0.5\exp(-j3\Omega/\Omega_0)\delta(\Omega+\Omega_0) * \frac{1}{j\Omega} \\
&\quad + 0.5\pi\exp(-j3\Omega/\Omega_0)\delta(\Omega+\Omega_0) * \delta(\Omega) \\
&\quad + 0.5\exp(-j3\Omega/\Omega_0)\delta(\Omega-\Omega_0) * \frac{1}{j\Omega} \\
&\quad + 0.5\pi\exp(-j3\Omega/\Omega_0)\delta(\Omega-\Omega_0) * \delta(\Omega) \\
&= \left. 0.5\exp(-j3\Omega/\Omega_0)\right|_{\Omega=-\Omega_0}\delta(\Omega+\Omega_0) * \frac{1}{j\Omega} \\
&\quad + \left. 0.5\pi\exp(-j3\Omega/\Omega_0)\right|_{\Omega=-\Omega_0}\delta(\Omega+\Omega_0) \\
&\quad + \left. 0.5\exp(-j3\Omega/\Omega_0)\right|_{\Omega=\Omega_0}\delta(\Omega-\Omega_0) * \frac{1}{j\Omega} \\
&\quad + \left. 0.5\pi\exp(-j3\Omega/\Omega_0)\right|_{\Omega=\Omega_0}\delta(\Omega-\Omega_0) \\
&= 0.5\exp(j3)\delta(\Omega+\Omega_0) * \frac{1}{j\Omega} \\
&\quad + 0.5\pi\exp(j3)\delta(\Omega+\Omega_0) \\
&\quad + 0.5\exp(-j3)\delta(\Omega-\Omega_0) * \frac{1}{j\Omega} \\
&\quad + 0.5\pi\exp(-j3)\delta(\Omega-\Omega_0) \\
&= 0.5\exp(j3)\frac{1}{j(\Omega+\Omega_0)} \\
&\quad + 0.5\pi\exp(j3)\delta(\Omega+\Omega_0) \\
&\quad + 0.5\exp(-j3)\frac{1}{j(\Omega-\Omega_0)} \\
&\quad + 0.5\pi\exp(-j3)\delta(\Omega-\Omega_0)
\end{aligned}
$$

We can combine the first and the third terms to get

$$
\begin{aligned}
\frac{1}{2\pi}G(\Omega) * U(\Omega) &= \frac{0.5\exp(j3)j(\Omega-\Omega_0)+0.5\exp(-j3)j(\Omega+\Omega_0)}{j(\Omega+\Omega_0)j(\Omega-\Omega_0)} \\
&\quad + 0.5\pi\exp(j3)\delta(\Omega+\Omega_0)+0.5\pi\exp(-j3)\delta(\Omega-\Omega_0) \\
&= \frac{j\Omega\cos(3)+\Omega_0\sin(3)}{\Omega_0^2-\Omega^2} \\
&\quad + 0.5\pi\exp(j3)\delta(\Omega+\Omega_0)+0.5\pi\exp(-j3)\delta(\Omega-\Omega_0)
\end{aligned}
$$

Thus, the final answer is given by

$$\mathcal{F}\left\{\cos(\Omega_0 t - 3)u(t)\right\} = 0.5\pi\exp(j3)\delta(\Omega+\Omega_0) + 0.5\pi\exp(-j3)\delta(\Omega-\Omega_0) + \frac{j\Omega\cos(3)+\Omega_0\sin(3)}{\Omega_0^2-\Omega^2}. \tag{4}$$

Validation using Mathematica
Using Mathematica, we only can evaluate the individual terms:

```
In[1]:= ExpandAll[FT[UnitStep[t]]]
Out[1]= -I/Omega + Pi DiracDelta[Omega]

In[2]:= ExpandAll[FT[Cos[Omega0 t- 3]]]
Out[2]= Exp [-3I] Pi DiracDelta[Omega-Omega0]
        + Exp[3I] Pi DiracDelta[Omega+Omega0]
```

Unfortunately, the generalized delta functions do not appear in the full expansion by Mathematica:

```
In[3]:= ExpandAll[FT [Cos[Omega0 t - 3] UnitStep[t]]]
Out[3]= I Omega Sqrt[1/Omega0^2] Sqrt[Omega0^2]Cos[3]/(-Omega^2+Omega0^2)
        Sqrt[Omega0^2] Sign[Omega0] Sin[3]/(-Omega^2+Omega0^2)
```

We do not really know why Mathematica misses these terms. However, there is strong evidence that Mathematica does not use convolution for the evaluation. To see this, note that Mathematica does not define continuous-space convolution. It also does not define convolutions with delta functions. It is possible to try to fix this, but it will take us well-beyond the scope of the assignment. I did provide some hints on how to fix this in class.

On the other hand, note that the terms that we do get from Mathematica are in agreement with our results. To verify this, note that `Sqrt[Omega0`^{2}`]Sign[Omega0]` is simply `Omega0` (Mathematica always returns the positive root).

2(d) Similarly for $\sin(\Omega_0 t - 5)u(t)$, we let $g(t) = \sin(\Omega_0 t - 5)$. We have (after simplifications):

$$\begin{aligned}\frac{1}{2\pi}G(\Omega) * U(\Omega) &= \frac{1}{2\pi}\left[-j\exp(-j5)\pi\delta(\Omega-\Omega_0) + j\exp(j5)\pi\delta(\Omega+\Omega_0)\right] * \left(\frac{1}{j\Omega} + \pi\delta(\Omega)\right)\\ &= 0.5\pi\left[-j\exp(-j5)\pi\delta(\Omega-\Omega_0) + j\exp(j5)\pi\delta(\Omega+\Omega_0)\right]\\ &\quad - j0.5\exp(-j5)\frac{1}{j(\Omega-\Omega_0)} + j0.5\exp(j5)\frac{1}{j(\Omega+\Omega_0)}\end{aligned}$$

For the last two terms, we have:

$$
\begin{aligned}
& -j0.5\exp(-j5)\frac{1}{j(\Omega-\Omega_0)} + j0.5\exp(j5)\frac{1}{j(\Omega+\Omega_0)} \\
& \quad = \frac{-0.5(\cos(5)-j\sin(5))(\Omega+\Omega_0)+0.5(\cos(5)+j\sin(5))(\Omega-\Omega_0)}{(\Omega-\Omega_0)(\Omega+\Omega_0)} \\
& \quad = \frac{-0.5\Omega\cos(5)-0.5\Omega_0\cos(5)+j0.5\sin(5)\Omega+j0.5\sin(5)\Omega_0}{\Omega^2-\Omega_0^2} \\
& \quad + \frac{0.5\cos(5)\Omega-0.5\cos(5)\Omega_0+j0.5\sin(5)\Omega-0.5j\sin(5)\Omega_0}{\Omega^2-\Omega_0^2} \\
& = \frac{-\Omega_0\cos(5)+j\Omega\sin(5)}{\Omega^2-\Omega_0^2}
\end{aligned}
$$

For the final answer we get:

$$
\begin{aligned}
\mathcal{F}\left\{\sin(\Omega_0 t-5)u(t)\right\} = -j0.5\pi\exp(-j5)\pi\delta(\Omega-\Omega_0)+j0.5\pi\exp(j5)\pi\delta(\Omega+\Omega_0) \\
+\frac{-\Omega_0\cos(5)+j\Omega\sin(5)}{\Omega^2-\Omega_0^2}
\end{aligned}
$$

Validation using Mathematica

Similar to 2(c), we can use Mathematica to verify the individual terms. For the sine-term, we correctly get:

```
In[1]:= ExpandAll[FT[Sin[Omega0 t - 5]]]
Out[1]= -I Exp[-5 I] Pi DiracDelta[Omega - Omega0]
       + I Exp[5 I]  Pi DiracDelta[Omega + Omega0]
```

For the full-transform, we again **only** get the non-generalized terms:

```
In[2]:= ExpandAll[FT[Sin[Omega0 t -5] UnitStep[t]]]
Out[2]= -Sqrt[Omega0^2] Cos[5] Sign[Omega0]/(Omega^2 - Omega0^2)
       + I Omega Sqrt[1/Omega0^2] Omega0^2 Sin[5]/(Omega^2 - Omega0^2)
```

Fortunately, the common terms appear to be in agreement with our evaluation.