ECE 539 Digital Signal Processing
Midterm Exam

Name: Solutions

Problem 1	30/**30**
Problem 2	30/**20**
Problem 3	20/**20**
Problem 4	30/**30**
Total	100/**100**

Good Luck!

Problem 1 (30 points total)

1(a)(10 points) Evaluate the continuous-time Fourier Transform of:

$$\cos(\Omega_0 t)\big(u(t)-u(t-t_0)\big).$$

A solution: $\cos(\Omega_0 t)\, u(t) \xleftrightarrow{F} \frac{1}{2}\left[U(\Omega-\Omega_0)+U(\Omega+\Omega_0)\right]$

$u(t) \xleftrightarrow{F} \frac{1}{j\Omega}+\pi\delta(\Omega)$, $u(t-t_0) \xleftrightarrow{F} \exp(-j\Omega t_0) \cdot U(\omega)$

$$F\{\cos(\Omega_0 t)(u(t)-u(t-t_0))\} = \frac{1}{2}\left[\frac{1}{j(\Omega-\Omega_0)}+\pi\delta(\Omega-\Omega_0) + \frac{1}{j(\Omega+\Omega_0)}+\pi\delta(\Omega+\Omega_0)\right] - \frac{\exp(-j\Omega t_0)}{2}\left[\frac{1}{j(\Omega-\Omega_0)}+\pi\delta(\Omega-\Omega_0) + \frac{1}{j(\Omega+\Omega_0)}+\pi\delta(\Omega+\Omega_0)\right]$$

which can be simplified!

1(b)(10 points) Evaluate the discrete-time Fourier Transform of:

$$\cos(\omega_0 n)\big(u(n)-u(n-n_0)\big).$$

Perhaps the easiest solution?

$$\cos(\omega_0 n) = \tfrac{1}{2}\left(\exp(j\omega_0 n)+\exp(-j\omega_0 n)\right)$$

$$u(n)-u(n-n_0) = \begin{cases} 1, & 0 \le n \le n_0-1 \\ 0, & \text{otherwise.} \end{cases}$$

$r = e^{j(\omega_0-\omega)}$ $\quad$ $r = e^{-j(\omega_0+\omega)}$

$$X(e^{j\omega}) = \frac{1}{2}\sum_{n=0}^{n_0} \exp(j\omega_0 n)\cdot\exp(-j\omega n) + \frac{1}{2}\sum_{n=0}^{n_0}\exp(-j\omega_0 n)\cdot\exp(-j\omega n)$$

Both are finite geometric series.

$$X(e^{j\omega}) = \frac{1}{2}\,\frac{1-\exp[j(n_0+1)(\omega_0-\omega)]}{1-\exp[j(\omega_0-\omega)]} + \frac{1}{2}\,\frac{1-\exp[-j(n_0+1)(\omega_0+\omega)]}{1-\exp[-j(\omega_0+\omega)]}$$

1(~~b~~c)(5 points) Evaluate the Z-Transform of:

$$\cos(\omega_0 \cancel{t}\, n)(u(n)-u(n-n_0))$$

and specify the ROC.

$$X(z) = \tfrac{1}{2}\sum_{n=0}^{n_0} \exp(jw_0 n)\, z^{-n} + \tfrac{1}{2}\sum_{n=0}^{n_0} \exp(-jw_0 n)\, z^{-n}$$

$$= \tfrac{1}{2}\sum_{n=0}^{n_0}\left[\exp(jw_0)\cdot z^{-1}\right]^n + \tfrac{1}{2}\sum_{n=0}^{n_0}\left[\exp(-jw_0)z^{-1}\right]^n$$

$$= \tfrac{1}{2}\,\frac{1-\left[\exp(jw_0)z^{-1}\right]^{n_0+1}}{1-\left[\exp(jw_0)z^{-1}\right]} + \tfrac{1}{2}\,\frac{1-\left[\exp(-jw_0)z^{-1}\right]^{n_0+1}}{1-\left[\exp(-jw_0)z^{-1}\right]}$$

$\text{ROC} = \{z : 0 < |z| \leq \infty\}$ for this finite length seq.

1(~~c~~d)(5 points) Comment (briefly) about the relationships among the answers in 1(a), 1(b) and 1(c).

Clearly, 1(b) is a discretization of 1(a) for unit interval sampling, while 1(c) reduces to 1(b) for $z = \exp(jw)$. Furthermore, due to sampling, 1(b) & 1(c) are periodic in w with $z = e^{jw}$ see problem 4 for more details on sampling.

Problem 2 (20 points total)

2(a)(10 points) Suppose that $H(z)$ represents the impulse response of a stable system (not necessarily causal) and it is given by

$$H(z) = \frac{1}{z^2 - 5z + 6}.$$

Find $h(n)$. Since $H(z)$ is stable, its ROC must contain the unit circle (BIBO).

$$H(z) = \frac{1}{z-3} - \frac{1}{z-2} = \frac{z^{-1}}{1-3z^{-1}} - \frac{z^{-1}}{1-2z^{-1}}$$

Since the poles are outside the unit circle, we have $|z|<2$ which implies that the system is anti-causal. We have:

$$Z^{-1}\left\{\frac{1}{1-3z^{-1}}\right\} = -3^n u(-n-1), \quad Z^{-1}\left\{\frac{1}{1-2z^{-1}}\right\} = -2^n u(-n-1)$$

$$\Rightarrow \boxed{h(n) = -3^{n-1} \cdot u(-n) + 2^{n-1} u(-n)}$$

Note: $\underline{\underline{-(n-1)-1 = -n}}$

2(b)(10 points) Compute the frequency response for:

$$y(n) - \frac{1}{2} y(n-1) = x(n) + 2x(n-1) + x(n-2).$$

$$Y(z) - \tfrac{1}{2} z^{-1} Y(z) = X(z) + 2z^{-1} X(z) + z^{-2} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2} z^{-1}}$$

For the frequency response:

$$\boxed{H(e^{j\omega}) = \frac{1 + 2e^{-j\omega} + e^{-2j\omega}}{1 - \frac{1}{2} e^{-j\omega}}}$$

Problem 3 (20 points) Suppose that the impulse response of an LTI system (**L**) is supported over a finite region. By this, we mean that:

$h(n) \neq 0$ for $-\infty < N_1 < n < N_2 < \infty$ and

$h(n) = 0$ for $n < N_1$ or $n > N_2$.

In this case, it is easy to show that **L** is BIBO stable **<u>if and only if</u>** it has an absolutely summable impulse response.

You are asked to prove this simple statement directly. You will receive no credit if you assume the more general statement that this holds for both finitely and infinitely supported impulse responses.

<u>Proof</u>: <u>Sufficiency</u>($\Leftarrow$) <u>Suppose</u> $\sum_{m=N_1}^{N_2} |h(m)| < \infty$ and $x(n)$ is bounded for all n: $|x(n)| < A < \infty$.

Then:

$$|y(n)| = \left|\sum_{m=N_1}^{N_2} h(m)\, x(n-m)\right| \leq \sum_{m=N_1}^{N_2} |h(m)| \cdot A = \left(\sum_{m=N_1}^{N_2} |h(m)|\right) \cdot A < \infty.$$

$\Rightarrow$ L is BIBO $\blacksquare$

<u>Necessity</u> ($\Rightarrow$)

Suppose that L is BIBO. Then, let $x(n) = \delta(n)$ which is bounded.

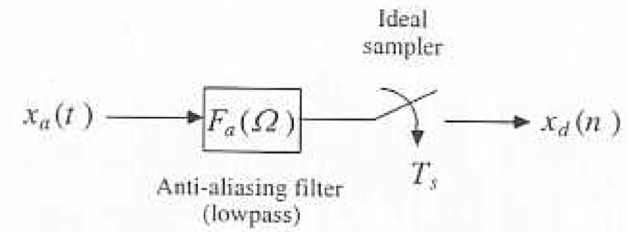
We have $|y(n)| = |h(n) * \delta(n)| < \infty$
$= |h(n)| < \infty$, all n.

<u>Say</u> $|h(n)| < B$. Then:

$$\sum_{m=N_1}^{N_2} |h(m)| < (N_2 - N_1)B < \infty$$

$\blacksquare$

Problem 4 (30 points total). Consider the simple sampling system with an anti-aliasing, lowpass filter:



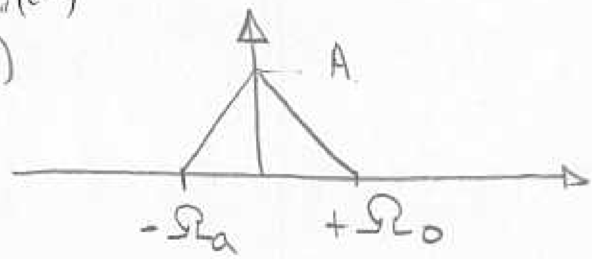
For the anti-aliasing filter, we have an ideal low-pass filter:

$$F_a(\Omega) = \begin{cases} 1; & |\Omega| \le \Omega_c \\ 0; & |\Omega| > \Omega_c \end{cases}$$

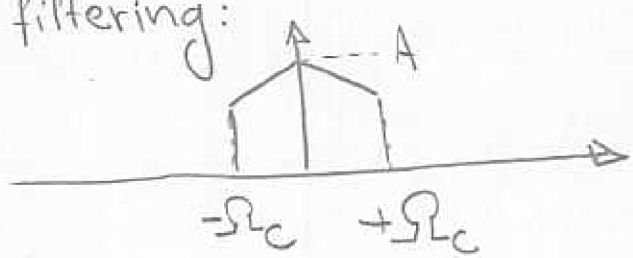
4(a)(7 points) Suppose that T_S is sufficiently small. Please sketch the magnitude spectra of:

- the analog signal
- the analog signal after lowpass filtering
- the sampled signal: $X_d(e^{j\omega})$

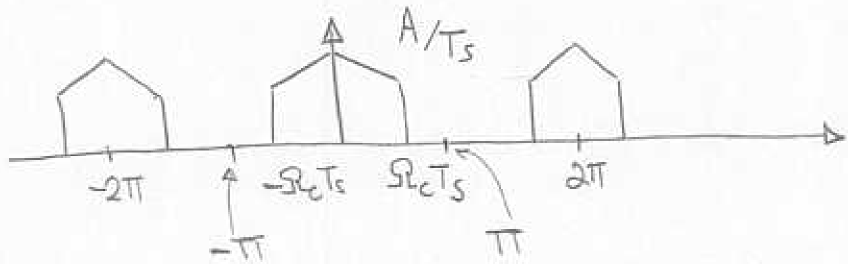
$X_a(\Omega)$



After low-pass filtering:



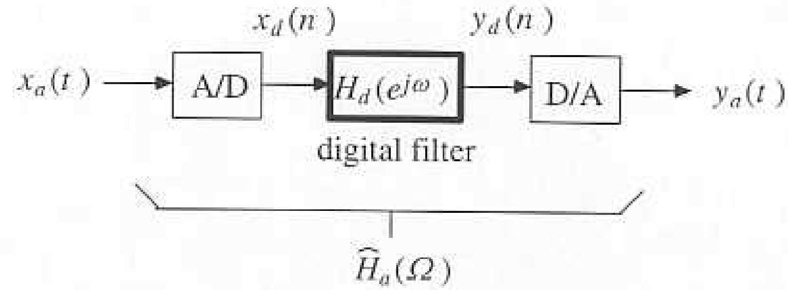
For sufficiently small T_s:



T_s must be such that: $\Omega_c T_s < 2\pi - \Omega_c T_s$

$\Rightarrow \boxed{T_s < \dfrac{\pi}{\Omega_c}} \Rightarrow \Omega_c < \dfrac{\pi}{T_s}$

4(b)(5 points)



For the system described in 4(a), recall that the response $Y_d\left(e^{j\omega}\right)$ of the **digital filter** with frequency response $H_d\left(e^{j\omega}\right)$ is:

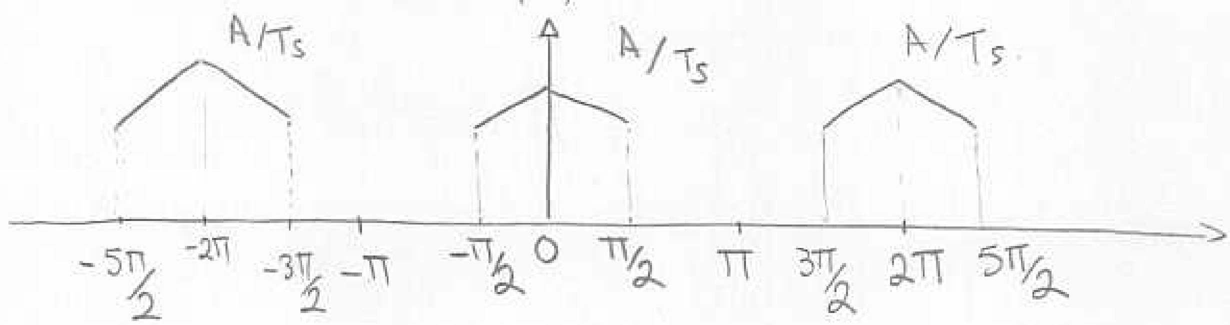
$$Y_d\left(e^{j\omega}\right) = \frac{1}{T_s}\sum_{m=-\infty}^{\infty} X_a\left(\frac{\omega-2\pi m}{T_s}\right)F_a\left(\frac{\omega-2\pi m}{T_s}\right)H_d\left(e^{j\omega}\right)$$

(with anti-aliasing).

Suppose that the frequency response $H_d\left(e^{j\omega}\right)$ is given by:

$$H_d\left(e^{j\omega}\right)=\begin{cases}1, & -\pi/2<\omega<\pi/2\\ 0, & \text{otherwise}\end{cases}.$$

Sketch the frequency response of $Y_d\left(e^{j\omega}\right)$.



This the plot for $\Omega_c T_s > \pi/2$.

If $\Omega_c T_s < \pi/2$, then the plot in 4(a) remains the same for this case.

4(c)(5 points) Give an expression for the maximum sampling period for 4(b) so that the output remains the same. Your expression should be in terms of the sampling period given in 4(a).

We want $2\pi - \Omega_c T_s' > \pi/2$.

$\Rightarrow \frac{3\pi}{2\Omega_c} > T_s'$. For critical sampling,

$\Omega_c < \frac{\pi}{T_s} \Rightarrow \frac{1}{\Omega_c} > \frac{T_s}{\pi} \Rightarrow \frac{3\cancel{\pi}}{2} \times \frac{T_s}{\cancel{\pi}} > T_s'$ $\Rightarrow$ $\boxed{\frac{3T_s}{2} > T_s'}$

NB: "Same" is misleading. If you thought same = no-aliasing $\Rightarrow T_s' = \frac{T_s}{2}$

4(d)(3 points) Suppose that

$$H_d\left(e^{j\omega}\right) = \begin{cases} 1, & -\pi/a < \omega < \pi/a \\ 0, & \text{otherwise.} \end{cases}$$

Repeat 4(c) for this case.

Similarly: $2\pi - \Omega_c T_s' > \pi/a$ & $\frac{1}{\Omega_c} > \frac{T_s}{\pi}$

$\Rightarrow (2 - 1/a)\cancel{\pi} \times \frac{T_s}{\cancel{\pi}} > T_s' \Rightarrow \boxed{(2 - 1/a) T_s > T_s'}$

4(e)(7 points) Suppose that:

$$H_d\left(e^{j\omega}\right) = \begin{cases} 1, & \omega_1 < \omega < \omega_2 \text{ and } -\omega_2 < \omega < -\omega_1 \\ 0, & \text{otherwise.} \end{cases}$$

Repeat 4(b) and 4(c) for this case.

If $\omega_1 = 0$, then from 4(d), we get:

$$\left(2 - 1/\omega_2\right) T_s > T_s'.$$

It is easy to see that this also holds for $\omega_1 > 0$ (try to redraw 4(a) if it is not obvious.

4(f)(3 points) For reconstructing the analog signal we use zero-order hold in the following system.

$$y_d(n) \longrightarrow \boxed{\Gamma_d(e^{j\omega})} \longrightarrow \boxed{\text{Z.O.H.}} \longrightarrow y_a(t)$$

Digital
compensating filter

we have:

$$\Gamma_d(e^{j\omega}) = \begin{cases} \dfrac{(\omega/2)}{\sin(\omega/2)}; & |\omega| < \pi \\ 0 & ; \text{else} \end{cases}$$

For the system in 4(d), indicate how you would modify $H_d\left(e^{j\omega}\right)$ so that we will not need to implement the digital compensating filter.

Simply set $H_d(e^{j\omega})$ to the product:

$$H_d(e^{j\omega}) = \begin{cases} \dfrac{\omega/2}{\sin(\omega/2)}; & |\omega| < \pi/a \\ 0, & \text{else} \end{cases}$$

where $a \geq 1$.