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Penetration of Electromagnetic Fields Through a Small Aperture into a Cavity

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ABSTRACT

The problem of determining the electromagnetic fields produced inside a perfectly conducting cavity by an external field when an aperture is present in the cavity wall is formulated. It is found that the fields excited within the cavity are proportional to the current density at the aperture that would be present if the aperture were completely shorted.

INTRODUCTION

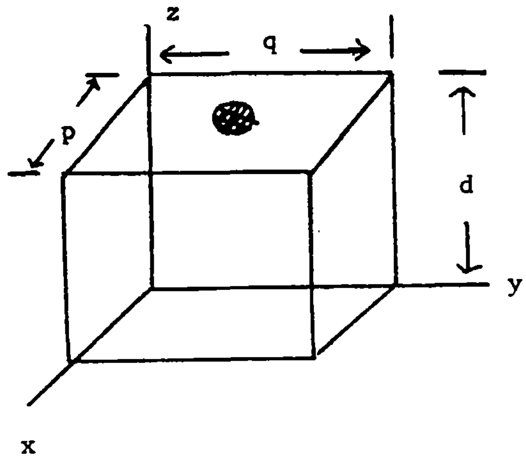
The problem formulated is that of determining the electromagnetic field produced inside a perfectly conducting cavity by an external field when an aperture is present in a cavity wall. For convenience, the cavity geometry is taken to be rectangular, and the dimensions of the aperture are considered small as compared to the wavelength of the incident radiation. General modal expansions for the field components inside the cavity are derived and the expansion coefficients are expressed in terms of the aperture fields for an arbitrarily shaped aperture. The theory of diffraction by small circular apertures is then used to obtain the aperture fields. However these fields are expressed in terms of the field components that would be present if the aperture were completely shorted. Two methods are proposed for determining the later.

The formulation yields the result that the fields excited within the cavity are proportional to the current density at the aperture that would be present is the aperture were completely shorted. This result has been observed for other cavities.

ANALYSIS

Interior Field

In general, the theory of wave guides and cavity resonators is well formulated (1,2). The geometry used is shown below. In general, the



medium of propagation is vacuum and the walls are perfectly conducting. The E field in the cavity can be written as

$$\vec{E} = \left[E_{ox}\,\hat{i} + E_{oy}\,\hat{j} + E_{oz}\,\hat{k} \right] e^{j(wt - k_g z)}$$

where E_{ox}, E_{oy}, E_{oz} are functions of x and y with the z dependence contained in the exponential, and k_g is the effective propagation constant in the z direction. In a straight forward manner Maxwell's equations lead to expressions for x and y components of $\vec{E}$ and $\vec{H}$ in terms of z components:

$$E_{ox} = \frac{-j\omega\mu}{(k^1)^2}\left(\frac{k_g}{\omega\mu}\frac{\partial E_{oz}}{\partial x} + \frac{\partial H_{oz}}{\partial y}\right)$$

$$E_{oy} = \frac{j\omega\mu}{(k^1)^2}\left(\frac{-k_g}{\omega\mu}\frac{\partial E_{oz}}{\partial y} + \frac{\partial H_{oz}}{\partial x}\right)$$

$$H_{ox} = \frac{j\omega\varepsilon}{(k^1)^2}\left[\frac{\partial E_{oz}}{\partial y} - \frac{k_g}{\omega\varepsilon}\frac{\partial H_{oz}}{\partial x}\right]$$

$$H_{oy} = \frac{-j\omega\varepsilon}{(k^1)^2}\left[\frac{\partial E_{oz}}{\partial x} + \frac{k_g}{\omega\varepsilon}\frac{\partial H_{oz}}{\partial y}\right]$$

where $(k^1)^2 = k^2 - k_g{}^2 = (\omega^2/c^2) - k_g{}^2$.

From the wave equation, it follows that

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + (k^2 - k_g{}^2)\ \Psi = 0$$

where Ψ is E_{oz} or H_{oz}. For the TE modes ($E_z = 0$) and TM modes ($H_z = 0$), the above equation and the boundary conditions ($E_{tang} = 0$ and ∇H_{oz} must be tangent to the surface at the surfaces) imply

$$\left.\begin{aligned} H_{oz} &= F_{nm} \cos \frac{n\pi x}{p} \cos \frac{m\pi y}{q} \\ E_{oz} &= G_{nm} \sin \frac{n\pi x}{p} \sin \frac{m\pi y}{q} \end{aligned}\right\} \qquad \begin{aligned} &(k^1)^2 = \left[\left(\frac{n}{p}\right)^2 + \left(\frac{m}{q}\right)^2\right]\pi^2 \\ &\text{m, n integers.} \end{aligned}$$

The general expressions for E_{oz} and H_{oz} including explicitly the z dependence are

$$E_{oz} = \sum_{nm} \left[G_{nm}\, e^{-jk_g z} + G'_{nm}\, e^{jk_g z}\right] \sin \frac{n\pi x}{p} \sin \frac{m\pi y}{q}$$

$$H_{oz} = \sum_{nm} \left[F_{nm}\, e^{-jk_g z} + F'_{nm}\, e^{jk_g z}\right] \cos \frac{n\pi x}{p} \cos \frac{m\pi y}{q} .$$

These expressions represent TM and TE modes propagating in both the +z and -z directions in the cavity. It is easy to show from previous results that

$$E_{ox} = \sum_{nm} \left\{ \frac{-jk_g}{(k^1)^2} \frac{n\pi}{p} \cos \frac{n\pi x}{p} \sin \frac{m\pi y}{q} \left[G_{nm}\, e^{-jk_g z} + G'_{nm}\, e^{jk_g z}\right] + \frac{j\omega\mu}{(k^1)^2} \frac{m\pi}{q} \cos \frac{n\pi x}{p} \sin \frac{m\pi y}{q} \left[F_{nm}\, e^{-jk_g z} + F'_{nm}\, e^{jk_g z}\right] \right\}$$

The G terms and H terms retain their identity with TM and TE modes, respectively; however, if coupling between modes exists, it should follow naturally from this general expression. E_{ox} can be rewritten as

$$E_{ox} = \sum_{nm} \cos \frac{n\pi x}{p} \sin \frac{m\pi y}{q} \left[\alpha_{nm}\, e^{-jk_g z} + \alpha'_{nm}\, e^{jk_g z}\right]$$

where

$$\alpha_{nm} = \frac{-jk_g}{(k^1)^2}\frac{n\pi}{p} G_{nm} + \frac{j\omega\mu}{(k^1)^2}\frac{m\pi}{q} F_{nm}$$

$$\alpha'_{nm} = \frac{-jk_g}{(k^1)^2}\frac{n\pi}{p} G'_{nm} + \frac{j\omega\mu}{(k^1)^2}\frac{m\pi}{q} F'_{nm} .$$

Similarly

$$E_{oy} = \sum_{nm} \sin\frac{n\pi x}{p} \cos\frac{m\pi y}{q} \left[\beta_{nm} e^{-jk_g z} + \beta'_{nm} e^{jk_g z}\right]$$

where

$$\beta_{nm} = \frac{-jk_g}{(k^1)^2}\frac{m\pi}{q} G_{nm} \frac{-j\omega\mu}{(k^1)^2}\frac{n\pi}{p} F_{nm}$$

$$\beta'_{nm} = \frac{-jk_g}{(k^1)^2}\frac{m\pi}{q} G'_{nm} \frac{-j\omega\mu}{(k^1)^2}\frac{n\pi}{p} F'_{nm} .$$

To complete the program, H_{ox} and H_{oy} become

$$H_{ox} = \sum_{nm}\left\{\frac{j\omega\varepsilon}{(k^1)^2}\frac{m\pi}{q} \sin\frac{n\pi x}{p} \cos\frac{m\pi y}{q}\left[G_{nm} e^{-jk_g z} + G'_{nm} e^{jk_g z}\right]\right.$$
$$\left.\frac{+jk_g}{(k^1)^2}\frac{n\pi}{p} \sin\frac{n\pi x}{p} \cos\frac{m\pi y}{q}\left[F_{nm} e^{-jk_g z} + F'_{nm} e^{jk_g z}\right]\right\}$$

$$H_{oy} = \left\{\sum_{nm} \frac{-j\omega\varepsilon}{(k^1)^2}\frac{n\pi}{p} \cos\frac{n\pi x}{p} \sin\frac{m\pi y}{q}\left[G_{nm} e^{-jk_g z} + G'_{nm} e^{jk_g z}\right]\right.$$
$$\left.\frac{+jk_g}{(k^1)^2}\frac{m\pi}{q} \cos\frac{n\pi x}{p} \sin\frac{m\pi y}{q}\left[F_{nm} e^{-jk_g z} + F'_{nm} e^{jkgz}\right]\right\}$$

Therefore if G_{nm}, G'_{nm}, F_{nm}, and F'_{nm} can be determined from boundary conditions then the fields in the cavity will be specified completely.

On the $z = 0$ surface $E_{ox} = E_{oy} = 0$; thus $\alpha_{nm} = -\alpha'_{nm}$ and $\beta_{nm} = -\beta'_{nm}$. From the two α and β relations, it is possible to show that

$$G_{nm} = -G'_{nm} \text{ and } F_{nm} = -F'_{nm} .$$

Thus, each square bracket in the field equations can be replaced by either $2jG_{nm} \sin k_g z$ or $2jF_{nm} \sin k_g z$. Perhaps an easier method is to find α'_{nm} and β'_{nm} using boundary conditions on the z = d surface and solve these equations for G_{nm} and F_{nm}. This approach will be followed.

$$E_{ox} = \sum_{nm} 2j\alpha'_{nm} \cos \frac{n\pi x}{p} \sin \frac{m\pi y}{q} \sin k_g z$$

$$E_{oy} = \sum_{nm} 2j\beta'_{nm} \sin \frac{n\pi x}{p} \cos \frac{m\pi y}{q} \sin k_g z$$

On the z = d surface (and since cos and sin form an orthogonal set of functions), it follows that

$$2j\alpha'_{nm} = \frac{4}{pq \sin k_g d} \int_o^p \int_o^q dxdy\ E_{ox}(x,y,d) \cos \frac{n\pi x}{p} \sin \frac{m\pi y}{q}$$

$$2j\beta'_{nm} = \frac{4}{pq \sin k_g d} \int_o^p \int_o^q dxdy\ E_{oy}(x,y,d) \sin \frac{n\pi x}{p} \cos \frac{m\pi y}{q} \quad .$$

Up to this stage, the formulation is general. It is necessary only to specify the tangential electric field components on the z = d surface and perform the above integrals. This will evaluate α'_{nm} and β'_{nm} from which G_{nm} and F_{nm} can be found. Once these are known the field expressions are known.

Aperture Field

To proceed further E_{ox} and E_{oy} on the z = d surface must be specified. Bethe (3) has considered a similar problem. The geometry of Bethe's problem is a hole in a single infinitely large flat conductor and he finds expressions for the field components in the hole and on the side of the plate opposite the impressed fields. For a circular aperature of radius a, these results may be approximated by

$$E_{ox} = \frac{1}{\pi} \frac{x}{(a^2 - x^2)^{\frac{1}{2}}} \quad E^{i}_{oz}$$

$$E_{oy} = \frac{1}{\pi} \frac{y}{(a^2 - r^2)^{\frac{1}{2}}} \quad E^{i}_{oz}$$

$$E_{oz} = \frac{1}{2} \; E^{i}_{oz}$$

where E^{i}_{oz}, the z component of the impressed field, is defined as the z component of the field that would be present if there were no hole in the screen. If the additional assumption is made that the dimensions of the hole are small compared to the wavelength of the impressed field, then E^{i}_{oz} can be evaluated at the center of the aperture and assumed constant throughout the aperture.

If the origin of the coordinate system is shifted to the center of the z = 0 surface and a small circular hole is assumed centered at the center of the z = d surface, the resulting approximations for the problem considered here become

$$2j\alpha'_{nm} = \frac{4}{pq \sin k_g d} \int_{-a}^{a} dy \left\{ (-1)^{\frac{m}{2}} \sin \frac{m\pi y}{q} \left[\int_{-x_1}^{x_1} dx \; E_{ox} (-1)^{\frac{n}{2}} \cos \frac{n\pi x}{p} \right] \right\}$$

$$2j\beta'_{nm} = \frac{4}{pq \sin k_g d} \int_{-a}^{a} dx \left\{ (-1)^{\frac{n}{2}} \sin \frac{n\pi x}{p} \left[\int_{-y_1}^{y_1} dy \; E_{oy} (-1)^{\frac{m}{2}} \cos \frac{m\pi y}{q} \right] \right\}$$

where m and n are even integers and

$$x_1^{\;2} + y_1^{\;2} = a^2 .$$

For m and n odd integers, it is true that

$$2j\alpha'_{nm} = \frac{4}{pq \sin k_g d} \int_{-a}^{a} dy \left\{ (-1)^{m+1} \cos \frac{m\pi y}{q} \left[\int_{-x_1}^{x_1} dx \; E_{ox} (-1)^{n} \sin \frac{m\pi y}{p} \right] \right\}$$

$$2j\beta'_{nm} = \frac{4}{pq \sin k_g d} \int_{-a}^{a} dx \left\{ (-1)^{n+1} \cos \frac{n\pi x}{p} \left[\int_{-y_1}^{y_1} dy \, E_{oy} (-1)^m \sin \frac{m\pi y}{p} \right] \right\}.$$

Appropriate expressions can be written when m(n) is even and n(m) is odd. Due to the algebraic form of the Bethe approximations, certain ones of the above terms are zero;

$$\alpha'_{nm} = 0 \qquad \text{for even } n$$

$$\beta'_{nm} = 0 \qquad \text{for even } m$$

and $G_{nm} = F_{nm} = 0$ for m and n even. To proceed further, it is necessary to have an explicit expression for the impressed field, E^i_{oz}.

As noted earlier Bethe's approximation considers only a single plate. In this problem, there will be reflected waves contributing to the fields in the aperture. A means for determining if this approximation is valid in this problem would be to compare the size of the E_z field induced in the cavity to the impressed field, E^i_{oz}. If the induced field is small compared to the impressed field, the approximation should be good.

Impressed Field

Obtaining the impressed field components is simplified somewhat by not having to consider the aperture present. Two procedures for obtaining these field components are proposed. The first is an experimental procedure and the second a theoretical-numerical approach.

According to the foregoing formulation, a knowledge of the z-component of the impressed electric field completely determines the field in the interior of the cavity. Provided the cavity walls are good conductors the impressed field is

$$E^i_{oz} = \rho/\varepsilon$$

where ρ is the surface charge density on the outside wall of the cavity at $z = d$. From the equation of continuity

$$\rho = j \frac{1}{\omega} \left[\frac{\partial}{\partial x} J_x(x,y) + \frac{\partial}{\partial y} J_y(x,y) \right]$$

where J_x and J_y are the components of the surface current density on the outer surface of the cavity at $z = d$. In general the current density is a periodic function of position. Therefore

$$E^i_{oz} \, \alpha \, |\vec{J}|_{\text{aperture}}$$

and hence the interior fields are proportional to the surface current density that would exist at the position of the aperture. Although this result is obtained for rectangular structures, it is expected that this result holds for an arbitrarily shaped closed shell with a small aperature. Indeed this has been observed for cylindrical shields.

According to the foregoing

$$E^i_{oz} = j \frac{1}{\omega\epsilon} \left[\frac{\partial}{\partial x} J_x(x,y) + \frac{\partial}{\partial y} J_y(x,y) \right]$$

expresses the impressed field in terms of surface current density, which is easily determined by experimental methods. The first method proposed for determining the impressed field is to experimentally determine J after shorting the aperture and use the result in the foregoing equation for E^i_{oz}.

The second method proposed for determining E^i_{oz} is to solve Maxwell's equations directly using finite-difference methods as suggested by K. S. Yee (4). This procedure necessitates the use of a high-speed digital computer. The authors plan for the near future to write a fortran program to compute the E^i_{oz} field component at the surface of a rectangular structure that is illuminated by a plane wave.

REFERENCES

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