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On Cavity Excitation Through Small Apertures

by

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ABSTRACT

The diffraction of plane waves by small apertures in a perfectly conducting plane screen is studied theoretically. A quasi-static solution technique is used for this problem. It is found that the far field may be expressed in terms of radiating electric and magnetic dipoles in the aperture. The transmission coefficient of the aperture is found to be simply related to ka, where a is the radius of aperture. Investigations are restricted to apertures which are of simple geometric shapes and which may be described in suitable coordinate systems, e.g., elliptical apertures. However, the diffraction screen may be finite.

It is shown that the electromagnetic field in a cavity when excited through a small hole can be represented in terms of the normal modes of the cavity and the dipole moments of the aperture. The associated Green's function and the expansion coefficients are written in terms of the magnetic and electric dipole moments which are associated with the fictitious magnetic charge and current distributions in the aperture used to satisfy the boundary conditions. Hence the field penetration through small apertures into cavities may be readily obtained.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

The problem of diffraction by an aperture in an infinite plane screen has received much attention within the last two decades. However, exact solutions are restricted to a few cases where the aperture is of simple geometric shape and may be described by a coordinate system in which the wave equation is separable. For example, the exact, but open-form, solution is now available for the diffraction of a linearly polarized electromagnetic plane wave by a circular disk or the equivalent circular hole (using Babinet's principle[1]) in a perfectly conducting screen. The most utilized theoretical methods for the more simple approximate treatments of this problem are the variational, the integral equation and the quasi-static approaches.

This thesis presents, first, an investigation of the quasi-static diffraction by circular and elliptical apertures using an approach similar to that of Bethe[2]. The procedure is based on solving Laplace's equation expressed in terms of oblate spheroidal and ellipsoidal coordinates. The diffracted field at large distances from the aperture is expressed in terms of the magnetic dipole moment and the electric dipole moment of equivalent aperture source distributions.

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[1]E. T. Copson, "An Integral-equation Method of Solving Plane Diffraction Problems," Proc. Roy. Soc. (London), Vol. A186, pp. 100-118, 1946; H. G. Booker, "Slot Aerials and Their Relationships to Complementary Wire Aerials," J. Inst. Elec. Engr. (London) Vo. 93, Part IIIA, pp. 620-626, 1946.

[2]H. A. Bethe, "Theory of Diffraction by Small Holes," Phys. Rev., Vol. 66, pp. 163-182, October 1944.

Second, the field distribution inside a cavity excited through a small aperture is obtained by using the aforementioned quasi-static diffraction results.

A cavity resonator usually takes the place of a tuned or resonant circuit in a microwave network. Since the resonant properties of the cavity are different for different modes, it is necessary in the study of the coupling of the cavity to another circuit to consider the excitation of a desired mode inside the cavity. Then the exciting source for the cavity can be properly designed. For instance, in order to excite a cavity for a particular mode by a loop, the plane of the loop is placed normal to the magnetic lines for the corresponding mode. Similarly, to excite in a cavity a desired mode with a probe or a short antenna, the axis of the antenna is placed parallel to the electric lines for the corresponding mode. The excitation of a cavity by a small aperture is also done in a similar manner requiring that the electric or magnetic lines inside the cavity for the desired mode coincide with the corresponding field lines of the exciting source at the aperture[3]. Thus, once the field distribution inside the cavity for a desired mode is known, the orientation and shape of the exciting aperture can be determined for an optimum excitation. Also, the study of field penetration through small apertures into cavities has application in the study of the degradation of electromagnetic shields resulting from the presence of small holes in the shield surface.

A generalized theory of cavities is developed for an arbitrary shaped cavity. In practice, however, simple geometries, such as

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[3] C. G. Montgomery, Technique of Microwave Measurements, McGraw-Hill, Inc., 1947, p. 859.

rectangular, cylinders or spheres, are commonly used. The diffracted field of a small aperture in simply connected cavities may be expanded in terms of the normal modes of the cavity. It is shown that the expansion coefficients for the lower order modes may be expressed in terms of an electric dipole moment and a magnetic dipole moment of the fictitious magnetic charge and current distributions in the aperture. Higher order multipole moments generally may be neglected for small apertures (i. e., small in terms of overall cavity dimensions).

CHAPTER II

GENERAL CONSIDERATIONS OF APERTURE DIFFRACTION

The theoretical analysis of high frequency or optical diffraction by apertures is usually based on Kirchhoff's mathematical formulation of Huygens' principle[1]. This approach considers that the aperture field is essentially the incident field, and the normal derivative of the scalar potential is assumed to vanish on the shadow face of the screen. The incident wave is considered a scalar wave, i. e., the electromagnetic field components are derivable from a scalar potential function so that polarization is ignored. However, the specific polarization effects cannot be ignored in electromagnetic wave problems when the wavelength is of the same order of magnitude as the aperture. The Kirchhoff formulation suitable for an electromagnetic field (harmonic time dependence) is given by Stratton and Chu[2], elaborated also by Kottle[3]. One way to account for polarization is by using a one-component Hertzian vector as the scalar wave function. On the other hand, if the scalar Kirchhoff formula is applied to each of the six rectangular components of the electric and magnetic vectors, then the six wave functions so obtained do not satisfy Maxwell's equation.

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[1]B. B. Baker and E. T. Copson, The Mathematical Theory of Huygens' Principle, Claredon Press, Oxford, 1950, 2nd ed., Chap. II.

[2]J. A. Stratton and L. J. Chu, "Diffraction Theory of Electromagnetic Waves," Phys. Rev., Vol. 56, pp. 99-107, 1939.

[3]F. Kottler, "Elektromagnetische Theorie der Beugung an Schwarzen Schirmen," Ann. Physik, Series 4, Vol. 71, pp. 457-508, 1923.

This difficulty can be obviated by introducing certain contour integrals. These integrals represent the effect of fictitious magnetic and electric line charges which are introduced along the rim of the aperture to ensure the fulfillment of the divergence conditions, $\nabla \cdot \vec{E} = \nabla \cdot \vec{H} = 0$. A defect in the Kirchhoff procedure is revealed by its failure to satisfy the assumed boundary values at the conducting screen. On the other hand, the transmitted field so determined from the incident field in the aperture does not vanish on the screen. Therefore, the results are valid only when the aperture dimension is large compared to the wave-length of the electromagnetic field. This technique is valid because the field on the shadow face of the screen generally will be relatively small.

Another approximate solution is due to Lord Rayleigh[4] who proposed a method of analysis of the long wavelength diffraction problem. The basic idea is that in the vicinity of the aperture the electromagnetic field can be calculated as if the wavelength were infinite (the conditions are essentially static). Then the problem is solved by the standard techniques of potential theory. Rayleigh treated the circular aperture with harmonic plane waves incident normally. After identifying the local field with a Hertzian oscillator, the known radiation characteristics of a dipole are used to find the diffracted field at large distances (as compared with aperture dimensions) from the aperture. With the foregoing approach, the tangential

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[4] Lord Rayleigh, "On the Incidence of Aerial and Electric Waves on Small Obstacles in the Form of Ellipsoids or Elliptic Cylinders, on the Passage of Electric Waves Through a Circular Aperture in a Conducting Screen," Phil. Mag., Vol. 44, p. 28, 1897.

electric field at the screen vanishes in this solution, as is required by the boundary conditions on a perfectly conducting surface. However, the evaluated transmission cross section (energy passing through the aperture per second/energy transported per unit area of the incident wave per second) represents the first term of an expansion in ascending powers of the ratio ka (circumference of aperture/wavelength) and, therefore, is accurate only for long wavelengths.

Bethe[5] reconsiders Rayleigh's approach and extends the theory to apply to an arbitrary spatial incident field. The problem is carried through with great skill. A new feature introduced is that the diffracted field behind the circular aperture is derived from fictitious magnetic currents and charges in the aperture. Then the fictitious distributions are obtained satisfying all boundary conditions. Bethe also points out that the Kirchhoff type of solution does not satisfy the conditions aforementioned. However, Bethe's derivation leads to an incorrect approximation for the field in the vicinity of the aperture although it supplies the correct field at large distances from the aperture. Included in his paper are some interesting predictions on the excitation of cavities coupled by a small hole. More recent investigations have corroborated Bethe's application of his theory to the coupling of cavities[6]. The error in Bethe's paper is pointed out

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[5] H. A. Bethe, "Theory of Diffraction by Small Holes," Phys. Rev., Vol. 66, pp. 163-182, October 1944.

[6] C. G. Montgomery, Technique of Microwave Measurements, Vol. 11 of MIT Radiation Laboratory Series, McGraw-Hill Book Co., New York, 1947, Chap. 14; N. Marcuvitz, Waveguide Handbook, Vol. 10 of the MIT Radiation Laboratory Series, McGraw-Hill Book Co., New York, 1951, Chap. 5.

in Bourgin's[7] criticism, which suggests that when the screen is not closed, it is necessary to add line singularities over the rim of the aperture. This is important for the second radiation condition and affects only the near field of the aperture.

Also, Bouwkamp[8] gives a detailed theoretical analysis of the diffraction of an electromagnetic wave by a circular hole small compared with wavelength. He determines the fictitious magnetic current and charge from a system of integro-differential equations that he derived. It is shown that Bethe's zeroth order solution for the magnetic current $\vec{K}_E$ is correct but the first order solution $\vec{K}_H$, where

$$\frac{1}{c}\ \vec{K}_H = \frac{ik}{\pi^2}\,(a^2 - \rho^2)^{1/2}\ \vec{H}_0 \quad ,$$

differs from the correct one which Bouwkamp expresses as K^1 ,

$$\frac{1}{c}\,\vec{K}^1 = \frac{1}{3\pi^2}\Big[(a^2 - \rho^2)^{1/2}\ \{2ik\ \vec{H}_o + \hat{n} \times \nabla(\hat{n} \cdot \vec{E}_o)\} + \vec{\rho} \cdot \{ik\ \vec{H}_o \times \hat{n} - \nabla(\hat{n} \cdot \vec{E}_o)\}\ \frac{\hat{n} \times \vec{\rho}}{(a^2 - \rho^2)^{1/2}}\ \Big] \quad .$$

Here $\vec{E}_o$ and $\vec{H}_o$ are the incident electric and magnetic fields evaluated at the center of the aperture. The difference is due to Bethe's $\vec{K}_H$ being correct except for the omission of a solenoidal vector[9].

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[7]D. G. Bourgin, "Bethe, H. A., Theory of Diffraction by Small Holes," Math. Rev., Vol. 6, p. 165, 1945.

[8]C. J. Bouwkamp, "Diffraction Theory", Philips Res. Rep., No. 5, pp. 401-442, December 1950.

[9]Evidently D. S. Jackson was not aware of this result when writing Section 9 of Chap. 9 in his text, Classical Electrodynamics, John Wiley & Sons, Inc., New York.

The explanation proferred by Meixner[10] is that the total field must have a singularity on the rim of the hole; that is, the electric field becomes infinite there and normal to the rim. Bethe's solution has no such singularity. However, Bouwkamp proves that Bethe's quasi-static approach does obtain correctly the dipole moments of the fictitious aperture distributions.

A higher-order approximation of the diffraction by a small circular disk is evaluated by Eggimann[11]. His approach is essentially an extension of the procedure described by Bouwkamp. Eggimann considers a solution for the vector potential which is expressed in a power-series in ka for an arbitrary incident field. The surface current density is calculated in terms of the electromagnetic field and its derivatives at the center of the disk. The induced electric and magnetic dipole moments and the distant fields are obtained directly.

The procedure for obtaining a rigorous, but open form, solution to the foregoing problem is first obtained by Meixner and Andrejewski[12]. The diffracted field is derived from the currents induced on a circular disk by an incident wave. The authors express the diffracted field in terms of the Hertzian vector potential, which is

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[10] J. Meixner, "Die Kantenbedingung in der Theorie der Beugung elektromagnetischer Welien an volldommen leitenden ebenen Schirmen", Ann. Physik, Vol. 6, pp. 1-9, 1949.

[11] W. H. Eggimann, "Higher-order Evaluation of Electromagnetic Diffraction by Circular Disks", IRE Trans. on Microwave Theory and Techniques, Vol. MIT-9, pp. 408-418, September 1961.

[12] J. Meixner and W. Andrejewski, "Strange Theorie der Beugung ebener Elektromagnetischer Wellen an der vollkommen leitenden Kreisscheibe und an der kreisformigen Offhung im vollkommen leitenden ebener Schirm," Ann. Physik, Vol. 7, pp. 157-168, 1950.

everywhere directed parallel to the plane of the disk. The two rectangular components of the vector potential are expanded in terms of the oblate spheroidal wave functions. The coefficients in these expansions are derived from the boundary conditions on the disk and its rim in a unique way. An approximate expression is then obtained for the scattering cross section, or echo area, of a small disk for the case of normal incidence. The complementary problem for the diffraction of a plane wave by a circular hole is solved by application of Babinet's principle.

This chapter considers a quasi-static theory of the electromagnetic wave diffraction by small apertures. This formulation is based on the work originally developed by Bethe[13]. This quasi-static procedure is presented because it is straightforward and applies for a finite screen as well as an infinite screen. The exact theories do not possess either of these features. The basic technique is that the fields in the neighborhood of a small aperture may be represented in the quasi-static approximation by the fields $\vec{E}_o$ and $\vec{H}_o$ impressed on the aperture before the aperture is cut, superimposed with the fields of electric and magnetic dipoles located at the center of the aperture. The dipole moments are proportional to $\vec{E}_o$ and $\vec{H}_o$ respectively and are functions of the shape and size of the aperture. Certain conditions, the so-called Bethe's conditions, will be presented regarding the symmetry and antisymmetry of the diffracted fields which are formulated to insure the continuity of the total field through the aperture.

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[13]H. A. Bethe, op. cit.

2.1 Fields and Coupling on Small Apertures

If harmonic electromagnetic waves impinge from the $z < 0$ space on an infinitesimally thin, perfectly conducting plane containing an aperture of any shape, the total field about the plate can be described as the sum of the incident, reflected, and perturbation fields. That is,

$$\vec{E}^T = \vec{E}^i + \vec{E}^r + \vec{E}^p \quad , \qquad \vec{H}^T = \vec{H}^i + \vec{H}^r + \vec{H}^p \quad . \tag{2.1}$$

Considering that $\vec{E}_o$ and $\vec{H}_o$ are fields on the lower surface of the screen and are uniform over the aperture before the aperture is cut, then

$$E_o = \hat{n} \cdot (\vec{E}^i + \vec{E}^r) \quad , \qquad |\vec{H}_o| = |\hat{n} \times (\vec{H}^i + \vec{H}^r)| \quad , \tag{2.2}$$

where $\hat{n}$ is unit vector normal to the screen. After the aperture is cut, the total fields are denoted by

$$\begin{aligned} \vec{E}_1^T &= \vec{E}_1^i + \vec{E}_1^r + \vec{E}_1^p \quad , & \vec{H}_1^T &= \vec{H}_1^i + \vec{H}_1^r + \vec{H}_1^p \quad , \\ \vec{E}_2^T &= \vec{E}_2^p \quad , & \vec{H}_2^T &= \vec{H}_2^p \quad . \end{aligned} \tag{2.3}$$

The subscripts "1" and "2" describe the $z < 0$ space and $z > 0$ space. All components of the electromagnetic field must be continuous across the aperture provided that the media on both sides of the aperture are the same. Therefore the fields satisfy the following boundary conditions:

On the hole surface,

$$\hat{n} \times \vec{E}_1^p = \hat{n} \times \vec{E}_2^p \quad , \quad \vec{E}_0 + \hat{n} \cdot \vec{E}_1^p = \hat{n} \cdot \vec{E}_2^p \quad ,$$

$$\vec{H}_0^p + \hat{n} \times \vec{H}_1^p = \hat{n} \times \vec{H}_2^p \, , \quad \hat{n} \cdot \vec{H}_1^p = \hat{n} \cdot \vec{H}_2^p \quad ; \tag{2.4}$$

On the screen surface,

$$\hat{n} \times \vec{E}_1^T = \hat{n} \times \vec{E}_2^T = 0 \quad , \quad \hat{n} \cdot \vec{H}_1^T = \hat{n} \cdot \vec{H}_2^T = 0 \quad . \tag{2.5}$$

The tangential component $|\hat{n} \times \vec{E}|$, of the electric field, as well as the normal component $\hat{n} \cdot \vec{H}$, of the magnetic field are symmetrical with regard to the screen; other components $\hat{n} \cdot \vec{E}$ and $|\hat{n} \times \vec{H}|$ are proved to be antisymmetrical[14]. Thus Bethe's conditions are obtained for the field components:

$$\hat{n} \cdot \vec{E}_1^p = -\frac{1}{2}\vec{E}_0 \quad , \quad \hat{n} \times \vec{H}_1^p = -\frac{1}{2}\vec{H}_0 \quad ,$$

$$\hat{n} \cdot \vec{E}_2^p = \frac{1}{2}\vec{E}_0 \quad , \quad \hat{n} \times \vec{H}_2^p = \frac{1}{2}\vec{H}_0 \quad . \tag{2.6}$$

As a result of above relations, we may conclude that the tangential electric field and the normal magnetic field are not equal to zero on the aperture and hence, if the aperture is closed by a magnetic wall, a magnetic current and a magnetic charge distribution must be given

$$\hat{n} \times (\vec{E}^i + \vec{E}^r) = \vec{K}^* \quad ,$$

$$\hat{n} \cdot (\vec{H}^i + \vec{H}^r) = \eta^* \quad , \tag{2.7}$$

[14] H. E. J. Neugenbauer, "Diffraction of Electromagnetic Waves Caused by Apertures in Absorbing Plane Screen," IRE Trans. on Antennas and Propagation, Vol. AP-4, pp. 115-119, April 1956.

since $\vec{E}_o = 0$, $\vec{H}_o = 0$ on the magnetic wall.

The dipole moments associated with the aperture current $\vec{K}^*$ and charge η^* are $\vec{P}$ and $\vec{M}$, respectively. One half the required discontinuity across the source is provided by $\vec{E}_2^p$, $\vec{H}_2^p$; the other half by $\vec{E}_1^p$, $\vec{H}_1^p$. The effective dipole moments for radiation into $z > 0$ space will be $\vec{P}/2$ and $\vec{M}/2$. The total field in the region $z < 0$ is equivalent to the sum of the incident field $\vec{E}_o$, $\vec{H}_o$ and the field radiated by dipoles with moments $\vec{P}/2$, $\vec{M}/2$, since the perturbation field of $-\vec{P}/2$ and $-\vec{M}/2$ cancels one-half the field radiated by $\vec{P}$ and $\vec{M}$.

CHAPTER III

ELECTROMAGNETIC WAVES DIFFRACTED BY SMALL APERTURES IN A PERFECTLY CONDUCTING PLANE--A QUASI-STATIC APPROACH

A general method[1] of solving the potential problem for a conducting disc or the complement problem of a conducting sheet with a hole is presented. This method requires the use of oblate spheroidal coordinates. There are two particular advantages in using oblate spheroidal coordinates. First, using such coordinates allows the separation of variables in solving Laplace's equation. Second, from the geometrical considerations one can visualize the field and equipotential surfaces, which must coincide with the surfaces of constant coordinates. In addition, the diffraction problem by an elliptic aperture is examined in the latter part of the chapter. However, attention will be confined to small apertures in a large, perfect, conducting plane.

3.1 Oblate Spheroidal Coordinates

Oblate spheroidal coordinates[2] are a degenerate case of the more general ellipsoidal doordinates that occurs when it is required that

[1] W. R. Smythe, Static and Dynamic Electricity, McGraw-Hill Book Company, Inc., 1950, 2nd., p. 191.

[2] T. M. MacRobert, Spherical Harmonics, Pergamon Press, 1967, 3 ed., p. 194.

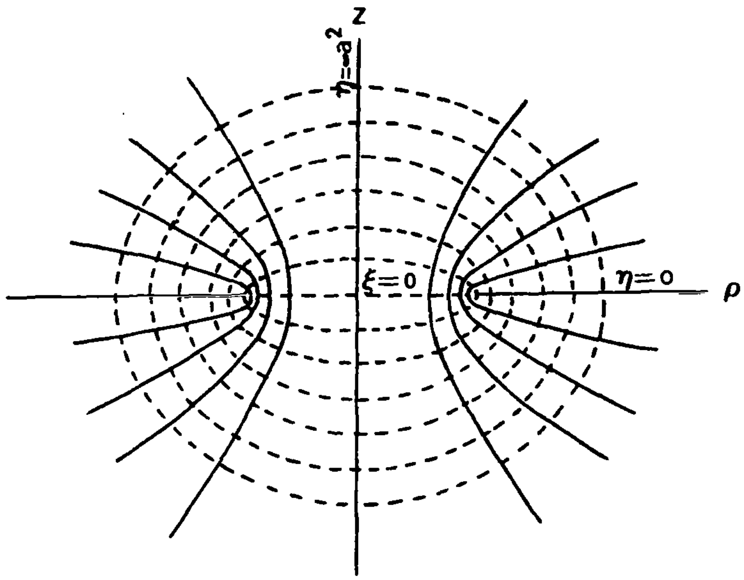

Figure 1. Oblate spheroidal coordinates.

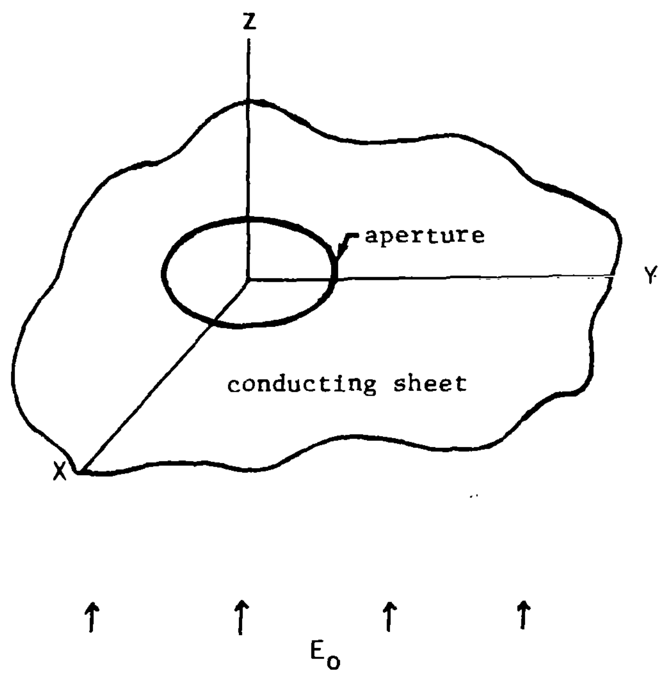

Figure 2. Conducting sheet with a circular aperture in a uniform applied field E_0

the three semi-axes of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

satisfy $a = b > c$. The relation between oblate spheroidal coordinates and cartesian coordinates can be written in a quadratic form

$$\frac{\rho^2}{a^2 + u} + \frac{z^2}{u} - 1 = 0 \quad , \tag{3.1}$$

together with

$$x = \rho \cos \phi \quad ,$$

$$y = \rho \sin \phi \quad ,$$

where ρ and ϕ are the usual polar coordinates in the xy-plane. The two roots of (3.1), i. e., the solutions for u, along with ϕ form the oblate spheroidal coordinates. The three independent variables ξ, η, and ϕ, Figure 1, have the following ranges:

$$\infty > \xi \geq 0 \quad , \quad 0 \geq \eta \geq -a^2 \quad , \quad 2\pi \geq \phi \geq 0 \quad . \tag{3.2}$$

Then the transformation from oblate spheroidal coordinates to cartesian coordinates is given by

$$z = \pm\sqrt{\frac{\xi\eta}{a^2}} \quad , \quad \rho = \sqrt{x^2 + y^2} = \sqrt{\frac{(\xi+a^2)(\eta+a^2)}{a^2}} \quad . \tag{3.3}$$

The surfaces of constant ξ and η become respectively the confocal oblate spheroids and hyperboloids of revolution of one sheet. A calculation of the metrical coefficients yields

$$h_1 = \sqrt{\frac{\xi - \eta}{4\xi(\xi+a^2)}} \quad , \quad h_2 = \sqrt{\frac{\xi - \eta}{4(-\eta)(\eta+a^2)}} \quad , \quad h_3 = \sqrt{\frac{(\xi+a^2)(\eta+a^2)}{a^2}} \tag{3.4}$$

Laplace's equation in terms of these coordinates is

$$\nabla^2\Phi = \frac{4}{(\xi-\eta)} [\sqrt{\xi} \frac{\partial}{\partial\xi} (R_\xi \frac{\partial\Phi}{\partial\xi}) + \sqrt{-\eta} \frac{\partial}{\partial\eta} (R_\eta \frac{\partial\Phi}{\partial\eta})] + \frac{a^2}{(\xi+a^2)(\eta+a^2)} \frac{\partial^2\Phi}{\partial\phi^2} \quad , \tag{3.5}$$

where $R_\xi = (\xi+a^2) \sqrt{\xi}$, $R_\eta = (\eta+a^2) \sqrt{-\eta}$. (3.6)

3.2 Conducting Sheet with a Circular Aperture

A uniform field $\vec{E}_0$ in the z-direction is considered in a region (the half space $z < 0$) bounded by a large perfectly conducting sheet containing a circular aperture, as indicated in Figure 2. The sheet is considered to be in the xy-plane for convenience. In terms of the spheroidal coordinates, the plane may be regarded as the limit of the hyperboloids of revolution of one sheet; the equation of the plane is expressed as

$$\frac{\rho^2}{a^2 - |\eta|} - \frac{z^2}{|\eta|} = 1 \quad , \tag{3.7}$$

as $|\eta| \to 0$.

To solve this provlem, a potential function must be found which satisfies (3.6), which behaves properly at infinity, and which is finite over the aperture. We presume, therefore, that Φ is a function of the form

$$\Phi = -E_o z F(\xi) = -E_o F(\xi) \sqrt{\frac{\xi\eta}{-a^2}} . \tag{3.8}$$

We need only substitute (3.8) into (3.5) and the equation satisfied by $F(\xi)$ is readily obtained as

$$\frac{d^2F}{d^2\xi} + \frac{dF}{d\xi} \frac{d}{d\xi} \ln[R_\xi \xi] = 0 . \tag{3.9}$$

Equation (3.7) is an ordinary differential equation of the second order and possesses two independent solutions; one solution we know already to be $F(\xi)$ = constant and the other is

$$F(\xi) = (\text{const.}) \int \frac{d\xi}{\xi R_\xi}$$

$$= (\text{const.}) \left[\frac{a}{\sqrt{\xi}} + \tan^{-1} \frac{\sqrt{\xi}}{a} - \frac{\pi}{2} \right] . \tag{3.10}$$

The term $(-\frac{\pi}{2})$ is an integration constant which is included to satisfy the boundary condition $\Phi(\xi,\eta) \xrightarrow[\sqrt{\xi} \to \infty]{} 0$. It is obvious that the diffracted field caused by a small aperture can be considered to be zero in the far zone. The constant factor is determined from the fact that, for $z \to -\infty$ (i.e., $\sqrt{\xi} \to -\infty$) , $\Phi(\xi,\eta) \longrightarrow -E_o z$. Thus

$$\text{const.} = -\frac{1}{\pi} . \tag{3.11}$$

Substituting (3.11) and (3.10) into (3.8), the potential function Φ is given by

$$\Phi = -\frac{E_o\sqrt{|\eta|}}{\pi} \left[\frac{\sqrt{\xi}}{a} \cot^{-1} \frac{\sqrt{\xi}}{a} - 1 \right] . \tag{3.12}$$

Note that (3.12) yields a zero potential, as it should be, on the conducting plane $\eta = 0$. At large distance $r >> a$ ($r = \sqrt{z^2 + \rho^2}$), equation (3.12) reduces to

$$\Phi \simeq -\frac{E_o\sqrt{|\eta|}}{\pi}\frac{\sqrt{\xi}}{a}\left[-\frac{a^3}{3\xi^{3/2}}\right] = \frac{E_o a^3}{3\pi}\frac{z}{r^3} \quad . \tag{3.13}$$

In the far zone, the electric field is identified as the dipole moment field of the aperture; the electric dipole moment is

$$\vec{P}_E = \frac{4\varepsilon_o E_o a^3}{3}\hat{z} \quad \text{coulomb-meters} \quad . \tag{3.14}$$

The potential inside the aperture is obtained from (3.10) by letting $\sqrt{\xi} = 0$. That is,

$$\Phi_i = \frac{E_o\sqrt{|\eta|}}{\pi} = \frac{E_o}{\pi}(a^2 - \rho^2)^{1/2} \quad . \tag{3.15}$$

By taking derivatives of (3.15) with respect to ρ (the polar radius), the tangential electric field inside the aperture is given as

$$\vec{E} = -\frac{\partial\Phi_i}{\partial\rho}\hat{\rho} = \frac{E_o\vec{\rho}}{\pi}(a^2 - \rho^2)^{1/2} \quad . \tag{3.16}$$

The case of the magnetic field diffracted by a circular aperture in a large conducting sheet carrying a surface current which is uniform before the aperture is cut may be treated with the same approach as in the foregoing paragraph. The magnetic scalar potential Φ^* is defined as

$$\vec{H} = -\operatorname{grad}\Phi^* \quad . \tag{3.17}$$

Without loss of generality, we may consider the constant surface current $\vec{K}_o$ which is induced by the uniformly incident magnetic field $\vec{H}_o$ (in the $z < 0$ space) as the hole is replaced by a conducting disc to be along the negative direction of x-axis, where

$$\hat{n} \times \vec{H}_o = -\vec{K}_o \quad ; \tag{3.18}$$

$\hat{n} = -\hat{z}$ is the unit vector normal to the lower surface of the screen. At large distances from the aperture (in the $z < 0$ space) the magnetic field is essentially undisturbed and the potential function is

$$\begin{aligned} \Phi_o{}^* &= \Phi^*_{\sqrt{\xi} \to -\infty} = H_o \, \rho \cos \phi \\ &= H_o \frac{1}{a} \sqrt{\eta+a^2} \sqrt{\xi+a^2} \cos \phi \quad . \end{aligned} \tag{3.19}$$

We may seek the solution for Φ^* in a form which involves the same factors depending on η and ϕ as in $\Phi^*_{\sqrt{\xi} \to -\infty}$ and a function depending only on ξ; thus

$$\Phi^* = H_o \frac{1}{a} \sqrt{\eta+a^2} \cos \phi \, G(\xi) \quad . \tag{3.20}$$

Now substituting (3.20) into Laplace's equation (3.5) yields

$$R_\xi \frac{d}{d\xi} \left(R_\xi \frac{dG}{d\xi}\right) - \frac{1}{4} (2\xi + a^2) \, G = 0 \quad . \tag{3.21}$$

Equation (3.21) is readily solved to give

$$G(\xi) = \sqrt{\xi+a^2} \int \frac{d\xi}{(\xi+a^2) R_\xi} \quad , \tag{3.22}$$

where $R_\xi = \sqrt{(\xi+a^2)^2 \, \xi} \quad .$ (3.23)

The limits of integration are arbitrary, but it is easily shown that the potential is given by

$$\Phi^*(\xi,\eta,\phi) = \frac{\Phi_o^*}{2} \int_{\sqrt{\xi'}=\sqrt{\xi}}^{\infty} \frac{d\xi'}{(\xi'+a^2)R_{\xi'}} \Big/ \int_0^{\infty} \frac{d\xi'}{(\xi'+a^2)R_{\xi'}}$$

$$= \frac{H_o}{\pi a} \sqrt{\eta+a^2}\,\sqrt{\xi+a^2}\,\left(\cot^{-1}\frac{\sqrt{\xi}}{a} - \frac{a\sqrt{\xi}}{\xi+a^2}\right)\cos\phi \quad . \tag{3.24}$$

At large distances from the aperture, where $\sqrt{\xi} \to r >> 1$, the potential becomes

$$\Phi^*(\xi,\eta) \simeq -\frac{a^3 H_o}{3\pi}\frac{x}{r^3} \quad , \tag{3.25}$$

where $\quad x = \rho \cos\phi \quad .$

Again the far field of the aperture is identified as the dipole field with magnetic dipole moment

$$\vec{P}_M = -\frac{8a^3\mu_o H_o}{3}\,\hat{x} \qquad \text{volt} \cdot \text{meter} \cdot \text{seconds} \quad . \tag{3.26}$$

The magnetic field inside the circular hole is obtained by taking the limit of the derivative of Φ^* in (3.24) as $\sqrt{\xi} \to 0$. It is

$$\vec{H}_z = -\frac{\partial\Phi^*}{\partial z}\,\hat{z}\,\Big|_{\sqrt{\xi}\to 0}$$

$$= \frac{-H_o\,\rho\cos\phi}{\pi\sqrt{a^2-\rho^2}}\,z \quad . \tag{3.27}$$

The transmission coefficient t of the aperture is defined as

$$t = \frac{\text{energy transmitted through aperture}}{\text{energy incident on aperture}}$$

$$= \frac{\mathrm{Re}\left\{ \int_{\Omega} \vec{E}_d \times \vec{H}_d \cdot \hat{r}\, ds \right\}}{\mathrm{Re}\left\{ \int_{S_a} \vec{E}_{inc} \times \vec{H}_{inc} \cdot \hat{z}\, ds \right\}} \quad , \tag{3.28}$$

where Ω is the surface of a large hemisphere in $z > 0$ space, S_a is the area of the aperture, and $\hat{r}$ is a unit vector perpendicular to the hemisphere. For normally incident plane waves, we have

$$\vec{H}_{inc} = \frac{1}{2} \vec{H}_o \hat{x} \quad ,$$

$$\vec{E}_{inc} = \zeta \vec{H}_{inc} \quad , \tag{3.29}$$

where $$\zeta = \sqrt{\frac{\mu_o}{\varepsilon_o}} \simeq 377 \text{ ohms} \quad . \tag{3.30}$$

If the aperture is small, the incident field vectors may be assumed constant over the aperture. The incident power density, or the magnitude of the poynting vector, is expressed as

$$|\vec{S}_{inc}| = \frac{1}{2} \left| R_e \{\vec{E}_{inc} \times \vec{H}_{inc}\} \right| = \frac{1}{8} (120\pi) \frac{\text{watts}}{\text{m}^2} \quad , \tag{3.31}$$

since $\vec{H}_o$, the magnetic field on the surface of aperture before the aperture is cut, is considered to have unit value (i. e., $H_o = 1$ Ampere/meter). Therefore, the total power incident on the aperture is given by

$$P_{inc} = |\vec{S}_{inc}| (\pi a^2)$$
$$= \frac{1}{8} (120\pi) (\pi a^2) \quad \text{watts} \quad . \tag{3.32}$$

The diffracted field may be considered as the superposition of the magnetic dipole and the electric dipole fields. The electric dipole does not contribute to the transmitted energy for normal incidence. The radiated power of the magnetic dipole moment[3] is

$$P_{mag} = \frac{\omega^4}{24\pi c^3 \mu_o} \left| \frac{8a^3\mu_o}{3} \right|^2$$

$$= \frac{8}{27} (120a^2) (k\ a)^4 \quad \text{watts} \quad , \tag{3.33}$$

where $k = \omega/c$.

The transmission coefficient is, therefore,

$$t = \frac{P_{mag}}{P_{inc}} = \frac{64}{27\pi^2} (k\ a)^4 \quad . \tag{3.34}$$

3.3 Conducting Sheet with an Elliptic Aperture

The evaluation of the dipole moments for an elliptic aperture may be obtained from the static dipole moments of an ellipsoid or the complement problem of an elliptic aperture placed in uniform static magnetic and electric fields.

In the preceding section, the potential problem of a conducting sheet with circular aperture was treated by using oblate spheroidal coordinates[4]. Now the problem of a conducting sheet with an elliptic aperture in a constant impressed field is solved in ellipsodial

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[3]See D. S. Jones, The Theory of Electromagnetism, Pergamon Press, New York, 1964, p. 156.

[4]J. A. Stratton, Electromagnetic Theory, McGraw-Hill Book Company, Inc., 1941, p. 207.

co-ordinates. The ellipsoidal coordinates are related to cartesian coordinates by the equations

$$\frac{x^2}{a^2+\xi}+\frac{y^2}{b^2+\xi}+\frac{z^2}{\xi} = 1 \quad ,$$

$$\frac{x^2}{a^2-|\eta|}+\frac{y^2}{b^2-|\eta|}-\frac{z^2}{|\eta|} = 1 \quad , \tag{3.35}$$

$$\frac{x^2}{a^2-|\zeta|}-\frac{y^2}{|\zeta|-b^2}-\frac{z^2}{|\zeta|} = 1 \quad ,$$

where $\quad a > b \;; \quad \xi \geq 0 \;, \quad 0 \geq \eta \geq -b^2 \;, \quad -b^2 \geq \zeta^2 \geq -a^2 \quad .$ (3.36)

The geometrical configurations of the surfaces of constant ξ, η, ζ are respectively, ellipsoids, hyperboloids of one sheet and hyperboloids of two sheets, all confocal with the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \tag{3.37}$$

in the xy-plane. As $|\eta| \to 0$, equation (3.35) degenerates into (3.37), which identifies the conducting plane. The transformation to rectangular coordinates is obtained by solving (3.35) simultaneously for x, y, and z. This gives

$$x = \pm\left[\frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{a^2(a^2-b^2)}\right]^{1/2} \quad ,$$

$$y = \pm\left[\frac{(\xi+b^2)(\eta+b^2)(\zeta+b^2)}{-b^2(a^2-b^2)}\right]^{1/2} \quad , \tag{3.38}$$

$$z = \pm\left[\frac{\xi\;\eta\;\zeta}{a^2\;b^2}\right]^{1/2} .$$

A calculation of the metrical coefficients gives

$$h_1 = \frac{1}{2}\left[\frac{(\xi-\eta)(\xi-\zeta)}{(\xi-a^2)(\xi+b^2)\xi^2}\right]^{1/2} ,$$

$$h_2 = \frac{1}{2}\left[\frac{(\eta-\zeta)(\eta-\xi)}{(\eta+a^2)(\eta+b^2)\eta^2}\right]^{1/2} , \tag{3.39}$$

$$h_3 = \frac{1}{2}\left[\frac{(\zeta-\xi)(\zeta-\eta)}{(\zeta+a^2)(\zeta+b^2)\zeta^2}\right]^{1/2} .$$

Laplace's equation in ellipsoidal coordinates is, therefore,

$$\nabla^2\Phi = \frac{4}{(\xi-\eta)(\zeta-\xi)(\eta-\zeta)}\left[(\eta-\zeta)R_\xi \frac{\partial}{\partial\xi}\left(R_\xi \frac{\partial\Phi}{\partial\xi}\right)\right.$$

$$\left. + (\zeta-\xi)R_\eta \frac{\partial}{\partial\eta}\left(R_\eta \frac{\partial\Phi}{\partial\eta}\right) + (\xi-\eta)R_\zeta \frac{\partial}{\partial\zeta}\left(R_\zeta \frac{\partial\Phi}{\partial\zeta}\right)\right] = 0 . \tag{3.40}$$

We shall first consider the potential problem of an elliptic aperture in a large conducting plane which is subjected to a constant magnetic field. Without loss of generality, we may take the magnetic field $\vec{H}$ to be along one of the axes of the ellipse. In any case, the field may be resolved into components along the axes, and the resultant field is a superposition of those arising from each component separately.

Considering the H-field along the x-axis (i.e., along the major axis a of the ellipse) the scalar potential function of the impressed magnetic field is

$$\Phi_o^* = - H_1 x = -H_1\left[\frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{a^2(a^2 - b^2)}\right]^{1/2} , \tag{3.41}$$

with x in ellipsoidal coordinates substituted from (3.38). The potential of the diffracted field $\Phi^*(\xi,\eta,\zeta)$ must vary functionally over every surface of the family ξ = constant in exactly the same manner as Φ_o^*. We consider Φ^* as a function of the form:

$$\Phi^*(\xi,\eta,\zeta) = -H_1\left[\frac{(\eta+a^2)(\zeta+a^2)}{a^2(a^2-b^2)}\right]^{1/2} F(\xi) \quad , \tag{3.42}$$

$F(\xi)$ is a function of ξ only and must satisfy Laplace's equation (3.40). Substitution of (3.42) into (3.40) yields

$$R_\xi \frac{d}{d\xi}\left(R_\xi \frac{dF}{d\xi}\right) - \frac{1}{4}(2\xi + a^2) F = 0 \quad , \tag{3.43}$$

where $$R_\xi = [\xi(\xi+a^2)(\xi+b^2)]^{1/2} \quad . \tag{3.44}$$

The solution of (3.32) is given by

$$F(\xi) = \sqrt{\xi+a^2} \int \frac{d\xi}{(\xi+a^2)R_\xi} \quad . \tag{3.45}$$

We may rewrite the integral of (3.45) in two parts which correspond to the two regions divided by the conducting plane (i.e. the z > 0 space and the z < 0 space).

$$\int \frac{d\xi}{(\xi+a^2)R_\xi} = \begin{cases} A\displaystyle\int_{\sqrt{S}=\sqrt{\xi}}^{\infty} \frac{dS}{(S+a^2)R_S} \, , & (\infty > \sqrt{\xi} \geq 0) , \\[2ex] 1 + A\displaystyle\int_{-\infty}^{\sqrt{S}=\sqrt{\xi}} \frac{dS}{(S+a^2)R_S} \, , & (0 \geq \sqrt{\xi} > -\infty) \quad , \end{cases} \tag{3.46}$$

where S is a dummy variable which replaces ξ to avoid confusion. The limits of integration are taken so that $\Phi^* \to 0$ as $\sqrt{\xi} \to \infty$; $\Phi^* \to \Phi_o{}^*$ as $\sqrt{\xi} \to -\infty$.

The constant factor A is determined from the condition that Φ^* is constant over the aperture. For this condition to be satisfied as $\xi \to 0$ and with arbitrary η and ζ, the value $\Phi^*_{\sqrt{\xi}\to o} = \Phi_a{}^*$ can be calculated in terms of the total charge which is supposed to be induced over the aperture. These fictitious charges correspond to a discontinuity of the normal magnetic field in the aperture; thus

$$\Phi_a{}^* = \Phi_o{}^* A \int_0^\infty \frac{dS}{(S+a^2)R_S} \quad . \tag{3.47}$$

It is easy to show $\Phi_a{}^* = \frac{1}{2}\Phi_o{}^*$, since the integrals (3.46) must approach the same value from both sides of the aperture. Consequently, the potential function of the radiating field is given by

$$\Phi^* = \frac{\Phi_o^*}{2}\left[\int_{\sqrt{S}=\sqrt{\xi}}^\infty \frac{dS}{(S+a^2)R_S} \Bigg/ \int_0^\infty \frac{dS}{(S+a^2)R_S}\right] \tag{3.48}$$

For large distances, the coordinate ξ is large and $\xi \simeq r^2$, hence

$$\int_{\sqrt{S}=\sqrt{\xi}}^\infty \frac{dS}{(S+a^2)R_S} \simeq \int_r^\infty \frac{2rdr}{r^5} = \frac{2}{3r^3} \quad . \tag{3.49}$$

The radiating potential due to the magnetic dipole moment in the aperture is

$$\Phi^* \simeq \frac{-H_1 x}{3r^3\alpha^{(x)}} \quad , \tag{3.50}$$

where

$$\alpha^{(x)} = \int_0^\infty \frac{dS}{(S+a^2)R_S} \quad .$$

The effective magnetic dipole moment for the diffracted field is, therefore,

$$\vec{M}_x = \frac{-4\ \pi\mu_0\vec{H}_1}{3\alpha^{(x)}} \quad \text{volt} \cdot \text{meter} \cdot \text{seconds} \quad . \tag{3.51}$$

Note that $\alpha^{(x)}$ is an elliptic integral of the second kind. If a new variable t is introduced, where $t^2 = S+a^2$, this integral may be split into two integrals:

$$\begin{aligned} \alpha^{(x)} &= \int_a^\infty \frac{2dt}{t^2[(t^2-a^2+b^2)(t^2-a^2)]^{1/2}} \\ &= \frac{2}{a^2-b^2}\left[\int_a^\infty \frac{dt}{(t^2-a^2+b^2)^{1/2}(t^2-a^2)^{1/2}}\right. \\ &\qquad \left. - \int_a^\infty \frac{(t^2-a^2+b^2)^{1/2}}{t^2(t^2-a^2)^{1/2}}\ dt\right] \quad . \end{aligned} \tag{3.52}$$

The above integrals have been tabulated by Jahnke & Emde[5]. The solution is

$$\alpha^{(x)} = \frac{2}{a^2-b^2}\left(\frac{1}{a}\right)[F(k,\phi) - E(k,\phi)] \quad , \tag{3.53}$$

where $k = \left[\frac{a^2-b^2}{a^2}\right]^{1/2} = e$, the eccentricity of the ellipse

[5]E. Jahnke and F. Emde, <u>Tables of Functions</u>, Dover Publications, Inc., New York, 1945, P. 58.

and $\sin\phi = 1$. (3.54)

Substituting (3.54) into (3.53),

$$\alpha^{(x)} = \frac{2}{e^2\ a^3}\,[F(e,\ \tfrac{\pi}{2}) - E(e,\ \tfrac{\pi}{2})] \quad . \tag{3.55}$$

$F(e,\frac{\pi}{2})$ and $E(e,\frac{\pi}{2})$ are called the complete elliptic integrals of the first and second kinds and are given as

$$F(e,\tfrac{\pi}{2}) = \frac{\pi}{2}\,[1 + \frac{1}{4}\,e^2 + \frac{9}{64}\,e^4 + \ldots\ ] \quad ,$$

$$F(e,\tfrac{\pi}{2}) = \frac{\pi}{2}\,[1 - \frac{1}{4}\,e^2 - \frac{3}{64}\,e^4 - \ldots\ ] \quad . \tag{3.56}$$

Furthermore, $\alpha^{(x)}$ of (3.51) may be replaced by (3.55) to give

$$\vec{M}_x = \frac{-\ 2\pi\mu_0\vec{H}_1\ a^3\ e^2}{3[F(e,\frac{\pi}{2}) - E(e,\frac{\pi}{2})]} \quad . \tag{3.57}$$

For small eccentricity, the magnetic dipole is reduced to

$$\vec{M}_x = \frac{-8\mu_0\vec{H}_1\ a^3}{3}\ (1 - \frac{3e^2}{4} - \text{higher order terms in } e^2) \quad . \tag{3.58}$$

It is straight forward to evaluate the transmission coefficient, which is defined as

$$t = \frac{\text{energy transmitted through aperture}}{\text{energy incident on aperture}}$$

$$= \frac{64(ka)^4}{27\pi^2}\ (\,\frac{a}{b}\,)\ [1 - \frac{3e^2}{2} + \ldots\ ] \quad . \tag{3.59}$$

Now, in the same fashion, the problem of a constant magnetic field along the minor axis b of the elliptic aperture (i.e. $\vec{H}_2$ along y-axis) may be readily solved. Here, the magnetic dipole moment of the aperture should be lying in the y direction, and an expression analogous to equation (3.56) is given by

$$\vec{M}_y = \frac{-4\pi\mu_0\vec{H}_2}{3\alpha^{(y)}} \quad , \tag{3.60}$$

where
$$\alpha^{(y)} = \int_0^\infty \frac{dS}{(S + b^2)R_S} \quad . \tag{3.61}$$

Again, let $t^2 = S + b^2$, $\alpha^{(y)}$ is readily expressed as

$$\begin{aligned}\alpha^{(y)} &= \int_b^\infty \frac{2dt}{t^2[(t^2+a^2-b^2)(t^2-b^2)]^{1/2}} \\ &= \frac{2}{a^2-b^2}\left[\int_b^\infty \frac{t^2+a^2-b^2)^{1/2}}{t^2(t^2-b^2)^{1/2}} \right.\\ &\qquad \left. - \int_b^\infty \frac{1}{(t^2-b^2)^{1/2}\ (t^2+a^2-b^2)^{1/2}}\right] .\end{aligned} \tag{3.62}$$

The solutions of the above integrals are also tabulated by Jahnke and Emde. Thus

$$\begin{aligned}\alpha^{(y)} &= \frac{2}{a^2-b^2}\ [\ \frac{a}{b^2}\ E(e,\frac{\pi}{2}) - \frac{1}{a}\ F(e,\frac{\pi}{2})] \\ &= \frac{2}{e^2\ a^3}\ [(1 - e^2)^{-1}\ E(e,\frac{\pi}{2}) - F(e,\frac{\pi}{2})] \quad .\end{aligned} \tag{3.63}$$

Therefore, equation (3.60) becomes

$$\vec{M}_y = \frac{-2\pi\mu_0 \vec{H}_2 a^3 e^2 (1-e^2)}{3[E(e,\frac{\pi}{2}) - (1-e^2)F(e,\frac{\pi}{2})]} \quad . \tag{3.64}$$

For small values of e, equation (3.64) reduces to

$$\vec{M}_y = \frac{-8a^3 \vec{H}_2}{3}[1 - \frac{9}{8} e^2 + \text{higher order terms in } e^2]. \tag{3.65}$$

Again, the transmission coefficient for small values of ka is expressed as (see equation (3.59))

$$t = \frac{64}{27\pi^2} \left(\frac{a}{b}\right)(ka)^4 \left[1 - \frac{9}{4} e^2 + \ldots\right] \quad . \tag{3.66}$$

Comparing the result, elaborated by C. Huang[6], of his treating the diffraction by elliptic apertures using the variational method, equation (3.66) agrees exactly with the lowest term of ka.

Again the electric dipole moment of an elliptic aperture may be calculated in the same manner of treating the static problem of a circular hole in a conducting plane placed in a uniform E-field. First, we seek a solution of the form:

$$\Phi = \Phi_o F(\xi) \tag{3.67}$$

Assume that a constant field $\vec{E}$ is directed along the z-axis, and consequently is perpendicular to the aperture. The potential of the applied field is

$$\Phi_o = -E_o z = -\frac{\sqrt{\xi \eta \zeta}}{a b} \quad . \tag{3.68}$$

[6]C. Haung and R. D. Kodis and H. Levine, "Diffraction by Apertures," J. Appl. Phys., Vol. 26, pp. 151-156, February 1954.

Substituting (3.68) into (3.40) yields

$$\frac{d^2F}{d\xi^2} + \frac{dF}{d\xi}\,\frac{d}{d\xi}\,\ln(R_\xi\,\xi) = 0 \quad , \tag{3.69}$$

where $$R_\xi = \sqrt{(\xi+a^2)(\xi+b^2)\,\xi} \quad . \tag{3.70}$$

The solution of (3.69) is

$$F(\xi) = A \int \frac{d\xi}{\xi^{3/2}\,\sqrt{(\xi+a^2)(\xi+b^2)}} \quad . \tag{3.71}$$

The integral is a form of elliptic integral and may be approximated by using the binomial expansion. To obtain a rigorous solution of the aperture problem, an approach is introduced here, however, which is similar to that used in the calculation of the dipole moment of a dielectric ellipsoid in a uniform E-field[7].

The problem for an oblate ellipsoid and also that for an elliptic aperture in the conducting screen lying perpendicular to the impressed field $\vec{E}_o$ is known to be trivial. In order that the calculation can be carried out, the infinitesmal value c, the semiaxis of ellipsoid along z-axis, is left within the integral (3.71). Formerly, c was assumed to be zero before the integration was carried out.

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[7] J. A. Stratton, loc. cit., p. 211.

The potential functions in different regions are

$$\Phi^{+} = \Phi_0 \, C_1 \int_{\sqrt{\xi}}^{\infty} \frac{dS}{(S+C^2)R_S} \quad , \qquad (\infty > \sqrt{\xi} \geq C)$$

$$\Phi_a = \Phi_0 \, C_2 \quad , \qquad (C \geq \sqrt{\xi} \geq -C) \tag{3.72}$$

$$\Phi^{-} = \Phi_0 \, [1 + C_3 \int_{-\infty}^{\sqrt{\xi}} \frac{dS}{(S+C^2)R_S}] , (0 \geq \sqrt{\xi} \geq -\infty)$$

where $\quad R_S = \sqrt{(S+a^2)(S+b^2)(S+c^2)} \quad , \quad S = \xi, \eta, \text{ or } \zeta \quad .$

The constants C_1, C_2, and C_3 are to be adjusted to satisfy the boundary conditions:

$$[\Phi^{+} = \Phi_a]_{\xi=0^{+}} \; , \quad \varepsilon_0 \left[\frac{\partial \Phi^{+}}{\partial \xi}\right]_{\xi=0^{+}} = \varepsilon_1 \left[\frac{\partial \Phi_a}{\partial \xi}\right]_{\xi=0^{+}} \; , \tag{3.73}$$

and

$$[\Phi_a = \Phi^{-}]_{\xi=0^{-}} \; , \quad \varepsilon_1 \left[\frac{\partial \Phi_a}{\partial \xi}\right]_{\xi=0^{-}} = \varepsilon_0 \left[\frac{\partial \Phi^{-}}{\partial \xi}\right]_{\xi=0^{-}} \; . \tag{3.74}$$

where ε_1 is the dielectric constant of the aperture. Equation (3.73) gives

$$C_1 = \frac{abc}{2} \; \frac{\varepsilon_0 - \varepsilon_1}{\varepsilon_0} \; C_2 \quad , \tag{3.75}$$

and (3.74) leads to

$$C_2 = 1 - C_3 \int_{0}^{\infty} \frac{dS}{(S+C^2)R_S} \quad ,$$

$$-C_3 = \frac{abc}{2} \; \frac{\varepsilon_0 - \varepsilon_1}{\varepsilon_0} \; C_2 \quad . \tag{3.76}$$

From (3.75) and (3.76), the constant C_1 is determined

$$C_1 = \left[\frac{abc}{2} \; \frac{\varepsilon_0 - \varepsilon_1}{\varepsilon_0} \right] \Bigg/ \left[1 - \frac{abc}{2} \; \frac{\varepsilon_0 - \varepsilon_1}{\varepsilon_0} \int_0^\infty \frac{dS}{(S+C^2)R_S} \right] . \tag{3.76}$$

The potential is, therefore,

$$\phi^+ = \frac{\Phi_0 \dfrac{abc}{2} \; \dfrac{\varepsilon_0 - \varepsilon_1}{\varepsilon}}{1 - \dfrac{abc}{2} \; \dfrac{\varepsilon_0 - \varepsilon_1}{\varepsilon_0} \displaystyle\int_0^\infty \frac{dS}{(S+C^2)R_S}} \int_{\sqrt{\xi}}^\infty \frac{dS}{(S+C^2)R_S} , \tag{3.78}$$

where ε_1 must be zero since the electric field inside the aperture is purely transverse. The electric dipole moment corresponding to (3.78) is given by

$$\vec{P} = \frac{-\varepsilon_0 E_0 V}{1 - \frac{abc}{2} \alpha^{(z)}} \hat{z} , \tag{3.79}$$

where $$V = \frac{4\pi}{3} abc , \quad \alpha^{(z)} = \int_0^\infty \frac{dS}{(S+a^2)R_S} . \tag{3.80}$$

The field diffracted by the aperture is described by the radiation of an electric dipole $\vec{P}/2$ located at the aperture, thus

$$\vec{P}_z = \frac{1}{2} \vec{P} . \tag{3.81}$$

It is easy to prove the relation

$$\frac{abc}{2} (\alpha^{(x)} + \alpha^{(y)} + \alpha^{(z)}) = 1 , \tag{3.82}$$

where

$$\alpha^{(x)} = \int_0^\infty \frac{dS}{(S+a^2)R_S} \quad , \quad \alpha^{(y)} = \int_0^\infty \frac{dS}{(S+b^2)R_S} \tag{3.83}$$

Substituting (3.82) into (3.81)yields

$$\vec{P}_z = \frac{\varepsilon_0 E_0 \; 4\pi \; \hat{z}}{3(\alpha^{(x)} + \alpha^{(y)})} \quad . \tag{3.84}$$

Now substitute (3.55) and (3.63) into (3.84) to obtain

$$\vec{P}_z = \frac{\varepsilon_0 E_0 \; 2\pi \; a^3 \; (1-e^2)}{3E(e,\frac{\pi}{2})} \; \hat{z} \quad . \tag{3.85}$$

For small values of e, $\vec{P}_z$ may be expanded in the form

$$\vec{P}_z = \frac{4\varepsilon_0 \vec{E}_0}{3} \; [1 - \frac{3}{4} e^2 - \frac{9}{64} e^4 + \dots] \quad . \tag{3.86}$$

In particular, for a narrow slit aperture, consider a >> b so that e ~ 1 . Considering the field as directed along the major axis, equation (3.51) is

$$\alpha^{(x)} = \int_0^\infty \frac{dS}{(S+a^2)^{3/2} \; (S+b^2)}$$

$$= \frac{2}{a^3 e^2} \; (\ln \sqrt{\frac{1+e}{1-e}} - e)$$

$$\simeq \frac{2}{a^3} \; (\ln \frac{2a}{b} - 1) \quad . \tag{3.87}$$

Comparison of (3.91) with (3.55) yields

$$F(e,\frac{\pi}{2}) = \ln\left(2\,\frac{a}{b}\right) \quad ,$$

$$E(e,\frac{\pi}{2}) = 1 \quad . \tag{3.88}$$

Consequently, the dipole moments of the elliptic slit are expressed as

$$\vec{P}_z = \frac{\varepsilon_0 \vec{E}_0 \; 2\pi \; ab^2}{3} \quad ,$$

$$\vec{M}_x = \frac{\mu_0 \vec{H}_1 \; 2\pi \; a^3}{3[\ln \frac{2a}{b} - 1]} \quad , \tag{3.89}$$

$$\vec{M}_y = \frac{\mu_0 \vec{H}_2 \; 2\pi \; ab^2}{3[1 - (1-e^2)\, \ln \frac{2a}{b}]} \quad .$$

CHAPTER IV

ON THE ELECTROMAGNETIC FIELD IN A CAVITY EXCITED THROUGH A SMALL APERTURE

For a cavity which is excited through an aperture it is well known that the solution to Maxwell's equations can be expressed in terms of the tangential electric and normal magnetic field components (or equivalent magnetic current and charge distributions) in the aperture.[1] However, it is shown here that for apertures whose linear dimensions are small compared with the wavelength and cavity dimensions, the cavity field may be expressed in terms of the electric dipole and magnetic dipole moments of the equivalent magnetic current and charge distributions in the aperture. The approach followed is similar to that used by Collin[2] in treating the excitation of a waveguide through small apertures. Static or quasi-static approximations may be used to determine the dipole moments of the equivalent source distributions in the aperture; and with the aforementioned it is possible to relate the fields penetrating a small aperture into an electromagnetic shield with the surface current and charge distributions on the outside of the shield.

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[1] S. A. Schelkunoff, "Field Equivalence Theorems," Commun. Pure. Appl. Math., Vol. 4, pp. 43-59, June 1951.

[2] R. E. Collin, Field Theory of Guided Waves, McGraw-Hill, Inc., 1960, p. 285.

4.1 Characteristic Fields of a Cavity Resonator

The general representation for electromagnetic field within a simply connected cavity T which is excited via a small aperture in a wall can be expressed in terms of a complete orthonormal set of eigenfunctions.[3] That is

$$\vec{E} = \sum_{m=1}^{\infty} A_m \vec{E}_m \quad , \tag{4.1}$$

$$\vec{H} = \text{grad } \Phi + i \omega \varepsilon_o \sum_{m=1}^{\infty} \frac{A_m}{k_m^2} \text{curl } \vec{E}_m \quad ,$$

where the $\{\vec{E}_m\}_{m=1}^{\infty}$ are the orthonormal short-circuit modes of the cavity and Φ satisfies Laplace's equation, i. e., $\nabla^2\Phi = 0$.

Consider $\vec{E}$, $\vec{H}$ to be harmonic time dependence $e^{i\omega t}$, then Maxwell's equations can be written in the following form:

$$\begin{aligned} \text{curl } \vec{E} &= -i \omega \mu_o \vec{H} \quad , & \text{div } \vec{E} &= 0 \quad , \\ \text{curl } \vec{H} &= i \omega \varepsilon_o \vec{E} \quad , & \text{div } \vec{H} &= 0 \quad . \end{aligned} \tag{4.2}$$

The equations in the first column give

$$\text{curl curl } \vec{E} = k_o^2 \vec{E} \quad , \tag{4.3}$$

where $k_o^2 = \omega^2 \mu_o \varepsilon_o$.

[3]For example, J. C. Slater, "Microwave Electronics", Rev. Mod. Phys., Vol. 18, pp. 441-512, 1946; D. S. Jones, <u>The Theory of Electromagnetism</u>, Pergamon Press, 1964, Chap. 4.

The coefficients in (4.1) can be calculated from the orthonormality relations:

$$A_m = \int_T \vec{E} \cdot \vec{E}_m \, d\tau = \frac{1}{k_m^2} \int_T \vec{E} \cdot \text{curl curl } \vec{E}_m \, d\tau \quad ,$$

and

$$A_m = \frac{1}{k_o^2} \int_T \text{curl } \vec{E} \cdot \text{curl } \vec{E}_m \, d\tau \quad .$$

By subtraction and use of the divergence theorem[4], we obtain

$$(k_o^2 - k_o^2) \, A_m = \int_{S_a+S_1} n \cdot \vec{E} \times \text{curl } \vec{E}_m \, ds \quad , \tag{4.4}$$

where "T" is the volume and "S" is the surface area of the cavity; and $\hat{n}$ is the unit vector normal to the surface. Let $S = S_a + S_1$, S_1 is expressed as perfectly conducting walls and S_a as the aperture.

We define the Green's function $G(\vec{r},\vec{r}')$ such that

$$\nabla^2 G = - \delta(\vec{r} - \vec{r}') \quad , \tag{4.5}$$

and $G(\vec{r},\vec{r}')$ satisfies the Neumann boundary condition. That is,

$$\left. \frac{\partial G}{\partial n} \right|_S = 0 \quad , \tag{4.6}$$

where n is the normal to the surface of the cavity. Then we have

$$\int_S \left(\frac{\partial \Phi}{\partial n} G - \Phi \frac{\partial G}{\partial n} \right) ds = \int_T (G\nabla^2\Phi - \Phi\nabla^2 G) d\tau \quad ,$$

---

[4] i.e., $\int_T \text{div}(\vec{E} \times \text{curl } \vec{E}_m) d\zeta = \int_S \hat{n} \times \vec{E} \cdot \text{curl } \vec{E}_m \, ds$

or

$$\int_S \left(\frac{\partial \Phi}{\partial n} G - \Phi \frac{\partial G}{\partial n} \right) ds = \Phi(\vec{r}') \quad , \tag{4.7}$$

and hence

$$\Phi(r') = \int_S \hat{n} \cdot \text{grad } \Phi \; G(\vec{r},\vec{r}')ds \quad . \tag{4.8}$$

Since $n \cdot \text{grad } \Phi = \hat{n} \cdot \vec{H}$ on S and $\hat{n} \cdot \vec{H} = 0$ on S_1, equation (4.8) becomes

$$\Phi(\vec{r}') = \int_{S_a} \hat{n} \cdot \vec{H} \; G(\vec{r},\vec{r}')ds \quad . \tag{4.9}$$

At the surface of a good, but not perfect, conductor there exists a small tangential electric field[5]

$$\hat{n} \times \vec{E} \Big|_{S_1} \simeq \frac{\omega \mu_0}{2\sigma} \; (1 + i) \; \vec{H} \Big|_{S_1}$$

or

$$\hat{n} \times \vec{E} \Big|_{S_1} \simeq \delta \; \omega \; \mu_0 (1 + i) \; \vec{H} \Big|_{S_1} \quad , \tag{4.10}$$

where $\delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$ is the skin depth; the magnetic field at the surface of the conductor should be essentially the same as in the perfectly conducting case so that

$$\vec{H} \Big|_{S_1} \simeq i \; \omega \; \varepsilon_0 \sum_{m=1}^{\infty} \frac{A_m}{k_m^2} \; \text{curl } \vec{E}_m \Big|_{S_1} \quad , \tag{4.11}$$

---

[5]See J. D. Jackson, Classical Electrodynamics, John Wiley & Sons, Inc., New York, 1962, Chap. 8.

where $\vec{H}\,|_S$ is the tangential magnetic field just outside the surface. With this approximation, (4.10)

$$\hat{n} \times \vec{E}\,\Big|_{S_1} \simeq \frac{i-1}{2}\,\delta\,k_o^2 \sum_{m=1}^{\infty} \frac{A_m}{k_m^2}\ \text{curl}\ \vec{E}_m\Big|_{S_1} \quad . \tag{4.12}$$

Now consider a cavity excited by a small aperture. Substituting (4.12) into (4.4) yields

$$(k_o^2 - k_m^2)\,A_m = \int_S \hat{n} \times \vec{E} \cdot \text{curl}\ \vec{E}_m\ ds$$

$$\simeq \frac{i-1}{2}\ \delta k_o^2\ \frac{A_m}{k_m^2} \int_{S_1} \left|\text{curl}\ \vec{E}_m\right|^2 ds$$

$$+ \int_{S_a} \hat{n} \times \vec{E} \cdot \text{curl}\ \vec{E}_m\ ds \quad . \tag{4.13}$$

The quality factor, Q , of a resonant cavity is defined as

$$Q = 2\pi\ \frac{\text{time average energy stored}}{\text{energy loss per cycle of oscillation}} \quad . \tag{4.14}$$

Then the Q of the cavity is given by (for the $m^{\underline{th}}$ mode)

$$\frac{1}{Q_m} \simeq \frac{1}{2}\ \frac{\delta}{k_m^2} \int_{S_1} \left|\ \text{curl}\ \vec{E}_m\right|^2 ds \quad . \tag{4.15}$$

For a lossy cavity and a very small aperture, we have

$$(k_o^2 - k_m^2)\ A_m \simeq \int_{S_a} \hat{n} \cdot \vec{E} \times \text{curl}\ \vec{E}_m\ ds$$

$$+ (i - 1)\ k_o^2\ A_m/Q_m \quad . \tag{4.16}$$

Since, in general, $\frac{1}{Q_m} << 1$ the equation (4.16) is, approximately:

$$[k_o^2(1 - \frac{i - 1}{Q_m}) - k_m^2]\, A_m \simeq \int_{S_a} \hat{n} \cdot \vec{E} \times \text{curl}\, \vec{E}_m \, ds \quad . \qquad (4.17)$$

4.2 Representation of Dipole Moments

An approximate theory, elaborated by H. A. Bethe, is available which states that the scattered field is considered to be caused by a combination of radiating electric and magnetic dipoles in the aperture. In section 4.1., the electromagnetic field inside a cavity is formulated by equation (4.1) which is determined with (4.9) and (4.17).

In order to represent the fields inside a cavity with related dipole moments, first examine equation (4.9). It is

$$\Phi(\vec{r}\,') = \int_{S_a} \hat{n} \cdot \vec{H}\, G(\vec{r},\vec{r}\,')\, ds \quad .$$

For small aperture[6], the Green's function $G(\vec{r},\vec{r}\,')$ is expanded in a Taylor's series about the center of the aperture, where $\vec{r} = \vec{r}_o$, to get

$$G(\vec{r},\vec{r}\,') = G(\vec{r}_o,\vec{r}\,') + \text{grad}_t\, G\Big|_{\vec{r}=\vec{r}_o} \cdot (\vec{r} - \vec{r}_o) + \ldots \quad ,$$

$$\simeq G(\vec{r}_o,\vec{r}\,') + \text{grad}_t\, G\Big|_{\vec{r}=\vec{r}_o} \cdot (\vec{r} - \vec{r}_o) \quad . \qquad (4.18)$$

---

[6]For apertures whose linear dimension are small compared with the wavelength.

Then (4.9) is given by

$$\Phi(\vec{r}\,') \approx G(\vec{r}_o,\vec{r}\,') \int_{S_a} \hat{n} \cdot \vec{H}\, ds$$

$$+ \operatorname{grad}_t G(\vec{r},\vec{r}\,')\Big|_{\vec{r}=\vec{r}_o} \int_{S_a} \hat{n} \cdot \vec{H}(\vec{r} - \vec{r}_o)\, ds \quad . \qquad (4.19)$$

The boundary conditions on the field may be satisfied with a surface magnetic current distribution and also a charge distribution in the aperture. On the other hand, Schelkunoff[7] has presented a field equivalence principle, in which the normal magnetic field and tangential electric field are not equal to zero on S_a and, hence, a magnetic charge and a magnetic current distribution is given by

$$\eta^* = \hat{n} \cdot \vec{H}$$
$$\vec{K}^* = \hat{n} \times \vec{E} \quad . \qquad (4.20)$$

Hence (4.19) becomes (for convenience r_o is taken to be the origin)

$$\Phi(\vec{r}\,') \approx G(\vec{r}_o,\vec{r}\,') \int_{S_a} \eta^*\, ds + \operatorname{grad}_t G(\vec{r},\vec{r}\,')\Big|_{\vec{r}=\vec{r}_o} \cdot \int_{S_a} \eta^* \vec{r}\, ds$$

$$= \operatorname{grad}_t G(\vec{r},\vec{r}\,')\Big|_{\vec{r}=\vec{r}_o} \cdot \vec{P}_M \quad , \qquad (4.21)$$

where

$$\vec{P}_M = \int_{S_a} \eta^* \vec{r}\, ds \quad ,$$

[7] S. A. Schelkunoff, "Field Equivalence Theorems", Commun. Pure Appl. Math., Vol. 4, pp. 43-59, June 1951.

which is defined as the magnetic dipole moment of the magnetic charge distribution. And

$$\int_{S_a} \eta^* \, ds = \int_{S_a} \hat{n} \cdot \vec{H} \, ds$$

$$= -\frac{1}{i\omega\mu_o} \int_{S_a} \hat{n} \cdot \text{curl } \vec{E} \, ds$$

$$= -\frac{1}{i\omega\mu_o} \oint_C \vec{E} \cdot d\vec{\ell} = 0 \quad , \tag{4.22}$$

since the field satisfies the boundary condition $\vec{E} \cdot d\ell = 0$ which shows that there is no net magnetic charge on S_a, where C is the aperture contour.

Second, examine the integral at the right side of (4.17),

$$I = \int_{S_a} \hat{n} \cdot \vec{E} \times \text{curl } \vec{E}_m \, ds$$

$$= \int_{S_a} \hat{n} \times \vec{E} \cdot \text{curl } \vec{E}_m \, ds \quad . \tag{4.23}$$

Now consider the magnetic surface current density $\vec{K}^*$, which is defined as

$$K^* = \hat{n} \times \vec{E} \quad .$$

Then (4.23) is given by

$$I = \int_{S_a} \vec{K}^* \cdot \text{curl } \vec{E}_m \, ds \quad . \tag{4.24}$$

By analogy, curl $\vec{E}_m$ is expanded in a Taylor's series about the center of the aperture:

$$\text{curl}\ \vec{E}_m = \text{curl}\ \vec{E}_m\Big|_{\vec{r}=\vec{r}_o} + \{(\vec{r} - \vec{r}_o) \cdot \text{grad}_t\}\ \text{curl}\ \vec{E}_m\Big|_{\vec{r}=\vec{r}_o} + \ldots \quad . \tag{4-25}$$

Next, redefine the co-ordinate system so that $r_o = 0$. Then,

$$\text{curl}\ \vec{E}_m \simeq \text{curl}\ \vec{E}_m\Big|_o + (\vec{r} \cdot \text{grad}_t)\ \text{curl}\ \vec{E}_m\Big|_o \quad . \tag{4-26}$$

Substitution of (4.26) into (4.24), gives the following approximation:

$$I \simeq \int_{S_a} \text{curl}\ \vec{E}_m\Big|_o \cdot \vec{K}^* \, ds + \int_{S_a} (\vec{r} \cdot \text{grad}_t)\ \text{curl}\ \vec{E}_m\Big|_o \cdot \vec{K}^* \, ds \quad . \tag{4-27}$$

To evaluate the first integral of (4.27), consider an equivalence expression for it obtained by using the fact that there is no net divergence $\vec{K}^*$ from the aperture. That is,

$$\begin{aligned} \text{div}_t(\psi\vec{K}^*)\ ds &= \int_{S_a} (\psi\ \text{div}_t\ \vec{K}^* + \vec{K}^* \cdot \text{grad}_t\psi)\ ds \\ &= \oint_C \psi\vec{K}^* \cdot \hat{n}_1\ d\ell = 0 \quad . \end{aligned} \tag{4-28}$$

Since the scattered field satisfies the boundary condition $\hat{n}_1 \cdot \vec{K}^* = 0$, there is no normal component of magnetic current at the boundary of the aperture. $\hat{n}_1$ is a unit vector perpendicular to the contour of the aperture and ψ is an arbitrary scalar function.

Let ψ be replaced first by x and then by y . A vector expression for $\int_{S_a} \vec{K}^* \, ds$ is obtained from (4.28) which can be evaluated in terms of magnetic charge density as following:

$$\int_{S_a} \vec{K}^* \, ds = - \int_{S_a} \vec{r} \, \mathrm{div}_t \, \vec{K}^* \, ds$$

$$= i \, \omega \int_{S_a} \vec{r} \, \eta^* \, ds$$

$$= i \, \omega \, \vec{P}_M \quad , \tag{4.29}$$

where

$$\mathrm{div}_t \, \vec{K}^* = - i \, \omega \, \eta^* \quad ,$$

and

$$\vec{P}_M = \int_{S_a} \vec{r} \, \eta^* \, ds \quad . \tag{4.30}$$

The second integral of (4.27) can be evaluated by converting the integrand into a form with rectangular components:

$$(\vec{r} \cdot \mathrm{grad}_t) \, \mathrm{curl} \, \vec{E}_m \Big|_o \cdot \vec{K}^*$$

$$= x \frac{\partial}{\partial x} (\mathrm{curl} \, \vec{E}_m \Big|_o)_x K_x^* + x \frac{\partial}{\partial x} (\mathrm{curl} \, \vec{E}_m \Big|_o)_y K_y^*$$

$$+ y \frac{\partial}{\partial y} (\mathrm{curl} \, \vec{E}_m \Big|_o)_x K_x^* + y \frac{\partial}{\partial y} (\mathrm{curl} \, \vec{E}_m \Big|_o)_y K_y^* \quad . \tag{4.31}$$

Subtracting and adding the same terms, (4.31) becomes

$$(\vec{r} \cdot \text{grad}_t) \text{ curl } \vec{E}_m\Big|_o \cdot \vec{K}^* =$$

$$\frac{x}{2} K_y^* \left[\frac{\partial}{\partial x} (\text{curl } \vec{E}_m\Big|_o)_y - \frac{\partial}{\partial y} (\text{curl } \vec{E}_m\Big|_o)_x\right]$$

$$- \frac{y}{2} K_x^* \left[\frac{\partial}{\partial x} (\text{curl } \vec{E}_m\Big|_o)_y - \frac{\partial}{\partial y} (\text{curl } \vec{E}_m\Big|_o)_x\right]$$

$$+ x \frac{\partial}{\partial x} (\text{curl } \vec{E}_m\Big|_o)_x K_x^* + y \frac{\partial}{\partial y} (\text{curl } \vec{E}_m\Big|_o)_y K_y^* + \frac{x}{2} \frac{\partial}{\partial x} (\text{curl } \vec{E}_m\Big|_o)_y K_y^*$$

$$+ \frac{x}{2} \frac{\partial}{\partial y} (\text{curl } \vec{E}_m\Big|_o)_x K_y^* + \frac{y}{2} \frac{\partial}{\partial x} (\text{curl } \vec{E}_m\Big|_o)_y K_x^*$$

$$+ \frac{y}{2} \frac{\partial}{\partial y} (\text{curl } \vec{E}_m\Big|_o)_x K_x^* \quad . \qquad (4.32)$$

The vector expression is given by

$$(\vec{r} \cdot \text{grad}_t) \text{ curl } \vec{E}_m\Big|_o \cdot \vec{K}^* = \frac{1}{2} \{(\vec{r} \times \vec{K}^*)_z \text{ (curl curl } \vec{E}_m\Big|_o)_z$$

$$+ \vec{r} \cdot (\vec{K}^* \cdot \text{grad}_t) \text{ curl } \vec{E}_m\Big|_o + \vec{K}^* \cdot (\vec{r} \cdot \text{grad}_t) \text{ curl } \vec{E}_m\Big|_o\} \quad .$$

(4.33)

Finally, equation (4.27) is readily expressed as

$$\int_{S_a} (\vec{r} \cdot \text{grad}_t) \text{ curl } \vec{E}_m\Big|_o \cdot \vec{K}^* \, ds$$

$$= \text{curl curl } \vec{E}_m\Big|_o \cdot \vec{P}_E + I_q \quad , \qquad (4.34)$$

where

$$\vec{P}_E = \frac{1}{2} \int_{S_a} \vec{r} \times \vec{K}^* \, ds \quad ,$$

which is defined as the electric dipole moment for magnetic current, $\vec{K}^*$, in analogy with the magnetic dipole moment. I_q represents the quadruple moments[8], i. e.,

$$I_q = \int_{S_a} \frac{1}{2} \{\vec{r} \cdot (\vec{K}^* \cdot \text{grad}_t) \text{ curl } \vec{E}_m\Big|_o$$

$$+ \vec{K}^* \cdot (\vec{r} \cdot \text{grad}_t) \text{ curl } \vec{E}_m\Big|_o \} ds \quad . \qquad (4.35)$$

In general, I_q is small compared with the dipole moment since the quadruple moments involve the square of the aperture dimension and can be neglected with small aperture. Then the integral (4.24) is expressed as

$$I = \int_{S_a} \vec{K}^* \cdot \text{curl } \vec{E}_m \, ds$$

$$\simeq i \, \omega \, \vec{P}_M \cdot \text{curl } \vec{E}_m\Big|_o + \text{curl curl } \vec{E}_m\Big|_o \cdot \vec{P}_E \quad . \qquad (4.36)$$

---

[8] I_q may be written in terms of the last six terms of (4.32) and use the identity of (4.28) to give

$$I_q = i \, \omega \, \{ \frac{\partial}{\partial x} (\text{curl } \vec{E}_m\Big|_o)_x Q_{xx} + \frac{\partial}{\partial y} (\text{curl } \vec{E}_m\Big|_o)_y Q_{yy}$$

$$+ [\frac{\partial}{\partial x} (\text{curl } \vec{E}_m\Big|_o)_y + \frac{\partial}{\partial y} (\text{curl } \vec{E}_m\Big|_o)_x] Q_{xy} \} \quad ,$$

since

$$Q_{xx} = \int_{S_a} x^2 \, \eta^* \, ds \quad , \quad Q_{yy} = \int_{S_a} y^2 \, \eta^* \, ds \quad ,$$

$$Q_{xy} = Q_{yx} = \int_{S_a} x \, y \, \eta^* \, ds \quad .$$

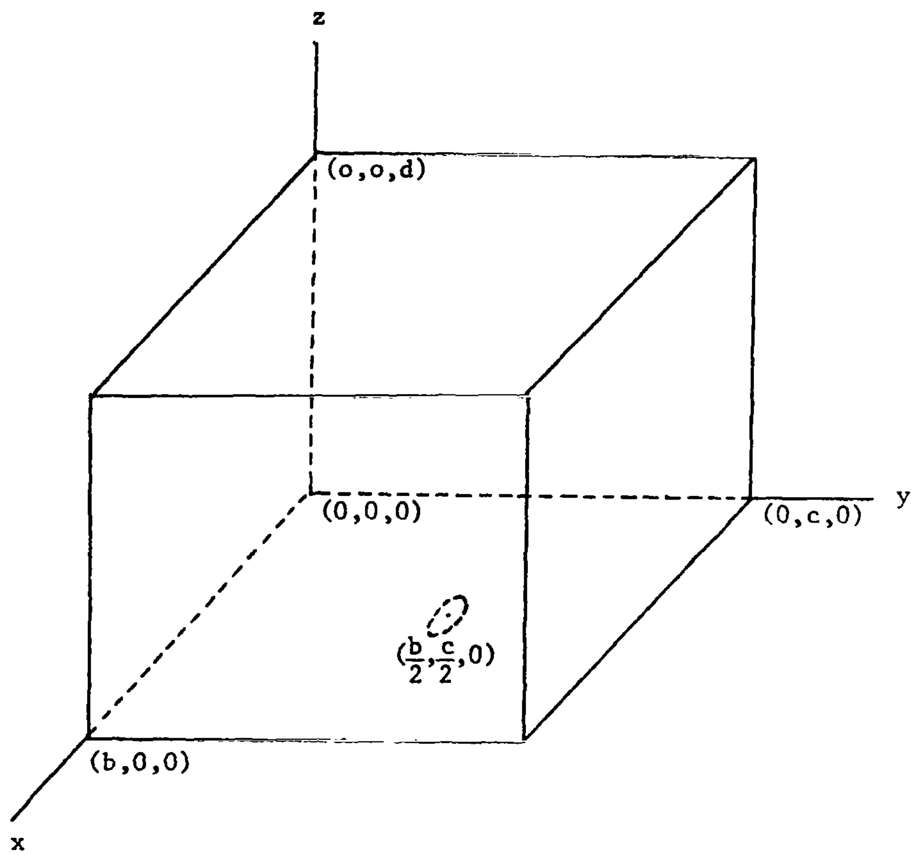

Figure 3. The configuration of rectangular cavity excited through a small aperture with center at (b/2, c/2, 0).

Consequently, equations (4.9) and (4.17), which determine the electromagnetic field inside a cavity in terms of the dipole moments in the aperture may be written as

$$\Phi(\vec{r}\,') \approx \text{grad}_t\, G(\vec{r},\vec{r}\,')\Big|_{\vec{r}=\vec{r}_o} \cdot \vec{P}_M \quad , \tag{4.37}$$

$$[k_o^2\,(1 - \frac{i-1}{Q_m}) - k_m^2]\, A_m \approx i\,\omega\, \vec{P}_M \cdot \text{curl}\, \vec{E}_m\Big|_o$$

$$+ \text{curl curl}\, \vec{E}_m\Big|_o \cdot \vec{P}_E \quad , \tag{4.38}$$

where

$$\text{curl curl}\, \vec{E}_m = k_m^2\, \vec{E}_m \quad . \tag{4.39}$$

4.3 Field Distribution Inside a Rectangular Cavity

The appropriate Green's function for a rectangular box is

$$G(\vec{r},\vec{r}\,') = \sum_{m,n,p} \frac{8/\pi bcd}{(\frac{m}{b})^2 + (\frac{n}{c})^2 + (\frac{p}{d})^2} \cos\frac{\pi m x'}{b} \cos\frac{\pi n y'}{c}$$

$$\cos\frac{\pi p z'}{d} \cos\frac{\pi m x}{b} \cos\frac{\pi n y}{c} \cos\frac{\pi p z}{d} \tag{4.40}$$

It is required that $G(\vec{r},\vec{r}\,')$ satisfies

$$G = -\,\delta(\vec{r} - \vec{r}\,') \quad ,$$

and

$$\frac{\partial G}{\partial n} = 0 \quad , \qquad \text{on } S.$$

The transverse gradient of G(r,r') is given by

$$\mathrm{grad}_t\, G(\vec{r},\vec{r}\,')\Big|_{\vec{r}=\vec{r}_o} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right) G(\vec{r},\vec{r}\,')\Big|_{\vec{r}=\vec{r}_o}$$

$$= \sum_{m,n,p} \frac{8/\pi bcd}{(\frac{m}{b})^2 + (\frac{n}{c})^2 + (\frac{p}{d})^2} \cos\frac{\pi m x'}{b} \cos\frac{\pi n y'}{c}$$

$$\cos\frac{\pi p z'}{d} \begin{cases} -(\frac{\pi m}{b})\,\hat{x} & \text{, for m odd and n even} \\ -(\frac{\pi n}{c})\,\hat{y} & \text{, for m even and m odd,} \\ 0 & \text{, for m, n both even or odd.} \end{cases} \tag{4.41}$$

From (4.37) and the foregoing equation, the scalar function $\Phi(\vec{r}\,')$ is found to be

$$\Phi(\vec{r}\,') = \sum_{m,n,p} \frac{8/\pi bcd}{(\frac{m}{b})^2 + (\frac{n}{c})^2 + (\frac{p}{d})^2} \cos\frac{\pi m x'}{b} \cos\frac{\pi n y'}{c}$$

$$\cos\frac{\pi p z'}{d} \begin{cases} -(\frac{\pi m}{b})\,\hat{x}\cdot\vec{P}_M & \text{, for m odd and n even,} \\ -(\frac{\pi n}{c})\,\hat{y}\cdot\vec{P}_M & \text{, for m even and n odd,} \\ 0 & \text{, for m,n both even or odd .} \end{cases} \tag{4.42}$$

Note that the center of the aperture is located at $(\frac{b}{2}, \frac{c}{2}, 0)$.

It is convenient to separate the total field inside the cavity into modes; transverse-electric (TE) modes and transverse-magnetic (TM) modes. These modes in normalized form are,

$TE_{m,n,p}$ modes

$$E_x = \frac{ik_2}{k_c} \sqrt{\frac{8}{bcd}} \cos k_1x \ \ \sin k_2y \ \ \sin k_3z \quad ,$$

$$E_y = \frac{-ik_1}{k_c} \sqrt{\frac{8}{bcd}} \sin k_1x \ \ \cos k_2y \ \ \sin k_3z \quad ,$$

$$E_z = 0 \quad , \tag{4.43}$$

$$H_x = \frac{-k_1k_3}{\omega\mu_0k_c} \sqrt{\frac{8}{bcd}} \sin k\ x \ \ \cos k\ y \ \ \cos k\ z \quad ,$$

$$H_y = \frac{k_2k_3}{\omega\mu_0k_c} \sqrt{\frac{8}{bcd}} \cos k_1x \ \ \sin k_2y \ \ \cos k_3z \quad ,$$

$$H_z = \frac{k_c}{\omega\mu_0} \sqrt{\frac{8}{bcd}} \cos k_1x \ \ \sin k_2y \ \ \sin k_3z \quad ,$$

$TM_{m,n,p}$ modes

$$E_x = \frac{-k_1k_3}{\gamma k_c} \sqrt{\frac{8}{bcd}} \cos k_1x \ \ \sin k_2y \ \ \sin k_3z \quad ,$$

$$E_y = \frac{-k_2k_3}{\gamma k_c} \sqrt{\frac{8}{bcd}} \sin k_1x \ \ \cos k_2y \ \ \sin k_3z \quad ,$$

$$E_z = \frac{k_c}{\gamma} \sqrt{\frac{8}{bcd}} \sin k_1x \ \ \sin k_2y \ \ \cos k_3z \quad , \tag{4.44}$$

$$H_x = \frac{i\omega\varepsilon_0k_2}{\gamma k_c} \sqrt{\frac{8}{bcd}} \sin k_1x \ \ \cos k_2y \ \ \cos k_3z \quad ,$$

$$H_y = \frac{-i\omega\varepsilon_0k_1}{\gamma k_c} \sqrt{\frac{8}{bcd}} \cos k_1x \ \ \sin k_2y \ \ \cos k_3z \quad ,$$

$$H_z = 0 \quad ,$$

where $k_1 = \frac{m\pi}{b}$, $k_2 = \frac{n\pi}{c}$, $k_3 = \frac{p\pi}{d}$,

$$k_c^2 = k_1^2 + k_2^2 \quad ,$$

and

$$\gamma^2 = k_1^2 + k_2^2 + k_3^2 \quad . \tag{4.45}$$

The integers m, n, and p may be viewed as the number of half-period variations of $\vec{E}$ and $\vec{H}$ along x, y, and z directions, respectively. The restrictions, m, n, or p > 0, for the TE-modes and for the TM-modes, are introduced to avoid trival cases in which $\vec{E}$ or $\vec{H}$ field components vanish.

In view of (4.43) and (4.44),

$$\vec{E}_m \Big|_{r=r_o} = \frac{k_c}{\gamma} \sqrt{\frac{8}{bcd}} \quad \sin\frac{m\pi}{2} \quad \sin\frac{n\pi}{2} \quad \hat{z}$$

$$= \begin{cases} \dfrac{k_c}{\gamma} \sqrt{\dfrac{8}{bcd}} & , \text{ for } m \text{ , } n \text{ odd} \\ 0 & , \text{ otherwise} \quad . \end{cases} \tag{4.46}$$

$$\text{curl } \vec{E}_m \Big|_{\vec{r}=\vec{r}_o} = - i\omega\mu_o \, \vec{H}_m \Big|_{\vec{r}=\vec{r}_o}$$

$$= \begin{cases} - i\omega\mu_o \sqrt{\dfrac{8}{bcd}} \left[\dfrac{-k_1 k_3}{\omega\mu_o k_o} + \dfrac{i\omega\varepsilon_0 k_2}{\gamma k_c} \right] , \text{ for m odd and n even,} \\ - i\omega\mu_o \sqrt{\dfrac{8}{bcd}} \left[\dfrac{-k_2 k_3}{\omega\mu_o k_o} \quad \dfrac{i\omega\varepsilon_0 k_1}{\gamma k_c} \right] , \text{ for m even and n odd,} \\ 0 \qquad , \text{ for m , n both even or odd.} \end{cases} \tag{4.47}$$

Substituting (4.46) and (4.47) into (4.38) yields, for m odd and n even,

$$[k_o^2\ (1 - \frac{i-1}{Q_m}) - \gamma^2]\ A_m \simeq \omega^2\mu_o \sqrt{\frac{8}{bcd}}\ \vec{P}_M \cdot \hat{x}\left[\frac{-k_1k_3}{\omega\mu_o k_c} + \frac{i\omega\varepsilon_0 k_2}{\gamma k_c}\right] , \tag{4.48}$$

for m even and n odd,

$$[k_o^2\ (1 - \frac{i-1}{Q_m}) - \gamma^2]\ A_m \simeq \omega^2\mu_o \sqrt{\frac{8}{bcd}}\ \vec{P}_M \cdot \hat{y}\left[\frac{-k_2k_3}{\omega\mu_o k_c} - \frac{i\omega\varepsilon_0 k_1}{\gamma k_c}\right] , \tag{4.49}$$

for m odd and n odd,

$$[k_o^2\ (1 - \frac{i-1}{Q_m}) - \gamma^2]\ A_m \simeq \gamma k_c \sqrt{\frac{8}{bcd}}\ \vec{P}_E \cdot \hat{z} \tag{4.50}$$

for m even and n even,

$$A_m = 0 \tag{4.51}$$

Note that the subscript m on the left side of (4.46) - (4.50) is symbolic of the three indices required of the expansion coefficients in the model expansion of rectangular cavity field.

Consider that $Q_m = 10$,

and

$$k_o b = k_o c = k_o d = 1 , \tag{4.52}$$

then (4.48) becomes ,

for m odd and n even,

$$A_m \simeq \frac{\frac{\omega}{k_o}\sqrt{\frac{8}{bcd}}\ \vec{P} \cdot \hat{x}\left[\frac{-mp}{\sqrt{m^2+n^2}} + \frac{in}{\sqrt{(m^2+n^2)(m^2+n^2+p^2)}}\right]}{[1 - \frac{i-1}{10} - (m^2+n^2+p^2)\pi^2]} \tag{4.53}$$

for m even and n odd,

$$A_m \simeq \frac{\frac{\omega}{k_o}\sqrt{\frac{8}{bcd}}\,\vec{P}_M\cdot\hat{y}\left[\frac{-np}{\sqrt{m^2+n^2}}-\frac{im}{\sqrt{(m^2+n^2)(m^2+n^2+p^2)}}\right]}{[1-\frac{i-1}{10}-(m^2+n^2+p^2)\pi^2]} \tag{4.54}$$

for m odd and n odd,

$$A_m \simeq \frac{\sqrt{(m^2+n^2)(m^2+n^2+p^2)}\;\pi^2\sqrt{\frac{8}{bcd}}\,\vec{P}_E\cdot\hat{z}}{[1-\frac{i-1}{10}-(m^2+n^2+p^2)\pi^2]} \tag{4.55}$$

for m even and n even,

$$A_m = 0 \quad .$$

Now consider contributions from the m = 1, n = 0, p = 1 modes inside a rectangular cavity when excited through a small elliptic aperture in the bottom wall of the cavity. From TE_{101} the contribution is

$$\begin{aligned} E_y &= -i\sqrt{\frac{8}{bcd}}\,\sin\frac{\pi x}{b}\,\sin\frac{\pi z}{d} \quad , \\ H_x &= -\frac{\pi}{\omega\mu_o d}\sqrt{\frac{8}{bcd}}\,\sin\frac{\pi x}{b}\,\cos\frac{\pi z}{d} \quad , \\ H_z &= \frac{\pi}{\omega\mu_o b}\sqrt{\frac{8}{bcd}}\,\cos\frac{\pi x}{b}\,\sin\frac{\pi z}{d} \quad , \end{aligned} \tag{4.56}$$

From (4.53) and (3.59) the model expansion coefficient is given by

$$A_{101} \simeq \frac{\omega}{k}\sqrt{\frac{8}{bcd}}\,\frac{\hat{x}\cdot\vec{P}_M}{2\pi^2} = -\frac{\omega}{k}\sqrt{\frac{8}{bcd}}\,\frac{\mu_0 H_1 a^3 e^2}{3\pi[F(e)-E(e)]} \quad . \tag{4.57}$$

Substituting (4.57) and (4.52) into (4.1), the contribution to the field inside the cavity from the m = 1, n = 0, p = 1 mode is

$$\vec{E} = A_{101}\,\vec{E}_y \simeq \frac{i\omega}{k_o}\sqrt{\frac{8}{bcd}}\;\frac{\hat{x}\cdot\vec{P}_M}{2\pi^2}\;\hat{y} \quad , \tag{4.58}$$

$$\vec{H} = \text{grad}\ \Phi + \frac{A_{101}}{2\pi^2}\;(\vec{H}_x + \vec{H}_z)$$

$$\simeq \sqrt{\frac{8}{bcd}}\;\frac{\hat{x}\cdot\vec{P}_M}{2\pi^2}\Big[(\pi^3 - \frac{1}{2\pi\mu_o})\ \sin\frac{\pi x}{b}\ \cos\frac{\pi z}{d}\ \hat{x}$$

$$+ (\pi^3 + \frac{1}{2\pi\mu_o})\ \cos\frac{\pi x}{b}\ \sin\frac{\pi z}{d}\ \hat{z}\ \Big] \quad , \tag{4.59}$$

where

$$\hat{x}\cdot\vec{P}_M = \frac{-2\pi\ \mu_0 H_1\ a^3 e^2}{3[F(e) - E(e)]} \quad . \tag{4.60}$$

CHAPTER V

SUMMARY AND CONCLUSIONS

This thesis presents a theoretical formulation treating electromagnetic waves diffracted by a small aperture. A modification of the small-hole coupling theory is used to obtain the field penetration into the cavity via an aperture. The basic considerations of the low-frequency diffraction formulation are

(a) to neglect retardation so that the phase changes in the field across the apertures (or on the surface of the complementary obstacles) are negligible and so that the quasi-static approximations may be used and

(b) to assume that the incident electromagnetic field is uniform over the aperture. Thus the obtained results are accurate only for sufficiently low frequency, i. e., ka << 1.

The dipole moments $\vec{M}_x$, $\vec{M}_y$, and $\vec{P}_z$ of the aperture distributions are calculated for small apertures in finite as well as infinite screens. After the dipole moments are obtained, the field transmitted through a hole in a cavity is calculated by finding the radiation from the dipoles into the cavity.

In Chapter III, a quasi-static approximation for solving diffraction by elliptic apertures is presented along with an investigation of circular aperture diffraction. The results obtained are in agreement with those obtained by other investigators who use different solution techniques for obtaining the diffraction by apertures in infinite screens.

In analogy with the treatment of transmission fields through small apertures into waveguides, Chapter IV presents a theoretical solution for the field penetration into cavities via a hole. In particular, rectangular cavities are investigated. It is proved that the field penetration into the cavity may be expressed in terms of the dipole moments of the aperture distribution. This result is particularly significant because quasi-static diffraction theory may be used to determine the dipole moments. Hence a very difficult boundary value problem may be solved directly by use of few approximations.

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