Interaction Notes
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A. TRANSIENT CURRENTS ON A PERFECTLY CONDUCTING CYLINDER IMMERSED IN A LOSSY TIME VARYING PLASMA (U)

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In a preceding note,[1] two equivalent time domain integrodifferential equations were obtained which govern the currents induced by transient fields incident on a perfectly conducting body immersed in a lossy time varying plasma. In this note, we will reduce one of these equations to the special geometrical case of a finite length, circular cylinder illuminated by a plane wave propagating perpendicular to the axis of the cylinder with the $\bar{E}$-field parallel to the axis. The time varying plasma will be represented as a transient pulse of ionization propagating at the speed of light in the same direction as the EM wave and characterized by a constant collision frequency ν and an electron density n which is uniform in a direction parallel to the z axis (Figure 1). On the basis of this equation, we will attempt to show that, for times earlier than the rise time of the current, the effect of the time varying electron density is to decrease the current induced on the cylinder to a level below that which would be produced by the same field incident on the same cylinder in free space.

Equation 42 of Reference 1 gives the following integrodifferential equation for the current density $\bar{J}_s$ as a function of the retarded time $\tau = t - R/c$ in terms of the magnetic field $\bar{H}_i$ incident on a perfect conductor immersed in a plasma with a time varying electron density.

$$\bar{J}_s(\tau, \bar{r}) = 2\hat{n} \times \bar{H}_i(\tau) + 2\hat{n} \times \nabla \times B[J(\tau, \bar{r})] \qquad (1)$$

$\hat{n}$ = unit vector normal to the surface

where all quantities are to be evaluated on the surface and the linear operator $B[\cdot]$ is given by

$$B[\cdot] = B_o[\cdot] + B_1[\cdot] + B_2[\cdot] \qquad (2)$$

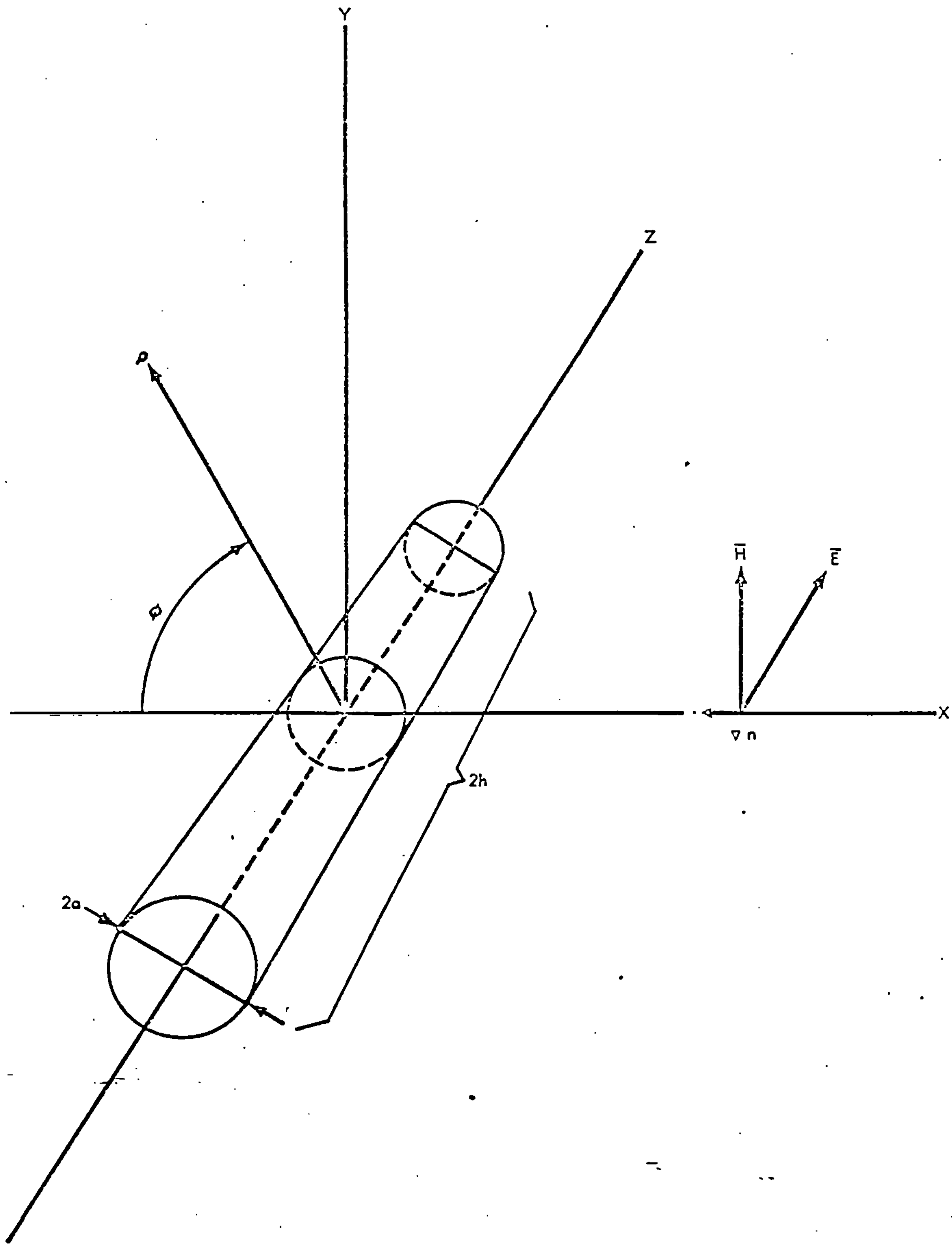

Figure 1.

and

$$B_o[\cdot] = \int_v \frac{dv[\cdot]}{4\pi R} \tag{3}$$

$$B_1[\cdot] = -\frac{\mu_o}{m}\left(\frac{e}{4\pi}\right)^2 \int_v \frac{dvn(\tau)}{R} \int_v \frac{dv'}{R'} \int_o^\tau d\lambda\, e^{-\nu(\tau-\lambda)} \frac{\partial[\cdot]}{\partial\lambda} \tag{4}$$

$$B_2[\cdot] = \frac{1}{m\epsilon_o}\left(\frac{e}{4\pi}\right)^2 \int_v \frac{dvn(\tau)}{R} \cdot \nabla \int_v \frac{dv'}{R'} \int_o^\tau d\lambda\, e^{-\nu(\tau-\lambda)} \int_o^\lambda d\lambda' \nabla\cdot[\cdot] \tag{5}$$

While Equation 1 involves some extremely complicated integral and differential operators, the relatively simple form of this expression permits a straightforward physical interpretation: The induced current density at a given point on the conductor is given as the sum of four terms. The first term is the primary contribution due to the incident magnetic field at the point. The second term is the current induced at the point due to EM-fields produced by currents flowing at all other points on the body. We note that this term does not depend explicitly on the plasma parameters, and thus it may be interpreted as the current which would have been induced by currents at other points on the conductor, <u>if the conductor were in free space</u>. The third and fourth terms are currents induced at the point due to fields produced by the interaction of the plasma with current and charge distributions along the conductor. The magnitudes of these terms depend most strongly on the electron density $n(\tau)$ and vanish as expected when $n(\tau) = 0$.

We can simplify Equation 1 somewhat by first noting that the effect of the operator B_2 will be negligible compared to B_1 except possibly at late times. B_2 involves integrals over a surface charge distribution which will require times on the order of the missile length divided by the velocity of light to become appreciable. Accordingly we will not include B_2 in the following considerations.

The remaining terms in Equation 1 can be simplified if we ignore time delays less than the diameter of the cylinder divided by the velocity of light. This is equivalent to assuming that the incident EM-field and the ionizing pulse are propagated instantaneously across the diameter of the cylinder: Thus, we have

$$\bar{H}_i(\tau) = \bar{H}_i(t) \tag{6}$$

$$n(\tau) = n(t) \tag{7}$$

Since most cylinders of interest are on the order of a few feet in diameter, this assumption will limit our consideration to events greater than 2 - 3 ns in duration.

Dropping the term involving $B_2[\cdot]$ and employing Equations 6 and 7, we can rewrite Equation 1 as follows:

$$\bar{J}_s(\tau, \bar{r}) = 2\hat{n} \times \bar{H}_i(t) + 2\hat{n} \times \nabla \times B_o[\bar{J}_s(\tau, \bar{r})] + 2\hat{n} \times \nabla \times B_1[\bar{J}_s(\tau, \bar{r})] \tag{8}$$

where

$$B_o[\cdot] = \int_v \frac{dv[\cdot]}{4\pi R} \tag{9}$$

$$B_1[\cdot] = -\frac{\omega_p^2(t)}{16\pi^2} \int_v \frac{dv}{R} \int_v \frac{dv'}{R'} \int_o^\tau d\lambda e^{-\nu(\tau-\lambda)} \frac{\partial[\cdot]}{\partial \lambda} \tag{10}$$

The quantity $\omega_p^2(t)$ defined as follows

$$\omega_p^2(t) \equiv \frac{e^2 n(t)}{\epsilon m} \tag{11}$$

may be interpreted as an instantaneous plasma frequency in analogy to the conventionally defined plasma frequency. The usual antenna theory assumptions of an axially directed current density[2]

$$\bar{J}_s(\tau, z)\, dv = \hat{i}_z\, I(\tau, z)\, dz \tag{12}$$

and

$$R = \left[(z - \zeta)^2 + \rho^2\right]^{\frac{1}{2}} \quad R' = \left[(\zeta - \zeta')^2 + \rho^2\right]^{\frac{1}{2}} \tag{13}$$

can now be employed to simplify (8) still further. Since $\bar{J}_s(\tau, z)$ consists of a single component, the curl operator in cylindrical coordinates reduces to[3]

$$\nabla \times [\cdot] = -\hat{i}_\phi \frac{\partial[\cdot]}{\partial \rho}$$

Hence

$$\hat{n} \times \nabla \times [\cdot] = \hat{i}_\rho \times \nabla \times [\cdot] = -\hat{i}_\rho \times \hat{i}_\phi \frac{\partial [\cdot]}{\partial \rho} = -\hat{i}_z \frac{\partial [\cdot]}{\partial \rho} \tag{14}$$

Using (12), (13), (14) and a considerable amount of algebra, we can reduce (8) to a single scalar equation in $I(\tau, z)$:

$$I(\tau, z) = 2H_\phi(t) + a^2 \int_{-h}^{h} \frac{d\zeta}{R^2} \left[\frac{1}{c} \frac{\partial I(\tau,\zeta)}{\partial \tau} + \frac{I(\tau,\zeta)}{R} \right]$$

$$- \frac{a^4 \omega_p^2(t)}{4\pi c^2} \iint_{-h}^{h} \frac{d\zeta d\zeta'}{R^2 R'} \left[\frac{1}{c} \frac{\partial I(\tau - \frac{R'}{c}, \zeta')}{\partial \tau} + \left(\frac{1}{R} - \frac{\nu}{c} \right) I'\left(\tau - \frac{R'}{c}, \zeta'\right) \right] \tag{15}$$

where

$$I'\left(\tau - \frac{R'}{c}, \zeta'\right) = \int_0^{\tau - \frac{R'}{c}} d\lambda \, e^{-\nu(\tau - \frac{R'}{c} - \lambda)} \frac{\partial I(\lambda, \zeta')}{\partial \lambda} \tag{16}$$

and

$$\left.\begin{aligned} R &= \left[(z-\zeta)^2 + a^2 \right]^{\frac{1}{2}} \\ R' &= \left[(\zeta - \zeta')^2 + a^2 \right]^{\frac{1}{2}} \end{aligned}\right\} \tag{17}$$

Equation 15 remains a very complicated expression; however, it is now in a form which is suitable for treatment by a modified form of Bennett's time domain method.[4,5] Without going into this method, let us attempt to gain some qualitative information about the effect of the plasma by estimating the relative effect of the terms on the right hand side of (15) for representative values of $\omega_p^2(t)$, ν, and a.

As noted previously, the second term in (15)

$$I_{fs} = a^2 \int_{-h}^{h} \frac{d\zeta}{R^2} \left[\frac{1}{c} \frac{\partial I(\tau,\zeta)}{\partial \tau} + \frac{I(\tau,\zeta)}{R} \right] \tag{18}$$

can be interpreted as the current density which would have been induced at a given point by currents at other points on the conductor, if the conductor were in free space. The third term

$$I_{pl} = -\frac{a^4 \omega_p^2(t)}{4\pi c^2} \int_{-h}^{h}\int \frac{d\zeta\, d\zeta'}{R^2 R'} \left[\frac{1}{c} \frac{\partial I(\tau - \frac{R'}{c}, \zeta')}{\partial \tau} + \left(\frac{1}{R} - \frac{\nu}{c}\right) I'\left(\tau - \frac{R'}{c}, \zeta'\right)\right] \quad (19)$$

is the induced current density due to the plasma.

Comparing these terms, we note that, in general, the free space and plasma contributions tend to cancel owing to the difference in sign. This will always be true in the early time regime, since the integrands in (18) and (19) will always have the same sign when τ is small. That is, in the neighborhood of $\tau = 0$ we have

$$\frac{1}{c}\frac{\partial I}{\partial \tau} + \frac{I}{R} > 0 \quad (20)$$

$$\frac{1}{c}\frac{\partial I}{\partial \tau} + \left(\frac{1}{R} - \frac{\nu}{c}\right) I' > 0 \quad (21)$$

since I and I' will vanish more rapidly than $\partial I/\partial \tau$ as $\tau \to 0$. In the intermediate regime where

$$\frac{\partial I}{\partial \tau} \approx 0, \quad I > 0,$$

that is, the time regime where the current reaches its first maximum, (20) and (21) reduce to

$$\frac{I}{R} > 0 \quad (22)$$

and

$$\left(\frac{1}{R} - \frac{\nu}{c}\right) I' \approx 0 \, . \quad (23)$$

Equation 23 is based on the assumption that

$$\nu\tau_{max} >> 1 \quad (24)$$

since in this case we have from (16)

$$I' = \int_0^{\tau_{max}} d\lambda \;\; e^{-\nu(\tau_{max}-\lambda)} \frac{\partial I(\lambda, \xi)}{\partial \lambda} \approx \int_{-\infty}^{\infty} d\lambda \;\; \delta(\tau_{max} - \lambda) \frac{\partial I(\lambda \xi)}{\partial \lambda}$$

$$\approx \frac{\partial I(\tau_{max}, \xi)}{\partial \tau}$$

$$\approx 0 \tag{25}$$

Thus, when (24) is satisfied, $I_{pl} << I_{fs}$ and the plasma contribution to the induced current is negligible compared to the free space contribution. This will occur, for example, when

$$\tau_{max} \geq 10^{-8} \text{ s and } \nu \geq 10^{9} \text{ s}^{-1} .$$

At later times $(\tau > \tau_{max})$ when

$$\frac{\partial I}{\partial \tau} < 0 \text{ and } I > 0$$

one or both of the integrands may change signs depending on the behavior of the exciting field and the magnitude of ν. In this regime, it is difficult to determine if the plasma contribution will add to, or subtract from, the free space contribution. However, we can estimate the magnitude of the plasma contribution relative to the free space contribution by forming the ratio

$$\frac{\overline{|I_{pl}|}}{\overline{|I_{fs}|}} \approx \frac{a^2 \overline{\omega_p^2(t)}}{4\pi c^2} \left[\frac{\overline{\left| \int_{-h}^{h}\int \frac{d\zeta d\zeta'}{R^2 R'} \left[\frac{1}{c} \frac{\partial I(\tau - \frac{R'}{c}, \zeta')}{\partial \tau} + \left(\frac{1}{R} - \frac{\nu}{c}\right) I'\left(\tau - \frac{R'}{c}, \zeta'\right)\right] \right|}}{\overline{\left| \int_{-h}^{h} \frac{d\zeta}{R^2} \left[\frac{1}{c} \frac{\partial I(\tau, \zeta)}{\partial \tau} + \frac{I(\tau, \zeta)}{R}\right] \right|}} \right] \tag{26}$$

where the bars indicate spatial and temporal averages. Since

$$\frac{\left| \int_{-h}^{h}\int \frac{d\zeta\, d\zeta'}{R^2 R'} \left[\frac{1}{c} \frac{\partial I(\tau - \frac{R'}{c}, \zeta')}{\partial \tau} + \left(\frac{1}{R} - \frac{\nu}{c}\right) I'\left(\tau - \frac{R'}{c}, \zeta'\right) \right] \right|}{\left| \int_{-h}^{h} \frac{d\zeta}{R^2} \left[\frac{1}{c} \frac{\partial I(\tau, \zeta)}{\partial \tau} - \frac{I(\tau, \zeta)}{R} \right] \right|} \approx 1 \qquad (27)$$

we have from (26)

$$\frac{\overline{|I_{pl}|}}{\overline{|I_{fs}|}} \approx \frac{a^2 \overline{\omega_p^2(t)}}{4\pi c^2} \qquad (28)$$

For a cylinder of radius 1 meter and an average electron density

$$\overline{n(t)} \approx 10^{14} \text{ m}^{-3}$$

we have

$$\frac{\overline{|I_{pl}|}}{\overline{|I_{fs}|}} \approx 0.28 \ .$$

In this case, we would expect the plasma contribution to be less than 28 percent of the free space contribution on the average.

From the preceding observations, we conclude that, for times earlier than the rise time of the current, the effect of a lossy, time varying plasma is to reduce the induced current below that which would occur if the cylinder were in free space. For any time later than the rise time, the plasma could lead to induced currents greater than the corresponding free space currents, which would occur at the same time. However, this effect, if it exists, will be small for most cylinders of interest, if the average electron density is less than 10^{14} m^{-3} .

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