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The Axial Current Induced on an Infinitely Long, Perfectly Conducting, Circular Cylinder in Free Space by a Transient Electromagnetic Plane Wave

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Abstract

The time behavior is obtained of the total axial current induced upon an infinitely long, perfectly conducting, circular cylinder located in free space by an incident electromagnetic plane wave. The current waveforms are calculated and graphed for incident fields with time histories of the Dirac delta function, step-function, and decaying exponential. Analytical expressions are derived for the early and late time asymptotic behavior of the current. Also, the small and large inverse time constant asymptotic expansions of the peak current response to an exponentially decaying electromagnetic pulse are obtained.

Acknowledgment

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I. Introduction

The results presented in this note are a first step in obtaining a knowledge of the current induced on a very long cylindrical structure, such as a long wire antenna used in very low-frequency communication systems, subjected to electromagnetic transients. The problem is idealized by calculating the transient axial current induced on the surface of an infinitely long, perfectly conducting, circular cylinder in free space by a transient incident plane wave. The incident plane wave has an arbitrary polarization and an arbitrary frequency dependence. The geometry of the problem is presented in fig. 1 where the axis of the cylinder of radius "a" coincides with the z-axis. The surface of the cylinder is defined by the surface $\Psi = a$ and the incident wave has a propagation vector $\vec{k}$ in the $\phi = \phi_1$ plane and the angle of incidence is defined by ϕ_1.

The problem is divided into two parts. First, the axial current induced on the cylinder by an incident vector plane wave polarized with the electric field vector perpendicular to the axis of the cylinder is considered. It is shown that the induced axial current is zero for this case. Secondly, the axial current induced on the cylinder by an incident vector plane wave polarized with the magnetic field vector perpendicular to the axis of the cylinder is calculated. The axial current induced on the cylinder by an incident wave of arbitrary polarization is the linear combination of the currents induced by the two fixed polarized waves contained in the wave of arbitrary polarization. The two fixed polarizations described above are shown in fig. 2.

The solution of the idealized problem will provide an early time solution for the real antenna since the assumption of an infinitely long structure is no limitation in the time interval of interest. That is, the time interval between the instant the leading edge of the incident wave reaches the cylindrical surface where the current is observed and the instant when the reflections from the ends of the structure have effect.

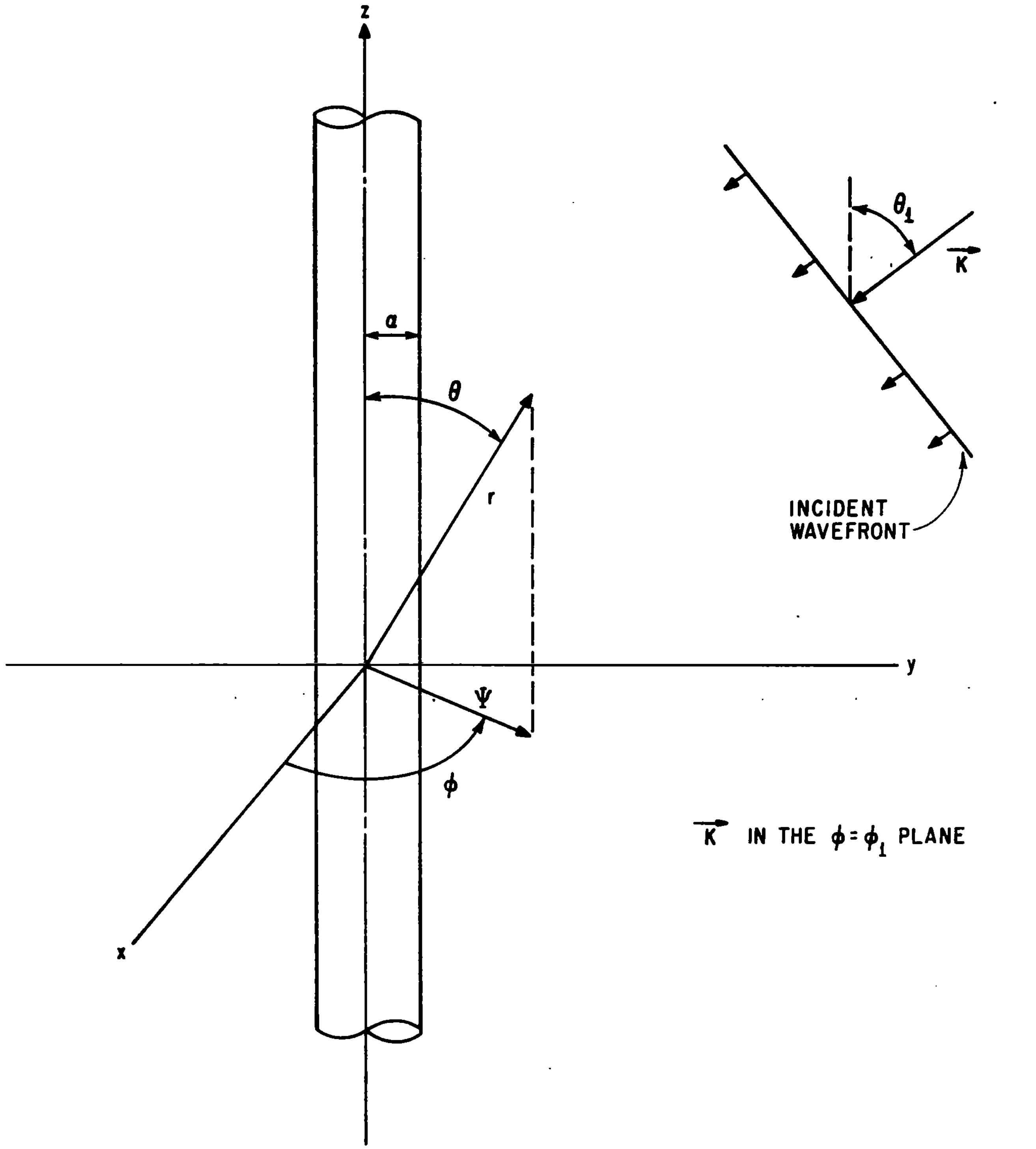
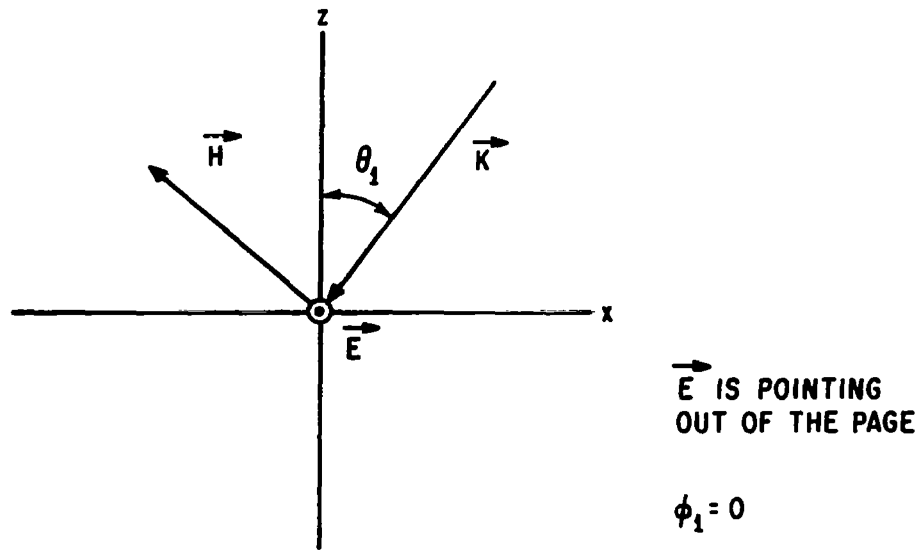
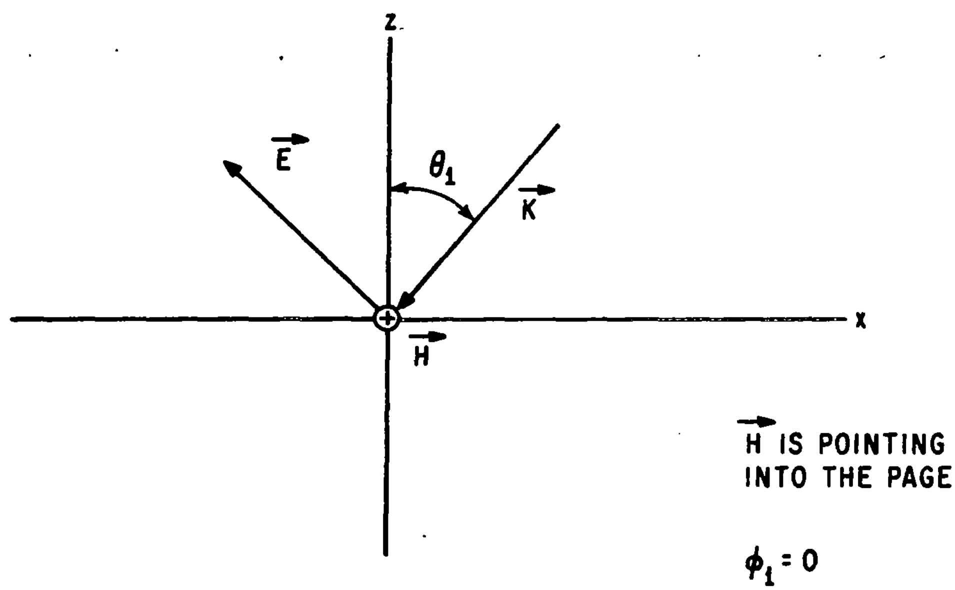


FIGURE 1. INFINITELY LONG CIRCULAR CYLINDER AND AN INCIDENT VECTOR PLANE WAVE.



A. ELECTRIC FIELD POLARIZED PERPENDICULAR TO THE z-AXIS



B. MAGNETIC FIELD POLARIZED PERPENDICULAR TO THE z-AXIS

FIGURE 2. TWO POLARIZATIONS TO CONSIDER FOR THE INCIDENT WAVE.

From fig. 3 we see that the applicable time interval is from zero time to the minimum value of

$$\left[\frac{\ell_1}{c}(1 - \cos\theta_1) \; , \quad \frac{\ell_2}{c}(1 + \cos\theta_1)\right]$$

Zero time is referenced to the time that the leading edge of the incident wave reaches the cylindrical surface where the current is observed and c is the speed of light.

II. Electromagnetic Fields in Cylindrical Coordinates

The general form for an electric field with zero divergence can be written as[1†]

$$\vec{E} = E_o \sum_{n=o}^{\infty} \left\{ \alpha_n \vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) + \beta_n \vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) \right\} \tag{1}$$

where E_o is some convenient constant with dimensions of volts per meter, α_n and β_n are appropriate coefficients, and $\vec{M}^{(\ell)}$ and $\vec{N}^{(\ell)}$ are vector eigenfunctions in cylindrical coordinates. Note that only a particular discrete ζ_1 is used in equation (1) thus simplifying the expression for the electric field as given by equation (28) in reference 1. ζ_1 will be clarified later in this section.

The magnetic field expansion can be obtained from the electric field expansion by replacing

$$\begin{aligned} \vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) &\rightarrow \frac{i}{Z}\,\vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) \\ \vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) &\rightarrow \frac{i}{Z}\,\vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) \end{aligned} \tag{2}$$

where Z is the wave impedance.

†The frequency dependence is suppressed in equation (1).

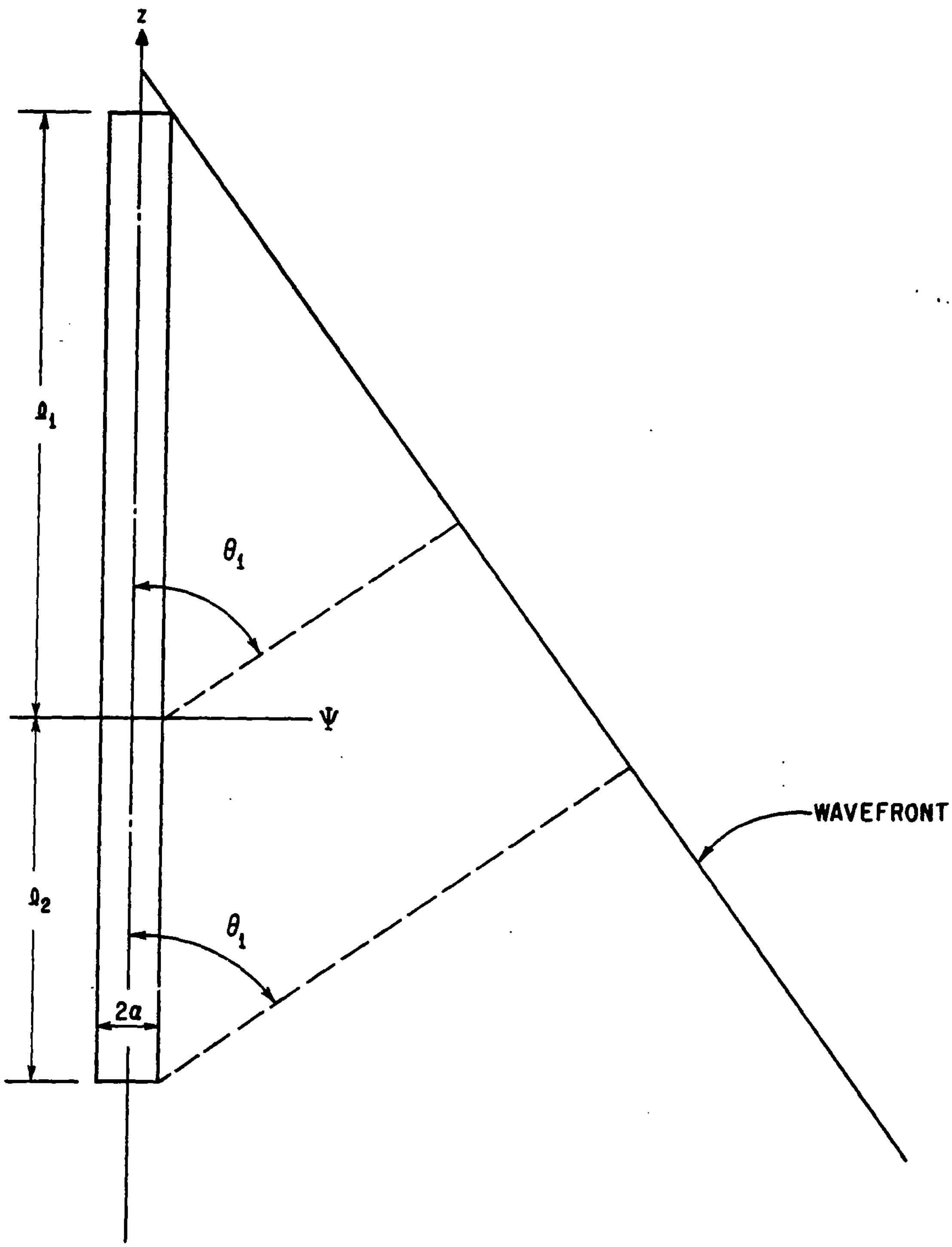


FIGURE 3. FINITE CYLINDRICAL STRUCTURE AND AN INCIDENT ELECTROMAGNETIC WAVE.

Similarly an expansion for $\vec{H}$ can be converted to one for $\vec{E}$ by substituting

$$\vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) \rightarrow -iZ\vec{N}^{(\ell)}(n,\zeta_1,{}^e_o)$$

$$\vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) \rightarrow -iZ\vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) \tag{3}$$

The $\vec{M}$ and $\vec{N}$ functions are given by

$$\vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) \equiv \frac{1}{k}\nabla \times \left[T^{(\ell)}(n,\zeta_1,{}^e_o)\vec{a}_z\right]$$

$$\vec{N}^{(\ell)}(n,\zeta_1,{}^e_o) \equiv \frac{1}{k}\nabla \times \vec{M}^{(\ell)}(n,\zeta_1,{}^e_o) \tag{4}$$

where k is the propagation constant, $\vec{a}_z$ is a unit vector in the z direction and T is a solution of the scalar wave equation given by

$$T^{(\ell)}(n,\zeta_1,{}^e_o) \equiv F_n^{(\ell)}(k\psi\zeta_2)e^{-ikz\zeta_1}\left\{\begin{matrix}\cos(n\phi)\\ \sin(n\phi)\end{matrix}\right\} \tag{5}$$

with

$$\zeta_1^2 + \zeta_2^2 = 1 \tag{6}$$

$F_n^{(\ell)}(k\psi\zeta_2)$ is one of the cylindrical Bessel functions $J_n(k\psi\zeta_2)$, $Y_n(k\psi\zeta_2)$, $H_n^{(1)}(k\psi\zeta_2)$, $H_n^{(2)}(k\psi\zeta_2)$ for ℓ = 1, 2, 3, 4 in that order. The third argument of T corresponds to using $\cos(n\phi)$ or $\sin(n\phi)$, respectively.

The $\vec{M}$ and $\vec{N}$ functions have the components

$$M_{\psi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \frac{F_n^{(\ell)}(k\psi\zeta_2)}{k\psi} e^{-ik\zeta_1} n \begin{Bmatrix} -\sin(n\phi) \\ \cos(n\phi) \end{Bmatrix}$$

$$M_{\phi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = -\zeta_2 F_n^{(\ell)\prime}(k\psi\zeta_2)\, e^{-ik\zeta_1} \begin{Bmatrix} \cos(n\phi) \\ \sin(n\phi) \end{Bmatrix}$$

$$M_{z}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = 0$$

(7)

$$N_{\psi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = -i\zeta_1\zeta_2 F_n^{(\ell)\prime}(k\psi\zeta_2)\, e^{-ikz\zeta_1} \begin{Bmatrix} \cos(n\phi) \\ \sin(n\phi) \end{Bmatrix}$$

$$N_{\phi}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = -i\zeta_1 \frac{F_n^{(\ell)}(k\psi\zeta_2)}{k\psi} e^{-ikz\zeta_1} n \begin{Bmatrix} -\sin(n\phi) \\ \cos(n\phi) \end{Bmatrix}$$

$$N_{z}^{(\ell)}(n,\zeta_1,{}^{e}_{o}) = \zeta_2^2 F_n^{(\ell)}(k\psi\zeta_2)\, e^{-ikz\zeta_1} \begin{Bmatrix} \cos(n\phi) \\ \sin(n\phi) \end{Bmatrix}$$

The general form of the electromagnetic field expansions in cylindrical coordinates is described in detail in reference 1. Having the general forms of $\vec{E}$ and $\vec{H}$ fields in cylindrical coordinates we now consider vector plane waves. The general forms of the electric and magnetic fields of a plane wave with direction propagation $\vec{e}_1$ in the $\phi = \phi_1$ plane are

$$\vec{E} = E_o\vec{e}_2 e^{-i\vec{k}\cdot\vec{r}}, \quad \vec{H} = H_o\vec{e}_3 e^{-i\vec{k}\cdot\vec{r}}$$

or (8)

$$\vec{E} = -E_o\vec{e}_3 e^{-i\vec{k}\cdot\vec{r}}, \quad \vec{H} = H_o\vec{e}_2 e^{-i\vec{k}\cdot\vec{r}}$$

where $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are mutually orthogonal unit vectors with the relations

$$\vec{k} \equiv k\vec{e}_1 , \quad \vec{e}_2 \times \vec{e}_3 = \vec{e}_1 \tag{9}$$

and k is the propagation constant.

If $\phi_1 = 0$, $\zeta_1 \equiv \cos(\theta_1)$, and $\zeta_2 \equiv \sin(\theta_1)$ we have the expansion

$$\vec{e}_2 e^{-i\vec{k}\cdot\vec{r}} = \frac{1}{\zeta_2} \sum_{n=o}^{\infty} [2-\delta_{n,o}] (-i)^n \vec{N}^{(1)}(n,\zeta_1,e) \tag{10}$$

and

$$\vec{e}_3 e^{-i\vec{k}\cdot\vec{r}} = \frac{i}{\zeta_2} \sum_{n=o}^{\infty} [2-\delta_{n,o}] (-i)^n \vec{M}^{(1)}(n,\zeta_1,e) \tag{11}$$

as given by equations (50) and (51) in reference 1. The function $\delta_{n,o}$ is the Kronecker delta function.

III. Axial Currents Induced in the Cylinder by an Incident Plane Wave with the Electric Field Polarized Perpendicular to the Axis of the Cylinder

Consider an incident vector plane wave with the electric field polarized perpendicular to the z axis as shown in figure 2A. The angle of incidence is θ_1 and the direction of propagation is given by the unit vector $\vec{k}$. Note that $\vec{k}$ is always in the $\phi = \phi_1$ plane where ϕ_1 is a constant. Since the cylinder geometry is independent of ϕ, ϕ_1 can be equal to zero without loss of generality.

The incident electric field is

$$\vec{E}_{inc} = -E_o F(s) \vec{e}_3 e^{-i\vec{k}\cdot\vec{r}}$$

$$= -\frac{E_o F(s)}{\zeta_2} \sum_{n=o}^{\infty} [2-\delta_{n,o}] (-i)^n \vec{M}^{(1)}(n,\zeta_1,e) \tag{12}$$

where F(s) is the Laplace transform of the incident field time history.

In order to satisfy the boundary conditions the tangential electric field on the perfectly conducting cylindrical surface must equal zero. And the reflected wave must be outward going and equal zero for $\psi\zeta_2 = \infty$. The second condition dictates that $\ell = 4$ for the reflected fields. The first condition requires that

$$\left[(\vec{E}_{inc})_z + (\vec{E}_{re})_z\right]\Big|_{\psi=a} = 0 \tag{13}$$

and

$$\left[(\vec{E}_{inc})_\phi + (\vec{E}_{re})_\phi\right]\Big|_{\psi=a} = 0 \tag{14}$$

The only expansion choice available for $\vec{E}_{re}$ that satisfies equation (13) is

$$\vec{E}_{re} = E_o F(s) \sum_{n=o}^{\infty} \alpha_n M^{(4)}(n,\zeta_1,e) \tag{15}$$

The substitution of equations (12) and (15) into equation (14) gives

$$\alpha_n = \frac{i}{\zeta_2}[2-\delta_{n,o}](-i)^n \frac{J_n'(ka\zeta_2)}{H_n^{(2)'}(ka\zeta_2)} \tag{16}$$

The total field distribution can now be written as

$$\vec{E} = \vec{E}_{inc} + \vec{E}_{re}$$

$$= \frac{E_o F(s) i}{\zeta_2} \sum_{n=o}^{\infty} [2-\delta_{n,o}] (i)^n \left[-\vec{M}^{(1)}(n,\zeta_1,e) + \frac{J_n'(ka\zeta_2)}{H_n^{(2)'}(ka\zeta_2)} \vec{M}^{(4)}(n,\zeta_1,e) \right] \quad (17)$$

and

$$\vec{H} = \frac{E_o F(s)}{Z\zeta_2} \sum_{n=o}^{\infty} [2-\delta_{n,o}] (i)^n \left[\vec{N}^{(1)}(n,\zeta_1,e) - \frac{J_n'(ka\zeta_2)}{H_n^{(2)'}(ka\zeta_2)} \vec{N}^{(4)}(n,\zeta_1,e) \right] \quad (18)$$

The total axial current in the cylinder can now be calculated as

$$I = a\int_o^{2\pi} H_\phi \Big|_{\psi=a} d\phi$$

$$= \frac{aE_o F(s)}{Z\zeta_2} \int_o^{2\pi} \sum_{n=o}^{\infty} [2-\delta_{n,o}] i^{n+1} n\zeta_1 \left[F_n^{(1)}(ka\zeta_2) - \frac{J_n'(ka\zeta_2)}{H_n^{(2)'}(ka\zeta_2)} F_n^{(4)}(ka\zeta_2) \right] \frac{e^{-ikz\zeta_1}}{ka} \sin(n\phi)\, d\phi$$

$$= 0 \quad (19)$$

IV. Axial Currents Induced in the Cylinder by an Incident Plane Wave with the Magnetic Field Polarized Perpendicular to the Axis of the Cylinder

Consider now an incident vector plane wave with the magnetic field polarized perpendicular to the z axis as shown in figure 2B. The incident fields are given by

$$\vec{E}_{inc} = E_o F(s) \vec{e}_2 \; e^{-i\vec{k}\cdot\vec{r}}$$

$$\vec{H}_{inc} = \frac{E_o F(s)}{Z} \vec{e}_3 \; e^{-i\vec{k}\cdot\vec{r}} \tag{20}$$

From equations (10) and (11) we see that $\vec{E}_{inc}$ can only be expanded with $\vec{N}$ functions and $\vec{H}_{inc}$ can be only expanded with $\vec{M}$ functions. The reflected electric field then can only be expanded in $\vec{N}$ functions to satisfy both equations (13) and (14). Since the reflected wave is outward going, $\ell = 4$. Thus, the reflected electric field is given by

$$\vec{E}_{re} = E_o F(s) \sum_{n=o}^{\infty} \beta_n \vec{N}^{(4)}(n, \zeta_1, e) \tag{21}$$

The substitution of equations (20) and (21) into equation (13) gives

$$\left(\vec{e}_2 \, e^{-i\vec{k}\cdot\vec{r}}\right)_z \Big|_{\psi=a} + \sum_{n=o}^{\infty} \beta_n N_z^{(4)}(n, \zeta_1, e) \Big|_{\psi=a} =$$

$$\frac{1}{\zeta_2} \sum_{n=o}^{\infty} [2-\delta_{n,o}] (-i)^n N_z^{(1)}(n, \zeta_1, e) + \sum_{n=o}^{\infty} \beta_n N_z^4(n, \zeta_1, e) = 0 \tag{22}$$

Solving for β_n gives

$$\beta_n = -\frac{1}{\zeta_2}[2-\delta_{n,o}](-i)^n \frac{J_n(ka\zeta_2)}{H_n^{(2)}(ka\zeta_2)} \tag{23}$$

The total field distribution can now be written as

$$\vec{E} = \vec{E}_{inc} + \vec{E}_{re}$$

$$= \frac{E_o F(s)}{\zeta_2} \sum_{n=o}^{\infty} [2-\delta_{n,o}](-i)^n \left[\vec{N}^{(1)}(n,\zeta_1,e) - \frac{J_n(ka\zeta_2)}{H_n^{(2)}(ka\zeta_2)} \vec{N}^{(4)}(n,\zeta_1,e) \right] \tag{24}$$

and

$$\vec{H} = \frac{E_o F(s) i}{Z\zeta_2} \sum_{n=o}^{\infty} [2-\delta_{n,o}](-i)^n \left[\vec{M}^{(1)}(n,\zeta_1,e) - \frac{J_n(ka\zeta_2)}{H_n^{(2)}(ka\zeta_2)} \vec{M}^{(4)}(n,\zeta_1,e) \right] \tag{25}$$

The ϕ component of $\vec{H}$ at $\psi = a$ is

$$H_\phi \Big|_{\psi=a} = i \frac{E_o F(s)}{Z\zeta_2} \sum_{n=o}^{\infty} [2-\delta_{n,o}](-i)^n e^{-ikz\zeta_1} \cos(n\phi) \cdot \left[-\zeta_2 J_n'(ka\zeta_2) + \zeta_2 \frac{J_n(ka\zeta_2)}{H_n^{(2)}(ka\zeta_2)} H_n^{(2)\prime}(ka\zeta_2) \right] \tag{26}$$

The total axial current in the cylinder is

$$I = a\int_0^{2\pi} H_\phi \Big|_{\psi=a} d\phi$$

$$= 2\pi a i \frac{E_o F(s)}{Z} e^{-ikz\zeta_1}\left[-J_o'(ka\zeta_2) + \frac{J_o(ka\zeta_2)H_o^{(2)'}(ka\zeta_2)}{H_o^{(2)}(ka\zeta_2)}\right] \quad (27)$$

By replacing $-J_o'(ka\zeta_2)$ with $J_1(ka\zeta_2)$ and $H_o^{(2)'}(ka\zeta_2)$ with $-H_1^{(2)}(ka\zeta_2)$ in equation (27), we have

$$I = 2\pi a i \frac{E_o F(s) e^{-ikz\zeta_1}}{Z}\left[J_1(ka\zeta_2) - \frac{J_o(ka\zeta_2)H_1^{(2)}(ka\zeta_2)}{H_o^{(2)}(ka\zeta_2)}\right] \quad (28)$$

Equation (28) can be reduced by the Wronskians given in equations 9.1.15, 9.1.16, and 9.1.17 in reference 2 to give

$$I = 4\,\frac{E_o F(s) e^{-ikz\zeta_1}}{Zk\zeta_2 H_o^{(2)}(ka\zeta_2)} \quad (29)$$

The substitution of $k = -is/c$ into equation (29) gives

$$I = \frac{2\pi c E_o F(s) e^{-sz\zeta_1/c}}{Z\zeta_2 s K_o(sa\zeta_2/c)} \quad (30)$$

where c is the speed of light and the relation $H_o^{(2)}(ix) = \frac{2i}{\pi} K_o(x)$ has been used. For the purpose of this note it is sufficient to consider only the current at z = 0 which is given by

$$I(s,\phi_1) = \frac{2\pi c E_o F(s)}{Z\zeta_2 s K_o(sa\zeta_2/c)} \quad (31)$$

Now define a normalized dimensionless Laplace transform variable as

$$\xi = sa\zeta_2/c \tag{32}$$

and a normalized dimensionless time as

$$q = \frac{ct}{a\zeta_2} \tag{33}$$

By the convolution theorem the inverse Laplace transform of the current can be written as

$$I = \frac{2\pi cE_o}{Z\zeta_2} \int_{-\infty}^{t} f(t-\tau) \left[\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}\, ds}{sK_o(sa\zeta_2/c)} \right] d\tau \tag{34}$$

where f(t) is the inverse Laplace transform of F(s).

In terms of the normalized variables, equation (34) becomes

$$I = \frac{2\pi aE_o}{Z} \int_{-\infty}^{q} f\left(\frac{a\zeta_2}{c}(q-\lambda)\right) \left[\frac{1}{2\pi i} \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{\xi\lambda}\, d\xi}{\xi K_o(\xi)} \right] d\lambda \tag{35}$$

where $\lambda = c\tau/a\zeta_2$ and $\gamma' = a\zeta_2\gamma/c$. In reference 3 it was deduced that

$$\frac{1}{2\pi i} \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{\xi\lambda}}{\xi K_o(\xi)}\, d\xi = 0 \qquad \text{if } \lambda < -1 \tag{36}$$

Therefore equation (35) can be written as

$$I = \frac{2\pi aE_o}{Z} \int_{o}^{q^*} f\left(\frac{a\zeta_2}{c}(q^*-\zeta)\right) \left[\frac{1}{2\pi i} \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{\xi(\zeta-1)}}{\xi K_o(\xi)}\, d\xi \right] d\zeta \tag{37}$$

where $\zeta = \lambda + 1$ and q^* is a shifted dimensionless time equal to $q + 1$. Equation (37) can be expressed as

$$I = \frac{2\pi a E_o}{Z} \int_o^{q^*} f\left(\frac{a\zeta_2}{c}(q^*-\zeta)\right) F(\zeta)\,d\zeta \tag{38}$$

where $F(\zeta)$ is a characterized function in reference 4 given by

$$F(\zeta) = \frac{1}{2\pi i} \int_{\gamma'=i\infty}^{\gamma'+i\infty} \frac{e^{\xi(\zeta-1)}}{\xi K_o(\xi)}\,d\xi = \int_o^{\infty} \frac{e^{-\xi(\zeta-1)} I_o(\xi)}{\xi\left[K_o^2(\xi)+\pi^2 I_o^2(\xi)\right]}\,d\xi \tag{39}$$

For convenience we can define a normalized dimensionless current as

$$\Lambda = \frac{ZI}{2\pi a E_o} \tag{40}$$

The substitution of equation (38) into (40) gives

$$\Lambda = \int_o^{q^*} f\left(\frac{a\zeta_2}{c}(q^*-\zeta)\right) F(\zeta)\,d\zeta \tag{41}$$

To calculate the impulse response, let

$$f(t) = \delta(t) = \delta\left(\frac{a\zeta_2 q}{c}\right) \tag{42}$$

where $\delta(t)$ is the Dirac delta function. Equation (41) becomes

$$\Lambda = \int_o^{q^*} \delta\left(\frac{a\zeta_2}{c}(q^*-\zeta)\right) F(\zeta)\,d\zeta$$

$$= F(q^*) \tag{43}$$

The normalized current response to a unit impulse incident vector plane wave is shown in figure 4. The first two terms of the small time asymptotic expansion of F(q*) developed in Appendix B of Sensor and Simulation Note 110 and the first six terms of the large time asymptotic expansion of F(q*) developed in Appendix A of this note are indicated by broken lines.

V. Response to an Exponentially Decaying Field

Consider an incident electric field with a time history given by

$$f(t) = e^{-\beta_\tau t} \tag{44}$$

The substitution of equation (44) into equation (41) gives

$$\Lambda = e^{-\beta q^*} \int_0^{q^*} e^{\beta\zeta} F(\zeta) d\zeta \tag{45}$$

where

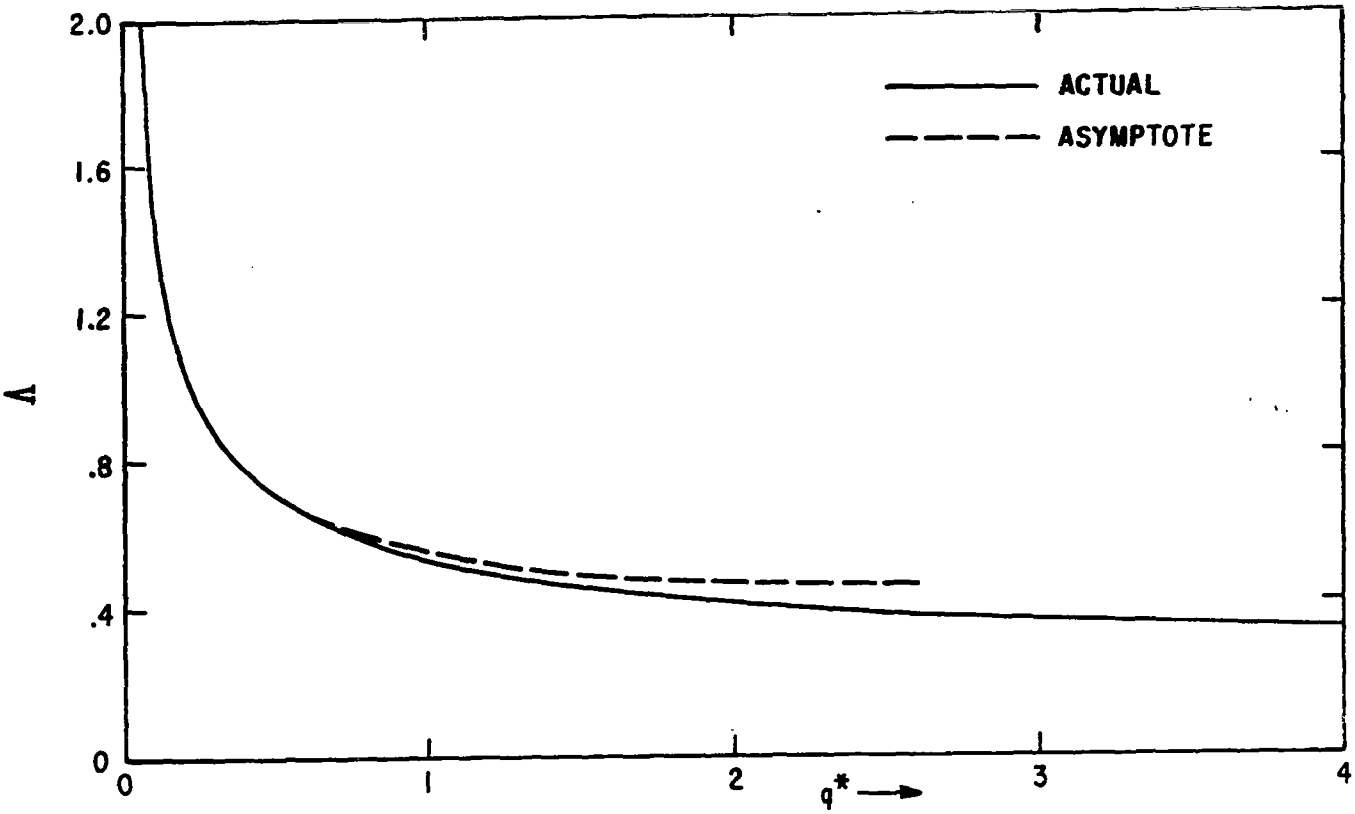
$$\beta = \frac{a\zeta_2}{c} \beta_\tau \tag{46}$$

In terms of the normalized transform variable ξ, Λ can be written as

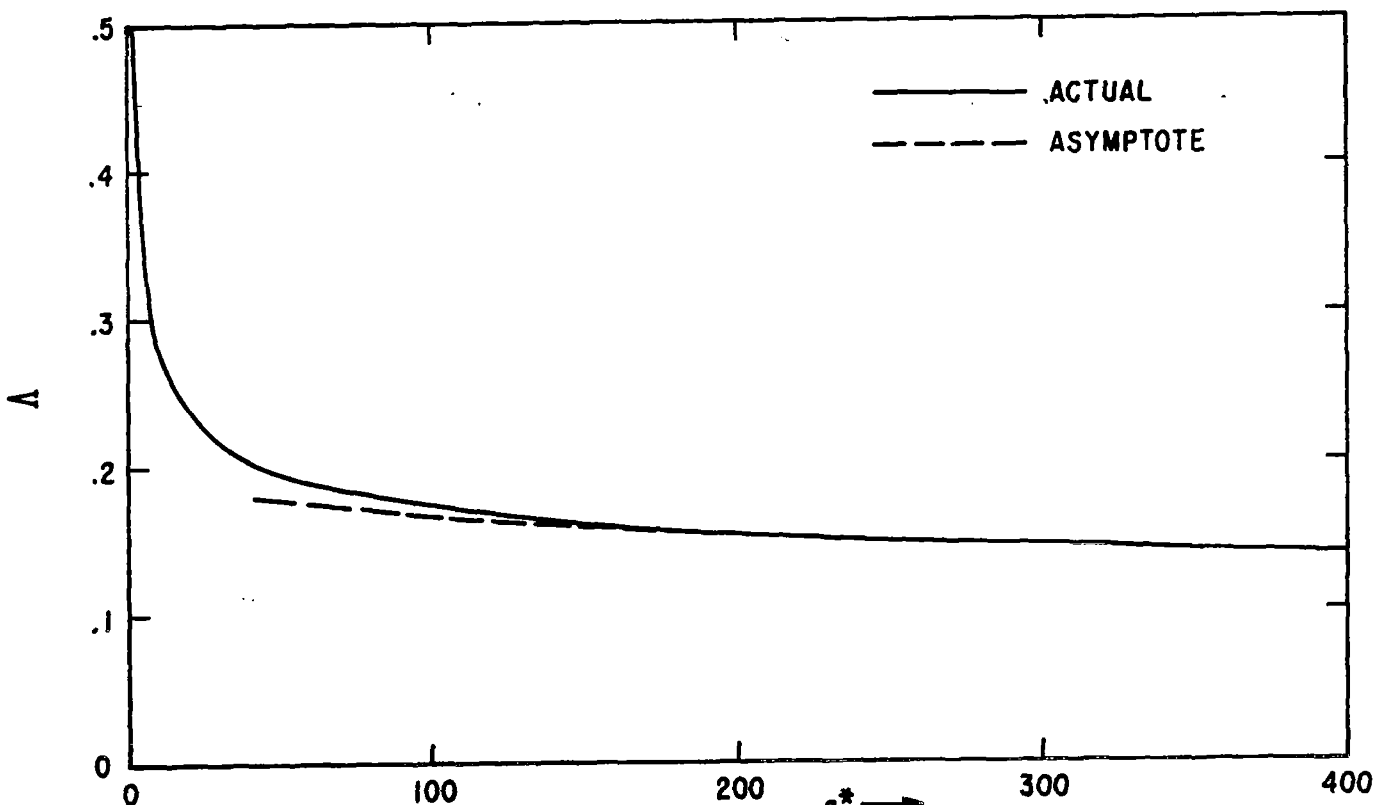
$$\Lambda = \frac{1}{2\pi i} \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{\xi q}}{(\xi+\beta)\xi k_o(\xi)} d\xi \tag{47}$$

Small time behavior

As $\xi \to \infty$ the Laplace transform of Λ can be written from equation (47) as



A. EARLY NORMALIZED SHIFTED TIME



B. LATE NORMALIZED SHIFTED TIME

FIGURE 4. NORMALIZED AXIAL CURRENT RESPONSE TO AN IMPULSE INCIDENT PLANE WAVE.

$$\mathcal{L}\{\Lambda\} \underset{q^*\to\xi}{=} \frac{\sqrt{2}}{\xi\sqrt{\xi\pi}\left(1+\frac{\beta}{\xi}\right)}\left[1+\frac{1}{8\xi}-\frac{7}{128\xi^2}+O(\xi^{-3})\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\frac{1}{\xi\sqrt{\xi}}\left[1+\frac{1-8\beta}{8\xi}+O(\xi^{-2})\right] \tag{48}$$

where the asymptotic expansion of $K_o(\xi)$ for $\xi \to \infty$ with $|\arg \xi| < (3/2)\pi$ as given by equation 9.7.2 of reference 2 and the Binomial series expansion of $(1+\beta/\xi)^{-1}$ have been used. By the application of the theorem in Appendix B of reference 4, the asymptotic expansion of Λ for $q^* \to 0$ is

$$\Lambda = \frac{2}{\pi}\sqrt{2q^*}\left[1+\frac{q^*}{12}(1-8\beta)+O((q^*)^2)\right]U(q^*) \tag{49}$$

As $q^* \to 0$ the asymptotic form of Λ can be written as

$$\Lambda \sim \frac{2}{\pi}\sqrt{2q^*}\; U(q^*) \tag{50}$$

Large time behavior

To obtain the large q^* behavior of Λ, equation (46); we first break the integral into three parts:

$$\Lambda = e^{-\beta q^*}\int_o^{q^*} e^{\beta\zeta}F(\zeta)d\zeta$$

$$= e^{-\beta q^*}\int_o^{\varepsilon}(\cdots\cdot)d\zeta + e^{-\beta q^*}\int_{\varepsilon}^{M}(\cdots\cdot)d\zeta + e^{-\beta q^*}\int_{M}^{q^*}(\cdots\cdot)d\zeta$$

$$= T_1(q^*) + T_2(q^*) + T_3(q^*) \tag{51}$$

where ε is a finite constant chosen such that $0 < \varepsilon << \Gamma/2$ and M is a finite constant chosen such that $\Gamma/2 << M < \infty$. Γ is the exponential of Euler's constant = 1.7810···. $F(\zeta)$ is finite for $\varepsilon \leq \zeta \leq M$, thus

$$T_2 = O(e^{-\beta q^*}) \tag{52}$$

The first integral can be written as

$$T_1 = e^{-\beta q^*} \frac{\sqrt{2}}{\pi} \int_0^{\varepsilon} \frac{e^{\beta\zeta}}{\sqrt{\zeta}} d\zeta + e^{-\beta q^*} \int_0^{\varepsilon} e^{\beta\zeta} \left[F(\zeta) - \frac{\sqrt{2}}{\pi\sqrt{\zeta}} \right] d\zeta \tag{53}$$

where $\sqrt{2}/\pi\sqrt{\zeta}$ is the first term of the asymptotic expansion of $F(\zeta)$ for small argument as given in Appendix B of reference 4 and

$$\left| F(\zeta) - \frac{\sqrt{2}}{\pi\sqrt{\zeta}} \right| < M_1 , \quad \zeta \leq \varepsilon \tag{54}$$

where M_1 is a finite constant. Thus, we may write

$$T_1 = O(e^{-\beta q^*}) \tag{55}$$

The third integral can be written as

$$T_3(q^*) = e^{-\beta q^*} \int_M^{q^*} \left\{ e^{\beta\zeta} \sum_{n=1}^{N} \frac{a_p}{[\ln(2\zeta/\Gamma)]^p} \right\} d\zeta + R_N \tag{56}$$

where the asymptotic expansion of $F(\zeta)$ for large argument as given in Appendix A has been used. And

$$R_N = e^{-\beta q^*} \int_M^{q^*} \left\{ e^{\beta\zeta} O\left([\ln(2\zeta/\Gamma)]^{-(N+1)} \right) \right\} d\zeta \tag{57}$$

The asymptotic expansion of $T_3(q^*)$ for $q^* \to \infty$ is developed in Appendix B as

$$T_3(q^*) = \sum_{p=1}^{N} \frac{a_p}{[\ln(2q^*/\Gamma)]^p} \sum_{n=0}^{N-p} \frac{\binom{-p}{n}(-1)^n n!}{\beta[\beta q^* \ln(2q^*/\Gamma)]^n}$$

$$+ O\left([\ln(2q^*/\Gamma)]^{-(N+1)}\right) \qquad \underline{\beta \neq 0} \tag{58}$$

And

$$T_3(q^*) = \sum_{p=1}^{N} \frac{a_p q^*}{[\ln(2q^*/\Gamma)]^p} \sum_{n=0}^{N-p} \frac{\binom{-p}{n}(-1)^n n!}{[\ln(2q^*/\Gamma)]^n}$$

$$+ O\left(q^*[\ln(2q^*/\Gamma)]^{-(N+1)}\right) \qquad \underline{\beta = 0} \tag{59}$$

Collecting the results in equations (52), (55), and (58) gives the asymptotic expansion of Λ for $q^* \to \infty$ with $\beta \neq 0$ as

$$\Lambda = \frac{1}{\beta \ln(2q^*/\Gamma)} + \frac{1 - \gamma_e q^*}{\beta(\beta q^*)\ln^2(2q^*/\Gamma)} + O\left(\frac{1}{\ln^3(2q^*/\Gamma)}\right) \tag{60}$$

where γ_e is Euler's constant.

For the case $\beta = 0$, collecting the results in equations (52), (55), and (59) gives the asymptotic expansion of Λ for $q^* \to \infty$ as

$$\Lambda = \frac{q^*}{\ln(2q^*/\Gamma)} + \frac{q^*(1-\gamma_e)}{\ln^2(2q^*/\Gamma)} + O\left(\frac{q^*}{\ln^3(2q^*/\Gamma)}\right) \tag{61}$$

Maximum value of the induced current

The maximum value of Λ at $q^* = q_o^*$ is given by

$$\Lambda_m = e^{-\beta q_o^*}\int_o^{q_o^*} e^{\beta\zeta}F(\zeta)d\zeta \tag{62}$$

and

$$\frac{\partial}{\partial q^*}(\Lambda(q^*,\beta))\Big|_{q^*=q_o^*} = \Lambda_m - \frac{F(q_o^*)}{\beta} = 0 \tag{63}$$

Rewriting equation (63) gives

$$\Lambda_m = \frac{F(q_o^*)}{\beta} \tag{64}$$

The asymptotic expansion of Λ_m for limiting cases of β can be obtained by the substitution of the small and large β behavior of q_o^* into equation (64).

To obtain the small β behavior of q_o^*, we assume q_o^* to be large. This assumption shall be justified by the result. The reader may easily verify that if q_o^* is assumed small a contradiction results. For q_o^* large equation (64) can be written as

$$\frac{\sqrt{2}}{\pi}\, e^{-\beta q_o^*}\int_o^{\varepsilon} e^{\beta\zeta}[\zeta^{-1/2}+O(\zeta^{1/2})]d\zeta + e^{-\beta q_o^*}\int_{\varepsilon}^{M} e^{\beta\zeta}F(\zeta)d\zeta$$

$$+ e^{-\beta q_o^*}\int_M^{q_o^*} e^{\beta\zeta}\Big[\ln^{-1}(\zeta)-\gamma_e\ln^{-2}(\zeta)+O(\ln^{-3}(\zeta))\Big]d\zeta$$

$$= \frac{1}{\beta\ln(q_o^*)} - \frac{\gamma_e}{\beta\ln^2(q_o^*)} + O\left(\frac{\ln^{-3}(q_o^*)}{\beta}\right) \tag{65}$$

where ε and M are constants such that $\varepsilon \ll 1$ and $M \gg 1$ and the asymptotic expansions of $F(\zeta)$ for large and small ζ have been used. Evaluating the first two integrals in equation (65) gives

$$\frac{\sqrt{2}}{\pi} e^{-\beta q_o^*} \int_0^{\varepsilon} e^{\beta\zeta}[\zeta^{-1/2}+O(\zeta^{1/2})]d\zeta + e^{-\beta q_o^*}\int_{\varepsilon}^{M} e^{\beta\zeta}F(\zeta)d\zeta = e^{-\beta q_o^*}O(1) \quad (66)$$

since $F(\zeta)$ is bounded for $\varepsilon \leq \zeta \leq M$. The substitution of $\tau = \beta q_o^* - \beta\zeta$ into equation (65) and using the result in equation (66) gives

$$\int_0^{\mu-\beta M} e^{-\tau}\left[\ln^{-1}\left(\frac{\mu-\tau}{\beta}\right) - \gamma_e \ln^{-2}\left(\frac{\mu-\tau}{\beta}\right) + O\ \ln^{-3}\left(\frac{\mu-\tau}{\beta}\right)\right]d\tau$$

$$= \frac{1}{\ln(\mu/\beta)} - \frac{\gamma_e}{\ln^2(\mu/\beta)} + O(\ln^{-3}(\mu/\beta)) \quad (67)$$

where $\mu = \beta q_o^*$. By the use of a Taylor's expansion of the non-exponential part of the integrand, equation (67) becomes

$$\int_0^{\mu-\beta M} e^{-\tau}\left[\frac{1}{\ln(\mu/\beta)} - \frac{\gamma_e}{\ln^2(\mu/\beta)} + \frac{}{\mu\ln^2(\mu/\beta)} + O(\ln^{-3}(\mu/\beta))\right]d\tau$$

$$= \frac{1}{\ln(\mu/\beta)} - \frac{\gamma_e}{\ln^2(\mu/\beta)} + O(\ln^{-3}(\mu/\beta)) \quad (68)$$

Evaluating the integrals gives

$$\frac{-e^{-\mu}}{\ln(\mu/\beta)} + \frac{1 - e^{-\mu}(1+\mu(1-\gamma_e))}{\mu\ln^2(\mu/\beta)} = o(\ln^{-3}(\mu/\beta)) \quad (69)$$

Equation (69) can be rewritten as

$$\mu = \ln(\mu\ln(\mu/\beta)) + O(\mu\ln^{-1}(\mu/\beta)) = g(\mu) \quad (70)$$

To solve for μ the iteration method can be used if[6]

$$\frac{\partial}{\partial\mu}(g(\mu)) = \frac{1}{\mu} + \frac{1}{\mu\ln(\mu/\beta)} < 1 \quad (71)$$

For $\mu \geq 1 + \varepsilon$ for all real $\varepsilon > 0$ there exists a β_o such that for $0 < \beta \leq \beta_o$ the condition in (71) is satisfied. We can choose a rough approximation $\mu_1 = 1$ and compute a second approximation

$$\mu_2 = g(\mu_1) = \ln\ln(\beta^{-1}) + O(\ln^{-1}(\beta^{-1})) \quad (72)$$

And a second iteration gives

$$\mu_3 = g(\mu_2) = \ln\ln\ln(\beta^{-1}) + \ln\ln\left[\frac{\ln\ln\beta^{-1}}{\beta}\right] + O\left(\frac{\ln\ln\beta^{-1}}{\ln\beta^{-1}}\right)$$

$$= \ln[(\ln\beta^{-1})\ln\ln\beta^{-1}] + O\left(\frac{\ln[(\ln\beta^{-1})\ln\ln\beta^{-1}}{\ln\beta^{-1}}\right) \quad (73)$$

Note that as $q_o^* = \mu/\beta \to \infty$, $\beta \to 0$ as assumed. The numerical comparison of μ_3 with μ confirms that μ_1 is sufficiently close to μ for the iteration method to be applicable.

The substitution of equation (73) into equation (64) gives the asymptotic expansion of Λ_m for $\beta \to 0$ as

$$\Lambda_m = \frac{F(\mu/\beta)}{\beta} = \frac{1}{\beta\ln\left(\frac{2\mu}{\beta\Gamma}\right)} - \frac{\gamma_e}{\beta\ln^2\left(\frac{2\mu}{\beta\Gamma}\right)} + O(\ln^{-3}(\mu/\beta)$$

$$= \frac{1}{\beta\ln[\beta^{-1}\ln\ln\beta^{-1}]} - \frac{\gamma_e}{\beta\ln^2[\beta^{-1}\ln\ln\beta^{-1}]} + O(\ln^{-3}[\beta^{-1}\ln\ln\beta^{-1}]) \quad (74)$$

To obtain the large β behavior of q_o^*, we assume q_o^* to be small. For small q_o^*, equation (64) can be written as

$$\beta e^{-\beta q_o^*}\int_o^{q_o^*} \frac{e^{\beta\zeta}}{\sqrt{\zeta}}\left[1+ \frac{\zeta}{4} + O(\zeta^2)\right] d\zeta = \frac{1}{\sqrt{q_o^*}} + \frac{\sqrt{q_o^*}}{4} + O(q_o^{*3/2}) \qquad (75)$$

where the asymptotic expansion of $F(\zeta)$ for $\zeta \rightarrow 0$ has been used. Now we can make a change of variable $\tau = \beta\zeta$ to obtain

$$e^{-\mu}\int_o^{\mu} \frac{e^{\tau}}{\sqrt{\tau}}\, d\tau + e^{-\mu}\int_o^{\mu} e^{\tau}\left[\frac{\sqrt{\tau}}{4\beta} + O\left(\frac{\tau^{3/2}}{\beta^2}\right)\right] d\tau$$

$$= \frac{1}{\sqrt{\mu}} + \frac{\sqrt{\mu}}{4\beta} + O\left(\frac{\mu^{3/2}}{\beta^2}\right) \qquad (76)$$

where again $\mu = \beta q_o^*$. Now let $\lambda^2 = \tau$ and $x^2 = \mu$, equation (76) becomes

$$2xe^{-x^2}\int_o^{x} e^{\lambda^2}\left[1+O\left(\frac{\lambda^2}{\beta}\right)\right] d\lambda = 1 + O\left(\frac{x^2}{\beta}\right) \qquad (77)$$

The substitution of Dawson's Integral

$$D(x) = e^{-x^2}\int_o^{x} e^{\lambda^2} d\lambda$$

gives

$$\frac{dD(x)}{dx} + O\left(\frac{xD(x)}{\beta}\right) = 0 \qquad (78)$$

The derivative of Dawson's Integral must equal zero in the limiting case for $\beta \rightarrow \infty$ since $xD(x)$ is bounded for all $x > 0$. Therefore the value of x is the maximum point for Dawson's Integral

$x_m = .9241388730\cdots$, as given by equation 7.1.17 in reference 2. The asymptotic form of q_o^* for $\beta \to \infty$ is

$$q_o^* \sim \frac{x_m^2}{\beta} = \frac{.85403265\cdots}{\beta} \tag{79}$$

The substitution of equation (79) into equation (64) gives the asymptotic expansion of Λ_m for $\beta \to \infty$ as

$$\Lambda_m = \frac{\sqrt{2}}{\pi x_m \sqrt{\beta}} \left\{ 1 + \frac{x_m^2}{4\beta} + O(\beta^{-2}) \right\} \tag{80}$$

Results

The normalized axial current response to an incident plane wave with a step-function time history, $\beta = 0$, is plotted in figure 5. In figure 6, the normalized axial current response to an incident plane wave with an exponentially decaying time history is plotted with β as a parameter. The solid curves in figures 5 and 6 were obtained by numerically integrating equation (45). The first two terms of the small time asymptotic expansion given by equation (49) and the first six terms of the large time asymptotic expansion given by equation (60) for $\beta \neq 0$ and equation (61) for $\beta = 0$ are plotted as broken lines.

In figure 7 the peak value of the normalized axial current response and the normalized shifted time of the peak are plotted against the normalized inverse decay constant of the incident wave. The solid curves were obtained by numerical solution of equation (63). In figure 7A the first two terms of the small β asymptotic expansion of βq_o^* given by equation (73) and the large β asymptote of βq_o^* given by equation (79) are plotted as broken lines. In figure 7B the first two terms of the small β asymptotic expansion for the peak value of the normalized current given by equation (74) are plotted as a broken line. And the first two terms of the large β asymptotic expansion for the

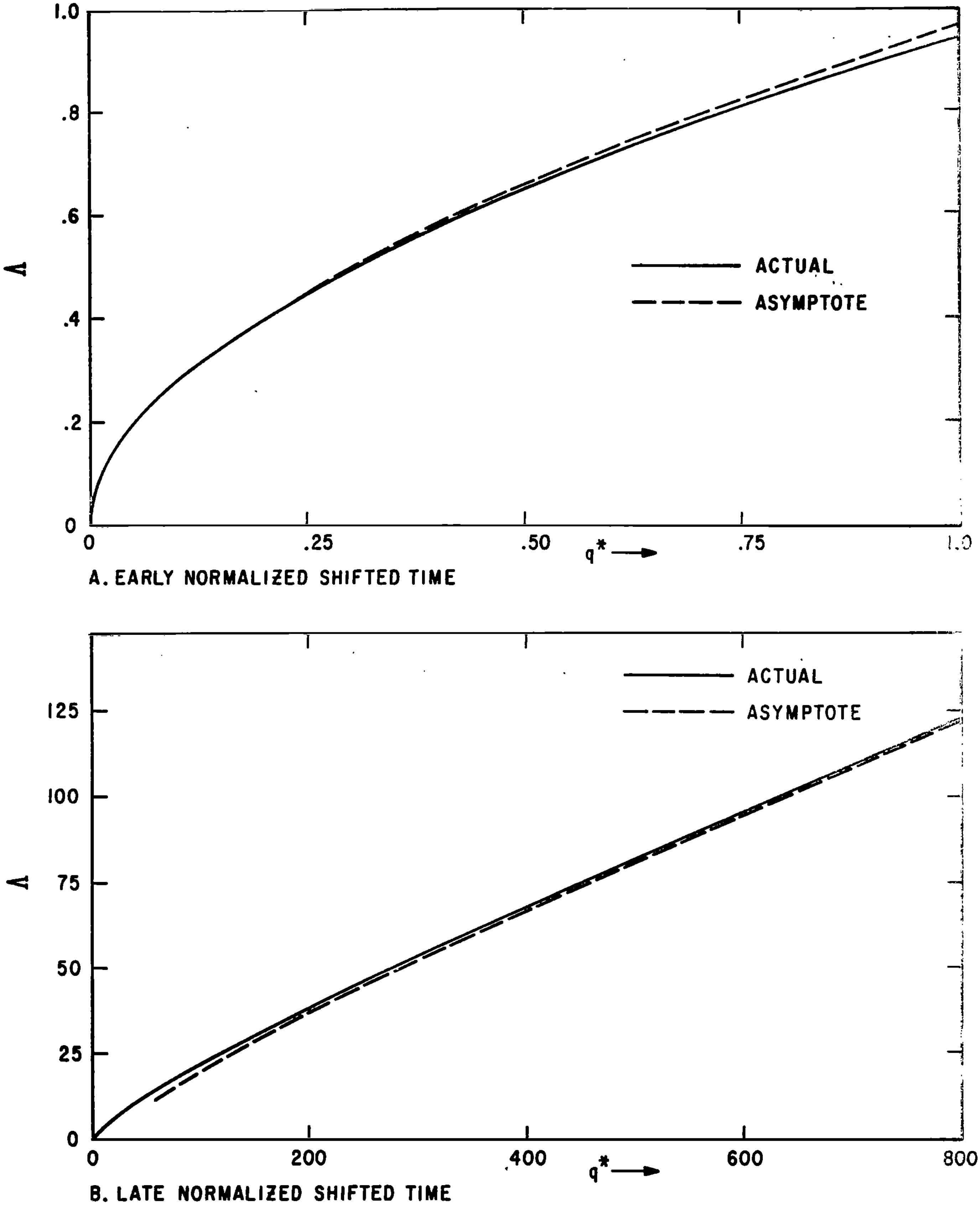


FIGURE 5. NORMALIZED AXIAL CURRENT RESPONSE TO AN INCIDENT PLANE WAVE WITH A STEP-FUNCTION TIME HISTORY.

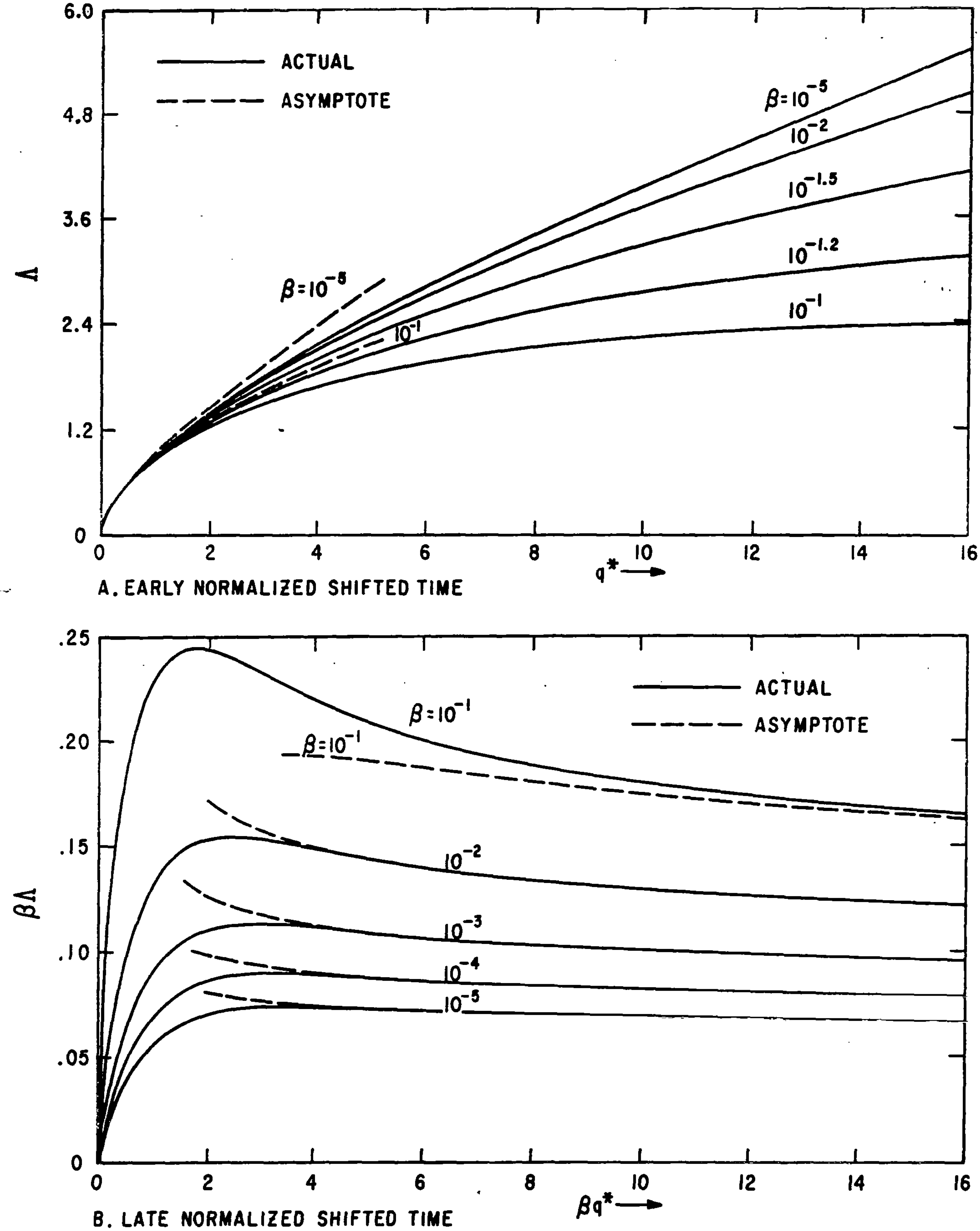
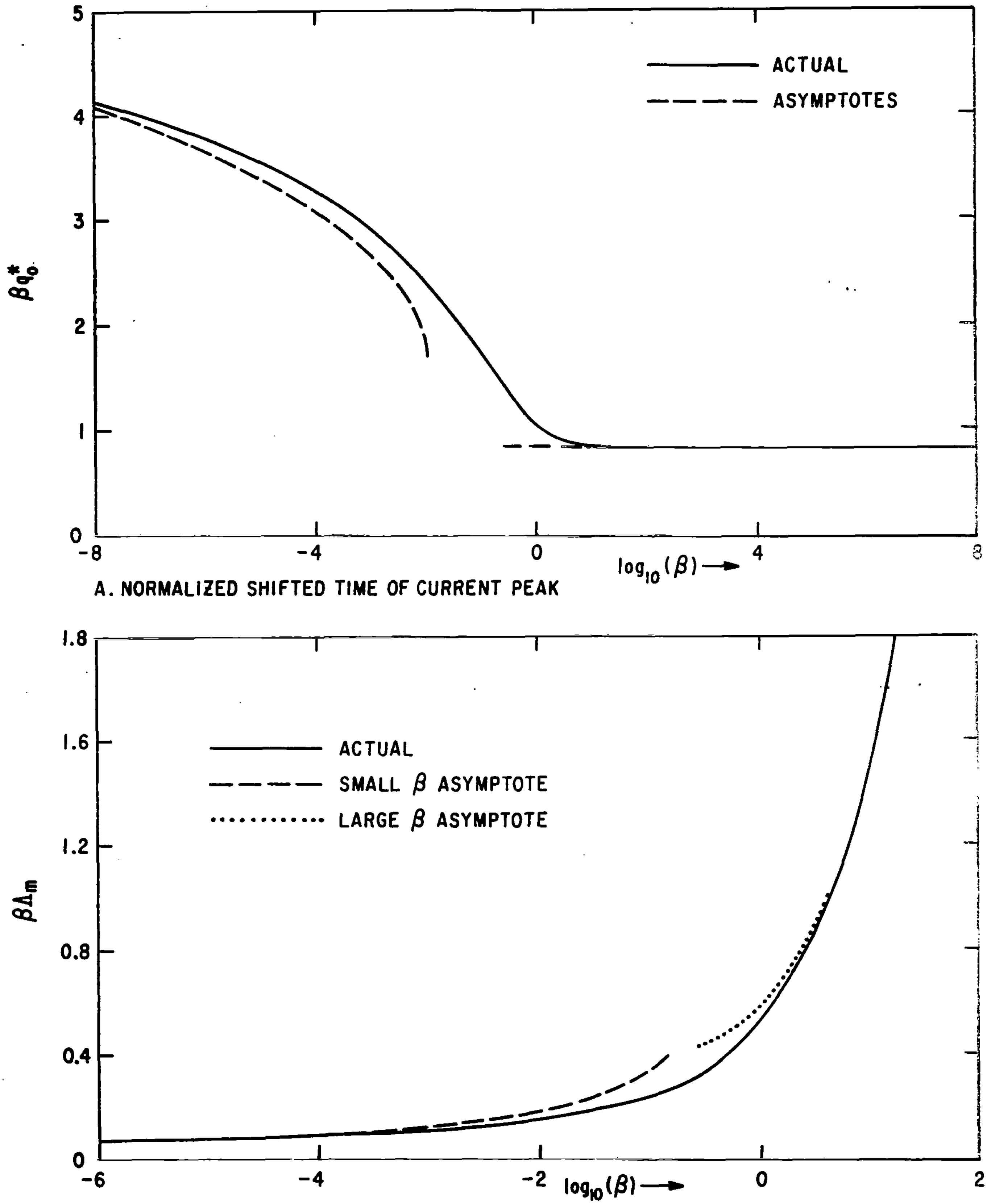


FIGURE 6. NORMALIZED AXIAL CURRENT RESPONSE TO AN EXPONENTIALLY DECAYING PLANE WAVE WITH β AS A PARAMETER.



A. NORMALIZED SHIFTED TIME OF CURRENT PEAK

B. MAXIMUM VALUE OF THE NORMALIZED AXIAL CURRENT

FIGURE 7. NORMALIZED PEAK AXIAL CURRENT RESPONSE TO AN EXPONENTIALLY DECAYING PLANE WAVE.

peak value of the normalized current given by equation (80) has been used to compute values for the dotted curve.

The relative error resulting from numerically integrating the integrand in equation (45) is less than 10^{-4}. The relative error of $F(\zeta)$ is approximately 8.1×10^{-4} as detailed on page 55 of reference 4. The maximum relative error of Λ can be estimated by the sum of the errors introduced by approximation of the integrand and numerical integration as in the order of 10^{-3}.

VI. Remarks

There are several conclusions which follow from the analytical analysis and the associated numerical computations from the preceding section. The following equations are given as "rules of thumb" for the induced current upon the surface of an infinitely long, perfectly conducting, circular cylinder in free space illuminated by an exponentially decaying plane wave.

The initial rise of the current is proportional to the square root of time and can be written from equation (50) as

$$I \sim \frac{2\pi a E_o}{Z} \frac{2\sqrt{2q^*}}{\pi} U(q^*)$$

$$= \frac{4aE_o}{Z}\sqrt{\frac{2ct^*}{a \sin\theta_1}} U(t^*) \text{ amperes} \qquad (81)$$

where E_o is the maximum value of the incident electric field with dimensions of volts per meter; a is the radius of the cylinder with dimensions of meters; c is the speed of light with dimensions of meters per second; θ_1 is the direction of the incident wave measured from the axis of the cylinder; and t^* is a shifted time given by

$$t^* = t + \frac{a \sin\theta_1}{c} \text{ seconds} \qquad (82)$$

The asymptotic form of the peak value of the induced current pulse for small β as given in equation (74) can be written as

$$I_m \sim \frac{2\pi a E_o}{Z\beta} \frac{1}{\ln[\beta^{-1}\ln\ln\beta^{-1}]}$$

$$= \frac{2\pi c E_o}{Z\beta_\tau \sin\theta_1} \frac{1}{\ln\left[\frac{c}{a\beta_\tau \sin\theta_1} \ln\ln\left(\frac{c}{a\beta_\tau \sin\theta_1}\right)\right]} \quad (83)$$

where β_τ is the inverse decay time constant of the incident wave with dimensions of per seconds. The asymptotic form of the shifted time the peak current occurs can be written from equation (73)

$$t_o^* \sim \frac{1}{\beta_\tau} \ln\ln\ln\left(\frac{c}{a\beta_\tau \sin\theta_1}\right) + \frac{1}{\beta_\tau} \ln\ln\left[\frac{c}{a\beta_\tau \sin\theta_1} \ln\ln\left(\frac{c}{a\beta_\tau \sin\theta_1}\right)\right] \quad (84)$$

For values of $\beta_\tau \leq c(10^{-5})/a \sin\theta_1$ the relative errors of equations (83) and (84) are less than five percent.

The decay of the induced current pulse is inversely proportional to the logarithm of time and can be written from equation (60) as

$$I \sim \frac{2\pi a E_o}{Z} \frac{1}{\beta \ln(q^*)}$$

$$= \frac{2\pi c E_o}{Z\beta_\tau \sin\theta_1} \frac{1}{\ln\left(\frac{ct^*}{a \sin\theta_1}\right)} \text{ amperes} \quad (85)$$

For values of $t^* \geq (a/c)(10^8)$ the relative error of equation (85) is less than five percent.

The analytical development in section five can be extended for an incident electric field with a time history given by

$$f(t) = \sum_{n=o}^{N} A_n e^{-\beta_n t} \tag{86}$$

The substitution of equation (86) into equation (41) gives

$$\Lambda = \sum_{n=o}^{N} A_n \Lambda_n \tag{87}$$

where Λ_n is found by the substitution of β_n for β in equation (45). Now consider an incident field with a double exponential time history given by

$$f(t) = e^{-\beta_\tau t} - e^{-\alpha_\tau t} \tag{88}$$

The initial rise of the current induced by a double exponential time history incident field is proportional to the 3/2 power of time and can be written as

$$I \sim \frac{2\pi a E_o 4q^* \sqrt{2q^*}}{3Z\pi}(\alpha-\beta)U(q^*)$$

$$= \frac{8E_o(\alpha_\tau-\beta_\tau)t^*}{3Z}\sqrt{\frac{2act^*}{\sin\theta_1}}\ U(t^*) \text{ amperes} \tag{89}$$

In figures 8 and 9, examples of the un-normalized induced current responses are plotted with the angle of incidence as a parameter, $Z = 120\pi$ ohms, $a = 3 \times 10^{-3}$ meters and $E_o = 10^5$ volts per meter. The incident wave time history considered in figure 8 is a single exponential with $\beta_\tau = 10^7$ per second and in figure 9 the incident wave time history is a double exponential with $\beta_\tau = 10^7$ per second and $\alpha_\tau = 5 \times 10^8$ per second. The

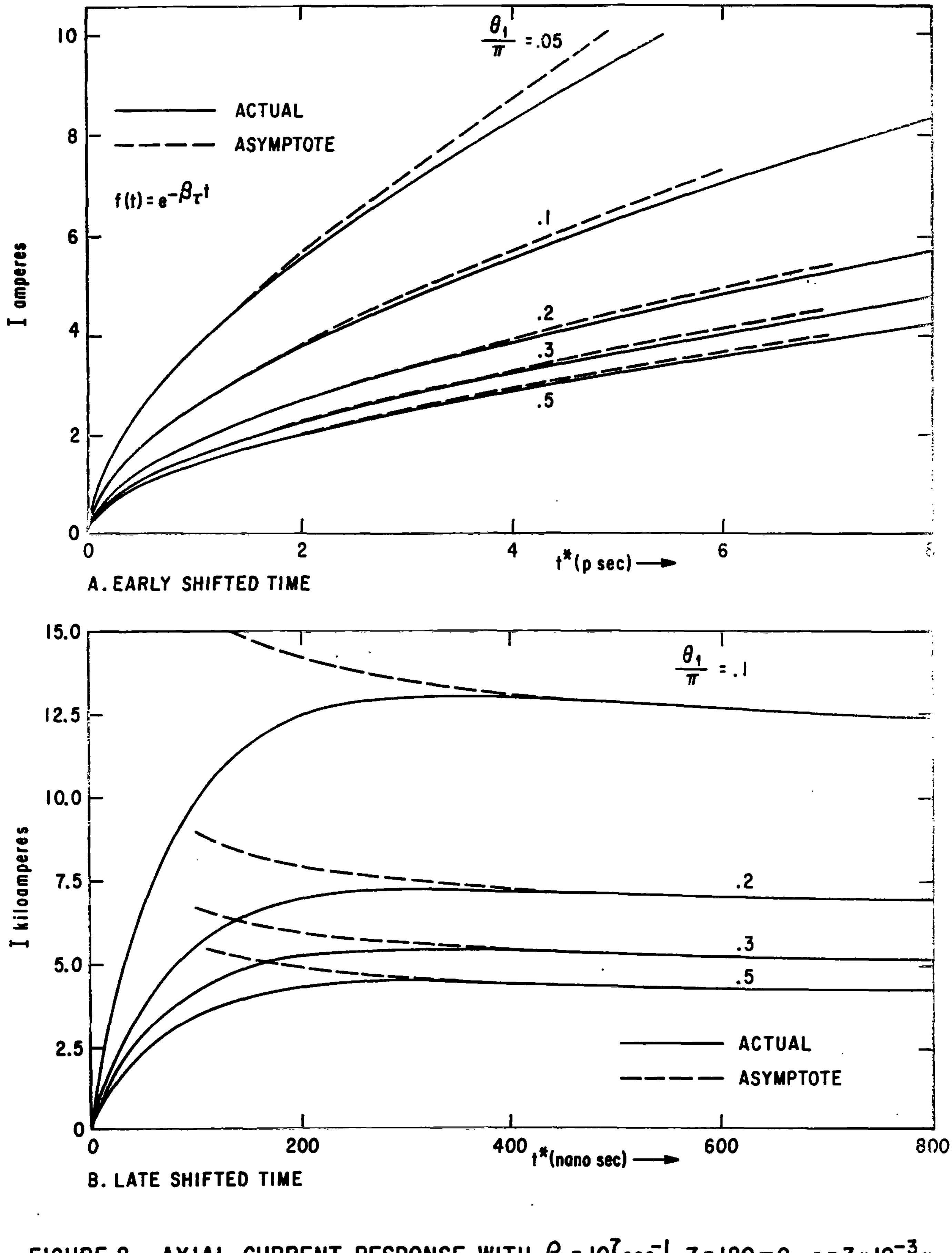


FIGURE 8. AXIAL CURRENT RESPONSE WITH $\beta_\tau = 10^7 \sec^{-1}$, $Z = 120\pi\Omega$, $a = 3 \times 10^{-3}$m, $E_0 = 10^5$ VOLTS PER METER, AND $\frac{\theta_1}{\pi}$ AS A PARAMETER.

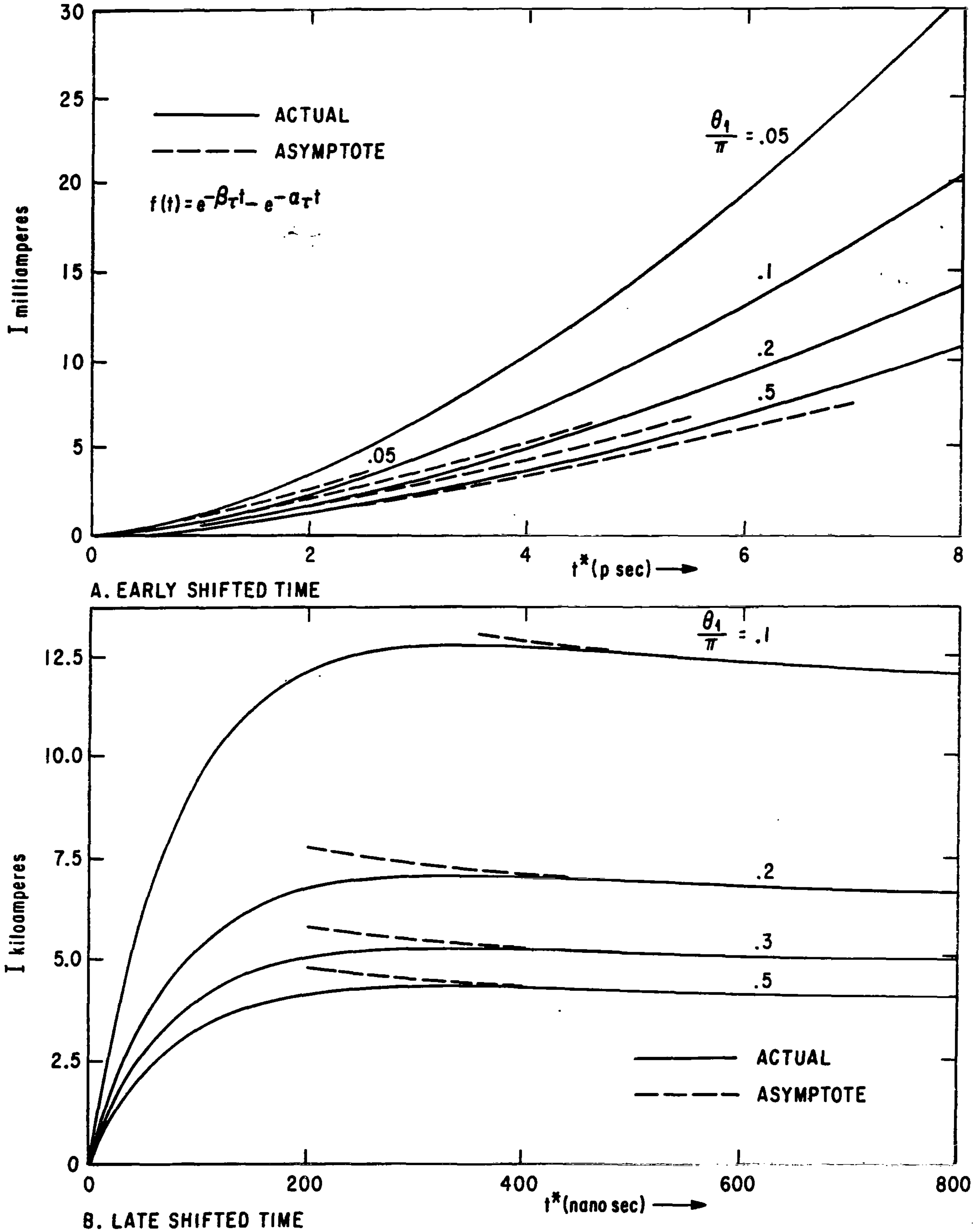


FIGURE 9. AXIAL CURRENT RESPONSE WITH $\beta_\tau = 10^7 \text{ sec}^{-1}$, $Z = 120\pi\Omega$, $a = 3\times10^{-3}$m, $a_\tau = 5\times10^8 \text{ sec}^{-1}$, $E_0 = 10^5$ V/m, AND $\frac{\theta_1}{\pi}$ AS A PARAMETER.

waveforms and asymptotic curves were computed by the same procedure used for the plots in figure 6. Notice that the initial rise of the current in figure 9 is much slower than the initial rise of the current in figure 8. But the late time responses are nearly identical since $\alpha_\tau = 50\beta_\tau$; actually a late time relative asymptotic difference in magnitude can be obtained from equations (85) and (87) as

$$D = \frac{I_{\beta_\tau} - I_{(\beta_\tau - \alpha_\tau)}}{I_{\beta_\tau}} \sim \frac{\beta_\tau}{\alpha_\tau} \tag{90}$$

where I_{β_τ} is the current response to an incident wave with a time history given by equation (44) and $I_{(\beta_\tau - \alpha_\tau)}$ is the current response to an incident wave with a time history given by equation (88). For $\alpha_\tau = 50\beta_\tau$, the relative asymptotic difference in the late time waveforms is 2%.

Finite cylindrical structure

The results developed for the infinitely long cylindrical structure can be applied to the finitely long cylindrical structure for the applicable time interval $0 \leq t^* \leq t_a^*$. The shifted time t_a^* is that time when the first reflection from the ends of the structure effects the induced current response. It was observed in section I that t_a^* equals the minimum value of

$$\left[\frac{\ell_1}{c}(1-\cos\theta_1) \;,\quad \frac{\ell_2}{c}(1+\cos\theta_1)\right]$$

Notice that as $\theta_1 \to 0$ the asymptotic behavior of t_a^* is

$$t_a^* = \frac{\ell_1}{2c}\,\theta_1^2 + O(\theta_1^4) \tag{91}$$

and as $\theta_1 \to \pi$ the asymptotic behavior of t_a^* is

$$t_a^* = \frac{\ell_2}{2c}(\pi-\theta_1)^2 + O((\pi-\theta_1)^4) \qquad (92)$$

The substitution of equations (91) and (92) into equation (81) gives the asymptotic form of the current response as $\theta_1 \to 0$ at time t_a^* as

$$I \sim \frac{4aE_o}{Z}\sqrt{\frac{\ell_1}{a}\,\theta_1} \qquad (93)$$

and as $\theta_1 \to \pi$, we have

$$I \sim \frac{4aE_o}{Z}\sqrt{\frac{\ell_2}{a}(\pi-\theta_1)} \qquad (94)$$

The shifted time of the peak value of the induced current, for $\beta_\tau \sin\theta_1 << c/a$, is proportional to a logarithmic function of θ_1 as given by equation (84). Therefore, for very long structures such that $(\ell_1,\ell_2) >> c/\beta_\tau$ the induced current will reach or approach its maximum value for a wide range of values for θ_1. From equation (83) it is evident that the peak current is inversely proportional to $\sin\theta_1$. Thus, the angle θ_m of the incident wave that will induce the largest current at $t^* = t_a^*$ is restricted to $0 < \theta_m \leq \theta_o$; where θ_o is the value of θ_1 such that $t_o^* = t_a^*$.

In figure 10 an example of the induced current response at $t^* = t_a^*$ is plotted against θ_1 for $\ell_1/c = 3 \times 10^{-6}$ second, $\ell_2/c = 1.5 \times 10^{-7}$ seconds, $E_o = 10^5$ volts per meter, $a = 3 \times 10^{-3}$ meters, $Z = 120\pi$ ohms, and $\beta_\tau = 10^7$ per second. In this example, $\theta_o \approx .139\pi$ and $\theta_m \approx .0874\pi$.

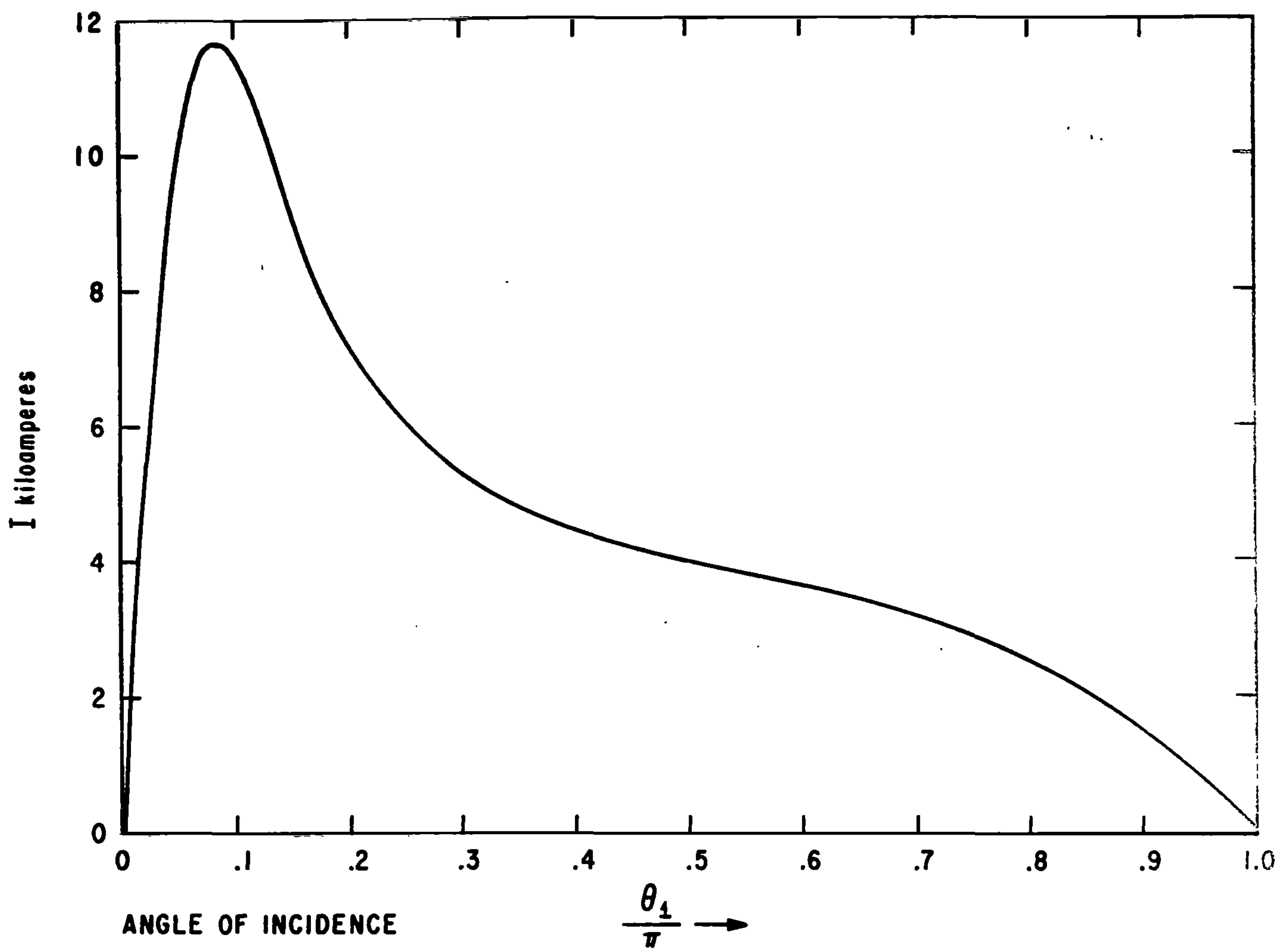


FIGURE 10. AXIAL CURRENT RESPONSE AT $t^* = t_a^*$ WITH $\beta_\tau = 10^7 \text{sec.}^{-1}$, $a = 3 \times 10^{-3}$m, $\frac{\ell_1}{c} = 3 \times 10^{-6}$s, $\frac{\ell_2}{c} = 1.5 \times 10^{-7}$s, $E_0 = 10^5$ V/m, AND $Z = 120\pi\Omega$.

Appendix A Asymptotic Expansion of $F(\zeta)$ for $\zeta \to \infty$

The function $F(\zeta)$ is given by

$$F(\zeta) = \int_o^\infty \frac{e^{-\xi(\zeta-1)} I_o(\xi)}{\xi\left[K_o^2(\xi)+\pi^2 I_o^2(\xi)\right]} d\xi \tag{A1}$$

As $\zeta \to \infty$, it was determined in Appendix A of reference 4 that, for $\eta = (\zeta-1) \to \infty$,

$$F(\zeta) = T(\zeta) + O(\zeta^{-1}) \tag{A2}$$

where

$$T(\zeta) = \int_o^\delta \frac{e^{-\xi(\zeta-1)}}{\xi[\ln^2(\xi\Gamma/2)+\pi^2]} d\xi \tag{A3}$$

where $\Gamma = 1.7810\cdots$, the exponential of Euler's constant and δ is chosen such that $0 < \delta < 2/\Gamma$. For convenience, replace $\zeta - 1$ by η

$$T = \int_o^\delta \frac{e^{-\xi\eta}}{\xi[\ln^2(\xi\Gamma/2)+\pi^2]} d\xi \tag{A4}$$

Integration by parts gives

$$T = -\frac{1}{2} + \int_o^\delta \eta e^{-\xi\eta} \frac{\arctan}{\pi}\left(\frac{\ln(\xi\Gamma/2)}{\pi}\right) d\xi + O(e^{-\delta\eta}) \tag{A5}$$

If we further restrict δ such that $0 < \delta < (2/\Gamma)e^{-\pi^2}$, then

$$T = O(e^{-\delta\eta}) - \int_0^\delta \frac{\eta e^{-\xi\eta}}{\ln(\xi\Gamma/2)}\, d\xi + \int_0^\delta \frac{\eta e^{-\xi\eta}\pi^2}{3\ln^3(\xi\Gamma/2)}\, d\xi$$

$$- \int_0^\delta \frac{\eta e^{-\xi\eta}\pi^4}{5\ln^5(\xi\Gamma/2)}\, d\xi + \cdots \qquad \text{(A6)}$$

where the series expansion of arctan z for $|z| > 1$ and $z^2 \neq -1$ as given in equation 4.4.42 of reference 2 has been used. Thus,

$$T = \sum_{n=o}^{N} T_n + O(e^{-\delta\eta}) + R_N \qquad \text{(A7)}$$

where, with $u = \xi\eta$,

$$T_n = \frac{\pi^{2n}(-1)^{(n+1)}}{(2n+1)} \int_0^{\delta\eta} \frac{e^{-u}}{[\ln u - \ln(2\eta/\Gamma)]^{2n+1}}\, du \qquad \text{(A8)}$$

T_n can be written as

$$T_n = \frac{\pi^{2n}(-1)^n}{(2n+1)[\ln(2\eta/\Gamma)]^{2n+1}} \int_0^{\delta\eta} \frac{e^{-u}}{\left[1 - \frac{\ln u}{\ln(2\eta/\Gamma)}\right]^{2n+1}}\, du \qquad \text{(A9)}$$

Expansion in equation (A9) of

$$\left[1 - \frac{\ln u}{\ln(2\eta/\Gamma)}\right]^{-(2n+1)}$$

by the binomial theorem gives

$$T_n = \frac{\pi^{2n}(-1)^n}{(2n+1)[\ln(2\eta/\Gamma)]^{2n+1}} \left\{ \sum_{k=o}^{M} P_k + R_M \right\} \qquad \text{(A10)}$$

where

$$P_k = \int_o^{\delta\eta} \binom{-(2n+1)}{k} \frac{e^{-u}\ln^k u(-1)^k du}{\ln^k(2\eta/\Gamma)} \tag{A11}$$

and

$$P_k = \binom{-2n-1}{k} \frac{(-1)^k}{\ln^k(2\eta/\Gamma)} \int_o^{\delta\eta} e^{-u}\ln^k u du$$

$$= \binom{-2n-1}{k} \frac{(-1)^k}{\ln^k(2\eta/\Gamma)} C_k \tag{A12}$$

where

$$C_k = \int_o^{\delta\eta} e^{-u}\ln^k u du \tag{A13}$$

C_k can be written as

$$C_k = \int_o^{\infty} e^{-u}\ln^k u du - \int_{\delta\eta}^{\infty} e^{-u}\ln^k u du$$

$$= C_k^{\infty} - C_k^{o} \tag{A14}$$

To bound C_k^o, we can write[5]

$$C_k^o = e^{-\delta\eta}\int_o^{\infty} e^{-x}\left[\ln(\delta\eta)+\ln\ 1+\left(\frac{x}{\delta\eta}\right)\right]^k dx \le E_k \tag{A15}$$

where

$$E_k = \frac{e^{-\delta\eta}}{(\delta\eta)^k}\int_o^{\infty} e^{-x}(\delta\eta\ln(\delta\eta)+x)^k dx \tag{A16}$$

The value of E_k is given by equation (68) of reference 5 as

$$E_k = \ln^k(\delta\eta)e^{-\delta\eta}\left[1 + \frac{k}{\delta\eta\ln(\delta\eta)} + \frac{k(k-1)}{[\delta\eta\ln(\delta\eta)]^2} + \cdots + \frac{k!}{[\delta\eta\ln(\delta\eta)]^k}\right] \tag{A17}$$

To bound R_M, we can write

$$R_M = \int_0^{\delta\eta} e^{-u}\left[\sum_{k=M+1}^{\infty}\binom{-(2n+1)}{k}\frac{\ln^k u}{\ln^k(2\eta/\Gamma)}\right]du \tag{A18}$$

The binomial series is bounded by a finite constant M_ε multiplied by the (M+1) term, thus

$$|R_M| \le \int_0^{\delta\eta}\frac{M_\varepsilon e^{-u}(\ln u)^{M+1}}{[\ln(2\eta/\Gamma)]^{M+1}}du$$

$$\le \frac{M_\varepsilon}{[\ln(2\eta/\Gamma)]^{M+1}}\left\{\int_0^1(\ln u)^{M+1}du + \int_1^\infty e^{-u}u^m du\right\}$$

$$= \frac{M_\varepsilon e^{-1}}{[\ln(2\eta/\Gamma)]^{M+1}}[1+m+m(m-1)+\cdots m!] \tag{A19}$$

where m is a finite integer $\ni u^m > (\ln u)^{M+1}$ for $1 \le u \le \infty$. From equation (A19) it follows that

$$R_M = O\left(\frac{1}{[\ln(2\eta/\Gamma)]^{M+1}}\right) \tag{A20}$$

To bound R_N, we can write

$$R_N = \int_0^{\delta\eta} e^{-u}\left\{\sum_{n=N+1}^{\infty} \frac{\pi^{2n}(-1)^{(n+1)}}{(2n+1)[\ln u-\ln(2\eta/\Gamma)]^{2n+1}}\right\}du \tag{A21}$$

And for a finite constant N_ε the series is bounded by N_ε multiplied by the (N+1) term, thus

$$|R_N| \leq \int_0^{\delta\eta} N_\varepsilon \frac{e^{-u}}{[\ln u-\ln(2\eta/\Gamma)]^{2n+1}}\,du$$

$$= N'_\varepsilon T_{N+1} \tag{A22}$$

where N'_ε is a finite constant. From equations (A10) and (A20), it follows that

$$R_N = O\left(\frac{1}{[\ln(2\eta/\Gamma)]^{2N+3}}\right) \tag{A23}$$

Collecting the results of equations (A7), (A10), (A12), (A14), (A17), (A20), and (A23) gives the asymptotic expansion of T for $\eta \to \infty$†

$$T = \sum_{n=o}^{N}\left\{\frac{\pi^{2n}(-1)^n}{(2n+1)[\ln(2\eta/\Gamma)]^{2n+1}}\left[\sum_{k=o}^{M}\binom{-2n-1}{k}\frac{C_k^\infty(-1)^k}{\ln^k(2\eta/\Gamma)}\right]\right\}$$

$$+ O([\ln(2\eta/\Gamma)]^{-2N-2}) \tag{A24}$$

where

$$C_k^\infty = \int_0^\infty e^{-u}\ln^k u\,du$$

†The terms for $(2n+1)M \geq 2N + 2$ are contained in the order symbol and are to be disregarded.

$$C_o^\infty = \int_o^\infty e^{-u}du = 1 \tag{A25}$$

$$C_1^\infty = \int_o^\infty e^{-u}\ln u du = -\gamma_e \text{ (Euler's constant)}$$

$$C_2^\infty = \int_o^\infty e^{-u}\ln^2 u du = \frac{\pi^2}{6} + \gamma_e^2$$

For N = 2 and M = 2, we have

$$T = \frac{1}{\ln(2\eta/\Gamma)} - \frac{\gamma_e}{\ln^2(2\eta/\Gamma)} - \frac{\left[(\pi^2/6)-\gamma_e^2\right]}{\ln^3(2\eta/\Gamma)} + \frac{\pi^2\gamma_e}{\ln^4(2\eta/\Gamma)}$$

$$- \frac{\left[(2\pi^4/15)+2\pi^2\gamma_e^2\right]}{\ln^5(2\eta/\Gamma)} + O(\ln^{-6}(2\eta/\Gamma)) \tag{A26}$$

The substitution of $\eta = \zeta - 1$ into equation (A26) and the substitution of equation (A26) into equation (A2) gives the asymptotic expansion of $F(\zeta)$ for $\zeta \to \infty$ as

$$F(\zeta) = \frac{1}{\ln(2\zeta/\Gamma)} - \frac{\gamma_e}{\ln^2(2\zeta/\Gamma)} - \frac{\left[(\pi^2/6)-\gamma_e^2\right]}{\ln^3(2\zeta/\Gamma)} + \frac{\pi^2\gamma_e}{\ln^4(2\zeta/\Gamma)}$$

$$+ O(\ln^{-5}(2\zeta/\Gamma)) \tag{A27}$$

$F(\zeta)$ can be written in closed form as

$$F(\zeta) = \sum_{p=1}^{4} \frac{a_p}{[\ln(2\zeta/\Gamma)]^p} + O(\ln^{-5}(2\zeta/\Gamma)) \tag{A28}$$

where

$$a_1 = 1$$

$$a_2 = -\gamma_e$$

$$a_3 = -\pi^2/6 + \gamma_e^2$$

$$a_4 = \pi^2\gamma_e$$

Appendix B Asymptotic Expansion of $T_3(q^*)$ for $q^* \to \infty$

The function $T_3(q^*)$ is given by

$$T_3(q^*) = e^{-\beta q^*}\int_M^{q^*} e^{\beta\zeta}F(\zeta)d\zeta \tag{B1}$$

where $M >> \Gamma/2$, Γ is the exponential of Euler's constant, and

$$F(\zeta) = \int_0^\infty \frac{e^{-\xi(\zeta-1)}I_o(\xi)}{\xi\left[K_o^2(\xi)+\pi^2 I_o^2(\xi)\right]}\,d\xi \tag{B2}$$

The asymptotic expansion of $F(\zeta)$ for $\zeta \to \infty$ is given in Appendix A as

$$F(\zeta) = \sum_{p=1}^{N} a_p \frac{1}{\ln^p(2\zeta/\Gamma)} + O([\ln(2\zeta/\Gamma)]^{-(N+1)}) \tag{B3}$$

where a_p is the pth coefficient. The p-coefficients for $N = 4$ are given in equation (A27) of Appendix A.

The substitution of equation (B3) into (B1) gives

$$T_3(q^*) = e^{-\beta q^*}\sum_{p=1}^{N} a_p \int_M^{q^*} \frac{e^{\beta\zeta}}{\ln^p(2\zeta/\Gamma)}\,d\zeta + R_N \tag{B4}$$

where, for a finite constant N_ε,

$$|R_N| \leq N_\varepsilon e^{-\beta q^*}\int_M^{q^*} \frac{e^{\beta\zeta}}{[\ln(2\zeta/\Gamma)]^{N+1}}\,d\zeta \tag{B5}$$

Now let $q^*u = \zeta$, equation (B4) becomes

$$T_3(q^*) = \sum_{p=1}^{N} a_p \int_{M/q^*}^{1} \frac{q^* e^{\beta q^*(u-1)}}{[\ln(2q^*/\Gamma)+\ln u]^p}\, du + R_N$$

$$= \sum_{p=1}^{N} a_p P_p + R_N \qquad \text{(B6)}$$

where

$$P_p = \frac{1}{[\ln(2q^*/\Gamma)]^p} \int_{M/q^*}^{1} \frac{q^* e^{\beta q^*(u-1)}}{\left[1+\frac{\ln u}{\ln(2q^*/\Gamma)}\right]^p}\, du \qquad \text{(B7)}$$

Expansion of

$$\left[1+\frac{\ln u}{\ln(2q^*/\Gamma)}\right]^{-p}$$

by the binomial theorem gives

$$P_p = \frac{1}{[\ln(2q^*/\Gamma)]^p} \left\{ \sum_{n=o}^{k} W_n + R_k \right\} \qquad \text{(B8)}$$

where

$$W_n = \binom{-p}{n} \int_{M/q^*}^{1} \frac{q^* e^{\beta q^*(u-1)} (\ln u)^n}{[\ln(2q^*/\Gamma)]^n}\, du \qquad \text{(B9)}$$

and

$$R_k = \int_{M/q^*}^{1} q^* e^{\beta q^*(u-1)} \sum_{n=k+1}^{\infty} \binom{-p}{n} \left[\frac{\ln u}{\ln(2q^*/\Gamma)}\right]^n du \qquad \text{(B10)}$$

Equation (B9) can be written in the more convenient form

$$W_n = \binom{-p}{n} \frac{1}{[\ln(2q^*/\Gamma)]^n} C_n \tag{B11}$$

where

$$C_n = \int_{M/q^*}^{1} q^* e^{\beta q^*(u-1)} (\ln u)^n du \tag{B12}$$

Let $v = u - 1$ and $v_o = (M/q^*) - 1$, equation (B12) becomes

$$C_n = q^* \int_{v_o}^{o} e^{\beta q^* v} [\ln(1+v)]^n dv \tag{B13}$$

The substitution of the series representation of $\ln(1+v)$ for $v^2 < 1.0$ as given by equation 601 in reference 7 into equation (B13) gives

$$C_n = q^* \int_{v_o}^{o} e^{\beta q^*} \left[v - \frac{v^2}{2} + \frac{v^3}{3} + \cdots \right]^n dv$$

$$= q^* \int_{v_o}^{o} e^{\beta q^*} [v + R(v)]^n dv \tag{B14}$$

where

$$R(v) = \sum_{m=2}^{\infty} (-1)^{m+1} \frac{v^m}{m} \tag{B15}$$

Expansion of $[v + R(v)]^n$ by the binomial theorem gives

$$[v + R(v)]^n = v^n\left[1 + \frac{R(v)}{v}\right]^n$$

$$= v^n\left[1 + \frac{nR(v)}{v} + \frac{n(n-1)}{2}\,\frac{R^2(v)}{v^2} + \cdots\right]$$

$$= v^n[1 + R'(v)] \qquad \text{(B16)}$$

where

$$R'(v) = \sum_{s=1}^{\infty}\binom{n}{s}\left(\frac{R(v)}{v}\right)^s \qquad \text{(B17)}$$

The substitution of equation (B16) into equation (B14) gives

$$C_n = q^*\int_{v_o}^{o} e^{\beta q^* v} v^n dv + q^*\int_{v_o}^{o} e^{\beta q^* v} R'(v) v^n dv \qquad \text{(B18)}$$

R'(v) is bounded by a finite constant R_ε and the first term of the series given by equation (B17) as

$$R'(v) \le R_\varepsilon\,\frac{R(v)}{v} \qquad \text{(B19)}$$

and it follows from equation (B15) that

$$R'(v) \le R'_\varepsilon v \qquad \text{(B20)}$$

where R'_ε is a finite constant. Now we can write

$$q^*\int_{v_o}^{o} e^{\beta q^* v} R'(v) v^n dv \le R'_\varepsilon q^*\int_{v_o}^{o} e^{\beta q^* v} v^{n+1} dv \qquad \text{(B21)}$$

Evaluation of the first integral in equation (B18) gives, for $\beta \neq 0$,

$$q^* \int_{v_o}^{o} e^{\beta q^*} v^n dv = \frac{(-1)^n n!}{\beta(\beta q^*)^n} + O(e^{-\beta q^*}) \tag{B22}$$

The substitution of equation (B22) into (B18) and (B21) gives

$$C_n = \frac{(-1)^n n!}{\beta(\beta q^*)^n} + O((q^*)^{-(n+1)}) \tag{B23}$$

R_k can now be bounded by writing equation (B10) as

$$|R_k| \leq k_\varepsilon \int_{M/q^*}^{1} q^* e^{\beta q^*(u-1)} \left[\frac{\ln u}{\ln(2q^*/\Gamma)}\right]^{k+1} du$$

$$= k'_\varepsilon \frac{C_{k+1}}{[\ln(2q^*/\Gamma)]^{k+1}} \tag{B24}$$

where k_ε and k'_ε are finite constants.

From equation (23) it follows that

$$R_k = O([q^* \ln(2q^*/\Gamma)]^{-(k+1)}) \tag{B25}$$

To bound R_N, we can write equation (B5) as

$$|R_N| \leq N_\varepsilon P_{N+1} = N'_\varepsilon \left[\frac{C_o + R_k}{[\ln(2q^*/\Gamma)]^{N+1}}\right]\Bigg|_{k=o} \tag{B26}$$

and it follows from equations (B23) and (B25) that

$$R_N = O([\ln(2q^*/\Gamma)]^{-(N+1)}) \tag{B27}$$

Collecting the results of equations (B6), (B8), (B11), (B12), (B23), (B25), and (B27) gives the asymptotic expansion of $T_3(q^*)$ for $q^* \to \infty$ with $\beta \neq 0$ as

$$T_3(q^*) = \sum_{p=1}^{N} a_p \frac{1}{(\ln(2q^*/\Gamma)]^p} \sum_{n=o}^{N-p} \binom{-p}{n} \frac{(-1)^n n!}{\beta[\beta q^* \ln(2q^*/\Gamma)]^n}$$

$$+ O([\ln(2q^*/\Gamma)]^{-(N+1)}) \tag{B28}$$

For k = 2, we have for $\beta \neq 0$

$$T_3(q^*) = \sum_{p=1}^{N} \frac{a_p}{\beta[\ln(2q^*/\Gamma)]^p} + \sum_{p=1}^{N-1} \frac{pa_p}{\beta(\beta q^*)[\ln(2q^*/\Gamma)]^{p+1}}$$

$$+ \sum_{p=1}^{N-2} \frac{p(p+1)a_p}{\beta(\beta q^*)^2[\ln(2q^*/\Gamma)]^{p+2}} + O([\ln(2q^*/\Gamma)]^{-(N+1)}) \tag{B29}$$

For the case $\beta = 0$, equation (B12) can be written as

$$C_n = q^* \int_{M/q^*}^{1} (\ln u)^n du = q^*(-1)^n \int_o^1 \left(\ln\left(\frac{1}{u}\right)\right)^n du = q^* \int_o^{M/q^*} (\ln u)^n du$$

$$= q^*(-1)^n \Gamma(n+1) - q^* \int_{-\infty}^{\ln(M/q^*)} e^x x^n dx \tag{B30}$$

where $x = \ln u$. Thus, it follows that C_n can be written as

$$C_n = (-1)^n n! q^* + O((\ln q^*)^n) \tag{B31}$$

The substitution of equation (B31) into (B24) gives

$$R_k = O(q^*[\ln(2q^*/\Gamma)]^{-(k+1)}) \tag{B32}$$

And the substitution of equation (B31) into (B26) gives

$$R_N = O(q^*[\ln(2q^*/\Gamma)]^{-(N+1)}) \tag{B33}$$

Collecting the results of equations (B6), (B8), (B11), (B31), (B32), and (B33) gives the asymptotic expansion of $T_3(q^*)$ for $q^* \to \infty$ with $\beta = 0$ as

$$T_3(q^*) = \sum_{p=1}^{N} a_p \frac{q^*}{[\ln(2q^*/\Gamma)]^p} \sum_{n=o}^{N-p} \binom{-p}{n} \frac{(-1)^n n!}{[\ln(2q^*/\Gamma)]^n}$$

$$+ O(q^*[\ln(2q^*/\Gamma)]^{-(n+1)}) \tag{B34}$$

For $k = 2$, we have for $\beta = 0$

$$T_3(q^*) = \sum_{p=1}^{N} \frac{a_p q^*}{[\ln(2q^*/\Gamma)]^p} + \sum_{p=1}^{N-1} \frac{p a_p q^*}{[\ln(2q^*/\Gamma)]^{p+1}}$$

$$+ \sum_{p=1}^{N-2} \frac{p(p+1) a_p q^*}{[\ln(2q^*/\Gamma)]^{p+2}} + O(q^*[\ln(2q^*/\Gamma)]^{-(N+1)}) \tag{B35}$$

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