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ELECTROMAGNETIC PULSE SCATTERING FROM HOLLOW CYLINDERS AND HALF-PLANES

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ABSTRACT

The electric-field integral equation in the time-domain has been derived for the electromagnetic pulse scattering from bodies of rotation. The equation has been solved for the semi-infinite hollow cylinder with near-axis incidence and for the half-plane as a limiting case. Numerical results have been obtained for the large hollow cylinder and half-plane problems. The results for the latter are verified by the exact results obtained by an analytical method.

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I. INTRODUCTION

With the advent of the large computers, many workers have studied electromagnetic scattering from various scattering bodies. The problem is to calculate the scattered fields from given scatterers with given incident waves. The problem can be categorized into two broad areas, i.e., time-dependent and time-harmonic cases.

In the time-harmonic cases, the problem can most often be formulated via the integral equations followed by their transformation into matrix equations. The coefficient matrices of these matrix equations are independent of the incident waves, and hence they represent the inherent characteristics of the scatterers. There are many good references available to this approach, and, in fact, the literature is too extensive to include all of them. However, the work by Poggio and Miller[1] is worth mentioning, which also includes a comprehensive bibliography. In addition to the standard integral equation approach, there are some variants to this method as well as some approximate techniques which are often useful for high frequencies or complex bodies.

In the case of time-dependent problems, one is interested in the scattered field due to the electromagnetic pulse (EMP) incident waves. There are basically two ways of investigating this problem. The first way is to take the Fourier transform of the data obtained from the time-harmonic solution. In this method, time-harmonic solutions are first computed for many discrete frequencies. These solutions are next weighted according to the frequency spectrum of the desired incident pulse wave, and then the time-dependent response is obtained by transforming the frequency domain response using a fast Fourier transform routine.

The second method is to attack directly the time-dependent integral equations and to obtain numerical solutions. As opposed to the first method, it is not necessary to perform a repeated calculation to obtain a discrete set of data. In the second method, the solution of the integral equation is dependent upon the incident wave. However, once the response to an impulse incident wave is obtained, the response to any other incident wave may be obtained by time-convolution.

There are also many useful references available to the EMP problems. In addition, it may be noted that recently a radically new approach to EMP problems was introduced by C. E. Baum.[2] His method is called the singularity expansion technique, and is essentially the analysis of the scattering behavior by investigating poles in the complex s-plane associated with the Laplace transform of time.

In this paper, the time-domain electric-field integral equations are used to solve the EMP problem for the large semi-infinite hollow cylinder. The magnetic field integral equation cannot be used because of the geometry of the scatterers to be considered, although this equation is believed to be numerically more stable for thick bodies as compared to the electric-field integral equation.

In Chapter II, the time-domain electric and magnetic integral equations are derived for general three-dimensional problems. In the next chapter, the electric-field integral equation is specialized to apply to the perfectly conducting scatterers possessing rotational symmetry. First, the integral equations are derived for near-axis incidence. Then, the E-field equation for the hollow cylinder is derived. If the analytical limit is properly taken, the equations can be used to handle the half plane problem. In Chapter IV, computational algorithms and some numerical results are presented for several structures. The last chapter is the conclusion of the present work and some comments pertaining to the future of the problem.

II. DERIVATION OF THE ELECTRIC AND MAGNETIC FIELD TIME-DOMAIN INTEGRAL EQUATIONS

Maxwell's equations may be used to derive two of the more familiar integral equations, the magnetic field (HFIE) and electric field (EFIE) integral equations. The derivation is based on the results of Poggio and Miller;[1] their work was based on that previously done by Stratton.[3] The results permit one to use the integral equations on surfaces that are not smooth, i.e., surfaces whose tangents are not analytic functions of position.

2.1 Maxwell's Equations and a Green's Identity

To derive the integral equations, we start with Maxwell's equations which may be written as

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} - \overline{K}$$

$$\nabla \times \overline{H} = \varepsilon \frac{\partial \overline{E}}{\partial t} + \overline{J}$$

$$\nabla \cdot \overline{E} = \rho/\varepsilon \qquad (2.1a)$$

$$\nabla \cdot \overline{H} = m/\mu$$

with corresponding continuity relations

$$\nabla \cdot \overline{J} = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \overline{K} = -\frac{\partial m}{\partial t} \quad . \qquad (2.2b)$$

The quantities are defined as follows: $\overline{E}$ and $\overline{H}$ are the electric and magnetic field intensities, $\overline{J}$ and $\overline{K}$ are the electric and magnetic vector current densities, and ρ and m are the electric and magnetic charge densities. In linear, isotropic, and homogeneous media, ε and μ are scalar constants and the first two Equations of (2.1a) may be written as

$$\nabla \times \nabla \times \overline{E} = -\mu\,\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} - \nabla \times \overline{K} - \mu \frac{\partial \overline{J}}{\partial t}$$

$$\nabla \times \nabla \times \overline{H} = -\mu\,\varepsilon \frac{\partial^2 \overline{H}}{\partial t^2} + \nabla \times \overline{J} - \varepsilon \frac{\partial \overline{K}}{\partial t} \quad . \qquad (2.2)$$

To develop the integral equations, we will use the Equations of (2.2) along with that of a scalar Green's function in a Green's identity. To obtain the Green's identity, we use the vector Green's theorem given by

$$\int_V (\overline{Q} \cdot \nabla \times \nabla \times \overline{P} - \overline{P} \cdot \nabla \times \nabla \times \overline{Q})\, dv =$$

$$\int_{\partial V} (\overline{P} \times \nabla \times \overline{Q} - \overline{Q} \times \nabla \times \overline{P}) \cdot d\overline{A} \qquad (2.3)$$

where ∂V represents the boundary of V. For the Green's theorem to be valid, it is sufficient for $\overline{P}$ and $\overline{Q}$ to have continuous second partial derivatives throughout V and for the surface integral to exist. Let $\overline{Q} = \hat{a}q$, then a scalar-vector Green's identity is written as

$$\int_V [q\nabla \times \nabla \times \overline{P} + \overline{P}\,\nabla^2 q + (\nabla q)\,(\nabla \cdot \overline{P})]\, dv =$$

$$\int_{\partial V} [-(\nabla q) \times (\hat{n} \times \overline{P}) + (\hat{n}\cdot\overline{P})\nabla q + \hat{n}q \times (\nabla \times \overline{P})] ds \qquad (2.4)$$

where the surface increment $d\overline{A}$ directed along the outward normal to ∂V is replaced by $\hat{n}$ ds, $\hat{n}$ being the outward normal to the surface.

Now let $\overline{P} = \overline{E}$ and $q = \phi$. The scalar Green's function, ϕ, corresponds to the vector potential, $\hat{a}\phi$, due to a point source with arbitrary direction $\hat{a}$. This function ϕ satisfies the scalar equation

$$\left(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2} \right) \phi(\overline{x}, t; \overline{x}', t') = -4\pi\, \delta(\overline{x} - \overline{x}')\, \delta(t - t') \qquad (2.5)$$

and is defined as

$$\phi(\overline{x}, t; \overline{x}', t') = \frac{\delta(t - t' - R/c)}{R}$$

in which $R = |\overline{x} - \overline{x}'|$ and $\overline{x}$ and $\overline{x}'$ are the position vectors of observation and source points, respectively. Substituting $\overline{E}$ and ϕ into Equation (2.4), we have

$$\int_V \left[\mu\varepsilon \left(\overline{E} \frac{\partial^2}{\partial t^2} \phi - \phi \frac{\partial^2}{\partial t'^2} \overline{E} \right) - \phi\nabla' \times \overline{K} - \mu \frac{\partial \overline{J}}{\partial t'} \phi \right.$$

$$\left. -4\pi\, \delta(\overline{x} - \overline{x}')\, \delta(t - t')\, \overline{E} + \nabla'\phi\, (\nabla' \cdot \overline{E}) \right] dv' =$$

$$\int_{\partial V} [-\nabla'\phi \times (\hat{n}' \times \overline{E}) + (\hat{n}' \cdot \overline{E})\, \nabla'\phi + \hat{n}'\phi \times (\nabla' \times \overline{E})]\, ds'$$

$$(2.6)$$

where the integration and differentiation are on the primed coordinates. We denote $\overline{E} = \overline{E}(\overline{x}', t')$ and $\phi = \phi(\overline{x}, t; \overline{x}', t')$. Substituting for $\nabla' \cdot E$ and $\nabla' \times E$ and using a vector relation* one obtains

$$\int_V \left[\mu\varepsilon \left(\overline{E} \frac{\partial^2}{\partial t^2} \phi - \phi \frac{\partial^2}{\partial t'^2} \overline{E} \right) - \overline{K} \times \nabla'\phi - \mu \frac{\partial \overline{J}}{\partial t'} \phi \right.$$

$$\left. -4\pi\, \delta(\overline{x} - \overline{x}')\, \delta(t - t')\overline{E} + (\rho/\varepsilon)\, \nabla'\phi \right] dv'$$

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* $\int_V \nabla \times \overline{F}\, dv = \int_{\partial V} d\overline{A} \times \overline{F}$.

$$= \int_{\partial V} \left[-\nabla'\phi \times (\hat{n}' \times \overline{E}) + (\hat{n}' \cdot \overline{E}) \nabla'\phi - \hat{n} \times \left(\mu\phi \frac{\partial \overline{H}}{\partial t} \right) \right] ds'. \tag{2.7}$$

We now integrate Equation (2.7) with respect to t' over $(-\infty, \infty)$ and change the order of space and time integration. Since ϕ has a finite time support, the terms containing the second time-derivatives cancel after integrating by parts. Calculating $\nabla'\phi$ in the distribution sense, we have

$$\nabla'\phi = \left[\frac{\delta(t - t' - R/c)}{R^2} + \frac{\delta'(t - t' - R/c)}{Rc} \right] \hat{R} \tag{2.8}$$

where $\hat{R} = \frac{\overline{x} - \overline{x}'}{R}$, and $\delta'(x)$ is defined in the sense that

$$\int \delta'(x)\; f(x)\; dx = - \int \delta(x) \frac{\partial f(x)}{\partial x}\; dx.$$

Substituting for ϕ and $\nabla'\phi$ in Equation (2.7) and integrating as indicated, we obtain

$$\int_V \Bigg\{ -\left[\frac{\overline{K}(\tau)}{R^2} + \frac{1}{Rc} \frac{\partial \overline{K}(\tau)}{\partial \tau} \right] \times \hat{R} - \frac{\mu}{R} \frac{\partial \overline{J}}{\partial \tau}(\tau) - 4\pi\delta(\overline{x} - \overline{x}')\, \overline{E}(\tau)$$

$$+ \left[\frac{\rho(\tau)}{\varepsilon R^2} + \frac{1}{\varepsilon Rc} \frac{\partial \rho(\tau)}{\partial \tau} \right] \hat{R} \Bigg\}_{\tau = t - R/c} dv' =$$

$$\int_{\partial V} \Bigg\{ -\hat{R} \times \left[\hat{n}' \times \left(\frac{\overline{E}(\tau)}{R^2} + \frac{1}{Rc} \frac{\partial \overline{E}(\tau)}{\partial \tau} \right) \right] + \hat{R} \left[\hat{n}' \cdot \left(\frac{\overline{E}(\tau)}{R^2} + \frac{1}{Rc} \frac{\partial \overline{E}(\tau)}{\partial \tau} \right) \right]$$

$$- \frac{\mu}{R} \hat{n}' \times \frac{\partial \overline{H}(\tau)}{\partial \tau} \Bigg\}_{\tau = t - R/c} ds'. \tag{2.9}$$

2.2 The Integral Equation Form

When $\overline{x}$ is not on the surface, ∂V, Equation (2.9) may be evaluated in a straight-forward manner. If $\overline{x}$ is on the bounding surface ∂V, then the surface integral must be interpreted in an appropriate manner. The surface integral is broken into three parts, as shown in Figure 1, where S_2 is a spherical shell about $\overline{x}$.

We define I to be the limit of the integral over S_2 as the radius of the shell becomes zero. Thus, as $R \to 0$, we only need to consider the $1/R^2$ terms which have the greatest singularities. It is noted that all the lower order singularities are integrable ones. Thus,

$$I = -\lim_{R \to 0} \int_{S_2} \overline{E}(\overline{x}', \tau) \frac{\hat{n}' \cdot \hat{n}'}{R^2} \, ds' = -\overline{E}(\overline{x}, t)\,\Omega \tag{2.10}$$

where Ω is the solid angle subtended by S_2 at $\overline{x}$ as R approaches zero.

If all the sources are located outside of S_o, then, because of the radiation condition, the integral over S_o may be replaced by $4\pi\overline{E}_{inc}(\overline{x}, t)$, where $\overline{E}_{inc}$ is due to the sources outside of S_o. Combining these results, and incorporating the integral notation for the Cauchy principal value, Equation (2.9) becomes

$$\overline{E}(\overline{x}, t) = T\,\overline{E}_{inc}(\overline{x}, t) - \frac{T}{4\pi} \int_V \left\{ \frac{\mu}{R} \frac{\partial \overline{J}(\tau)}{\partial \tau} + \left[\frac{1}{R^2} + \frac{1}{Rc} \frac{\partial}{\partial \tau} \right] \overline{K}(\tau) \right.$$

$$\left. \times\, \hat{R} - \left[\frac{1}{R^2} + \frac{1}{Rc} \frac{\partial}{\partial \tau} \right] \frac{\rho(\tau)}{\varepsilon} \hat{R} \right\}_{\tau = t - R/c} dv' -$$

$$\frac{T}{4\pi} \fint_S \left\{ \frac{\mu}{R} \hat{n}' \times \frac{\partial \overline{H}(\tau)}{\partial \tau} + \hat{R} \times \left[\hat{n}' \times \left(\frac{1}{R^2} + \frac{1}{Rc} \frac{\partial}{\partial \tau} \right) \overline{E}(\tau) \right] \right.$$

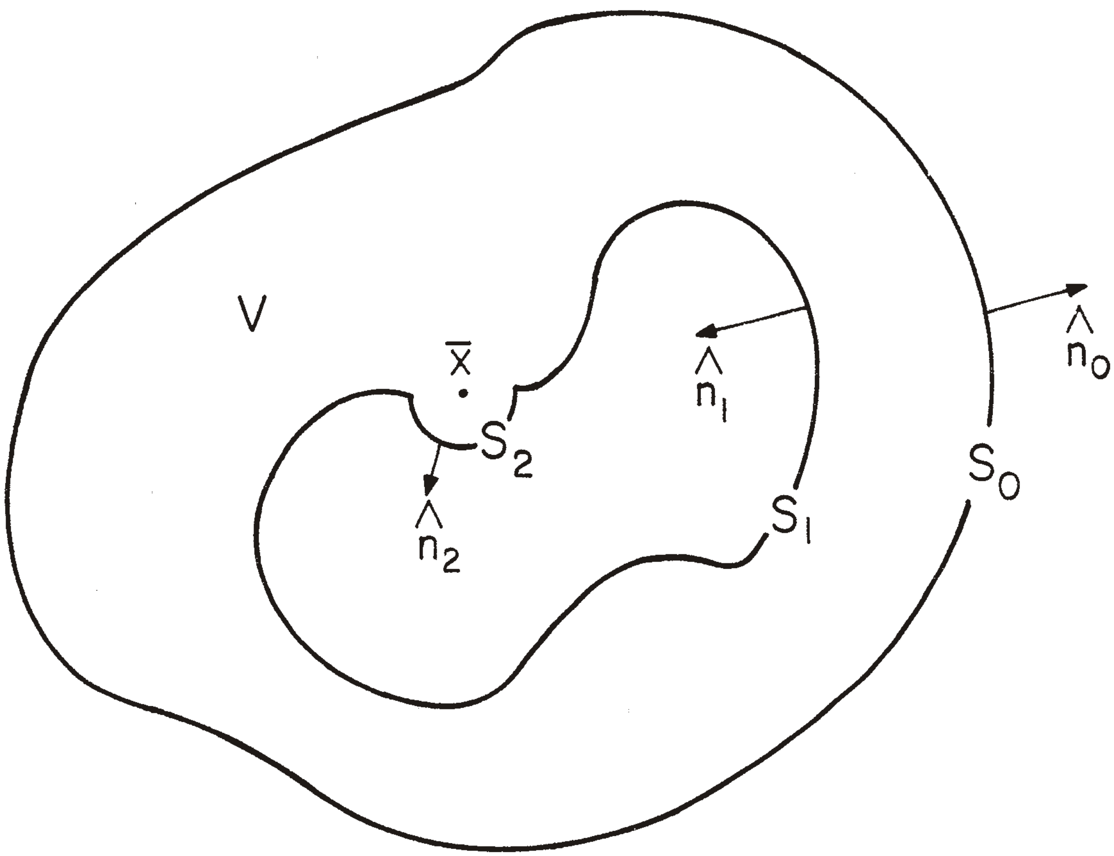

Figure 1. Integration surfaces for derivation of integral equations.

$$- \hat{R}\left[\hat{n}' \cdot \left(\frac{1}{R^2} + \frac{1}{Rc}\frac{\partial}{\partial \tau}\right)\overline{E}(\tau)\right]\Bigg\}_{\tau=t-R/c} ds' \qquad (2.11a)$$

where S is the combined scattering surface $(S_1 + S_2)$, $\hat{n}'$ is redefined as the inward normal to V, and $T = \left[1 - \frac{\Omega}{4\pi}\right]^{-1}$. This is the electric field integral representation which will be used in the rest of this paper. It should be noted that the magnetic integral representation can be obtained simply by taking the dual,

$$\overline{H}(\overline{x}, t) = T\overline{H}_{inc}(\overline{x}, t) + \frac{T}{4\pi}\int_V \Bigg\{ -\frac{\varepsilon}{R}\frac{\partial \overline{K}(\tau)}{\partial \tau} + \left[\frac{1}{R^2} + \frac{1}{Rc}\frac{\partial}{\partial \tau}\right]\overline{J}(\tau) X \hat{R}$$

$$+\left[\frac{1}{R^2} + \frac{1}{Rc}\frac{\partial}{\partial \tau}\right]\frac{m(\tau)}{\mu}\hat{R}\Bigg\}_{\tau=t-R/c} dv' + \frac{T}{4\pi}\oint_S \Bigg\{\frac{\varepsilon}{R}\hat{n}'$$

$$X \frac{\partial \overline{E}(\tau)}{\partial \tau} - \hat{R} X\left[\hat{n}' X\left(\frac{1}{R^2} + \frac{1}{Rc}\frac{\partial}{\partial \tau}\right)\overline{H}(\tau)\right] + \hat{R}\Bigg[\hat{n}'$$

$$\cdot \left(\frac{1}{R^2} + \frac{1}{Rc}\frac{\partial}{\partial \tau}\right)\overline{H}(\tau)\Bigg]\Bigg\}_{\tau=t-R/c} ds' \quad . \qquad (2.11b)$$

2.3 Perfectly Conducting Scatterers

Let us now restrict ourselves to a volume containing no sources and only perfectly conducting scatterers. In such a case, the volume integral drops out and $\hat{n}' X \overline{E} = \hat{n}' \cdot \overline{H} = 0$ on the surface of the scatterer. Rewriting Equations (2.11), we have

$$\overline{E}(\overline{x}, t) = T\overline{E}_{inc}(\overline{x}, t) - \frac{T}{4\pi}\oint_S \Bigg\{\frac{\mu}{R}\hat{n}' X \frac{\partial \overline{H}(\tau)}{\partial \tau} - \hat{R}\Bigg[\hat{n}' \cdot \Bigg(\frac{1}{R^2} + \frac{1}{Rc}$$

$$\cdot \frac{\partial}{\partial \tau}\Bigg)\overline{E}(\tau)\Bigg]\Bigg\}_{\tau=t-R/c} ds' \qquad (2.12a)$$

$$\overline{H}(\overline{x}, t) = T\overline{H}_{inc}(\overline{x}, t) + \frac{T}{4\pi} \oint_S \left\{ \left[\hat{n}' X \left(\frac{1}{R^2} + \frac{1}{Rc} \frac{\partial}{\partial \tau} \right) \overline{H}(\tau) \right] X \hat{R} \right\}_{\tau = t - R/c} ds'. \tag{2.12b}$$

Thus, the fields at a point $\overline{x}$ and time t may be written as a sum of the incident term plus an integral of the equivalent sources on the surface of the scatterer. The Equations (2.12) can be written in terms of the equivalent sources defined by

$$\begin{aligned} \overline{K}_s &= - \hat{n}' X \overline{E} = 0 \\ \overline{J}_s &= \hat{n}' X \overline{H} \end{aligned} \tag{2.13}$$

on the surface of the scatterer. To express $\hat{n}' \cdot \overline{E}$ in terms of the equivalent sources, Maxwell's equations are used to obtain

$$\begin{aligned} \hat{n}' \cdot \varepsilon \frac{\partial \overline{E}}{\partial t} &= \hat{n}' \cdot \nabla X \overline{H} \\ &= - \nabla \cdot (\hat{n}' X \overline{H}) \\ &= - \nabla \cdot \overline{J}_s. \end{aligned} \tag{2.14}$$

Let us now derive the convenient forms of the integral equations for the scatterer. To this end, first let $\overline{x}$ go to the surface of the scatterer and then take a cross product of the Equations (2.12) with $\hat{n}$, the surface normal at $\overline{x}$. The resultant equations read

$$\hat{n} X \overline{E}_{inc}(\overline{x}, t) = \frac{\hat{n} x}{4\pi} \oint_S \left[\frac{\mu}{R} \frac{\partial \overline{J}_s(\overline{x}', \tau)}{\partial \tau} + \hat{R} \left(\frac{1}{R^2} + \frac{1}{Rc} \frac{\partial}{\partial \tau} \right) \frac{1}{\varepsilon} \int_{-\infty}^{\tau} \nabla \cdot \overline{J}_s \, d\tau' \right]_{\tau = t - R/c} ds' \tag{2.15a}$$

$$\overline{J}_s(\overline{x}, t) = T\,\hat{n} \times \overline{H}_{inc}(\overline{x}, t) + \frac{T}{4\pi}\,\hat{n} \times \int_S \left[\left(\frac{1}{R^2} + \frac{1}{Rc}\frac{\partial}{\partial \tau} \right) \cdot \overline{J}_s(\overline{x}', \tau) \times \hat{R} \right]_{\tau = t - R/c} ds'. \quad (2.15b)$$

Equations (2.15) are of the proper form to solve for the equivalent currents on the scattering surface. These results may then be used in Equation (2.12) to calculate the electric and magnetic field intensities anywhere in space.

III. THEORETICAL DEVELOPMENT FOR BODIES OF ROTATION WITH NEAR-AXIS INCIDENCE

Several different types of scattering bodies have been investigated in the past. Of these, the half-plane, cylinder, thin wire, and sphere can all be classified as bodies of rotation, the half-plane being a special case.

In this chapter we will develop the theory for bodies of rotation restricted to the case of near-axis incidence, in other words, incident fields that appear almost uniform in the surface cross-section. The development will be done using the electric field integral equation (EFIE) since the magnetic field integral equation is not appropriate for such surfaces as hollow cylinders with thin walls. In addition, we will analytically extend the theory for the bodies of rotation to obtain the integral equation for the half-plane.

3.1 Formulation of the Problem

In this section we decompose the electric field integral equation, (2.15a), into its components which are more tractable for actual calculation. To this end, first define time in light meters (i.e. $ct \to t$) for ease of computation. We may then write Equation (2.15a) as

$$\hat{n} \times \overline{E}_{inc}(\overline{x}, t) = \frac{Z_o}{4\pi} \hat{n} \times \int_S \left[\frac{1}{R} \frac{\partial \overline{J}_s(\overline{x}', \tau)}{\partial \tau} + \hat{R} \left(\frac{1}{R^2} + \frac{1}{R} \frac{\partial}{\partial \tau} \right) \int_{-\infty}^{\tau} \nabla \cdot \overline{J}_s(x', \tau')d\tau' \right]_{\tau = t-R} ds' \quad . \tag{3.1}$$

Since $\overline{E}_{inc}$ is a causal function, it is incident upon the scatterer only after some specific time t_o before which the surface currents must be zero on the scatterer. The problem thus turns out to be an initial value problem and, due to the retarded potential effect of the fields, it is a Volterra integral equation of the first kind.

The geometry under consideration is shown in Figure 2. We define the direction of the incident field to be (ϕ_i, θ_i) with the electric field contained entirely in the x-z plane without loss of generality. The surface coordinates are defined by

$$\hat{a}_\phi = -\sin\phi\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{a}_n = \cos\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) + \sin\theta\hat{z} = \hat{n}$$

$$\hat{a}_p = -\sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) + \cos\theta\hat{z} \quad . \tag{3.2}$$

The incident electric field may be written

$$\overline{E}_{inc}(\overline{x}, t) = \frac{\cos\theta_i\,\hat{x} - \cos\phi_i \sin\theta_i\,\hat{z}}{\sqrt{\cos^2\theta_i + \cos^2\phi_i\,\sin^2\theta_i}}$$

$$\cdot\ E_{inc}(t - x\cos\phi_i\,\sin\theta_i - y\sin\phi_i\,\sin\theta_i - z\cos\theta_i).$$

To specify near-axis incidence, we require that $2r_{max}\sin\theta_i$ be such that the incident field spatial variation be small over this distance where r_{max} is the maximum radius of the scatterer. In such cases, $\overline{E}_{inc}$ can be approximated by

$$\overline{E}_{inc}(\overline{x}, t) \simeq \frac{\cos\theta_i\,\hat{x} - \cos\phi_i\,\sin\theta_i\,\hat{z}}{\sqrt{\cos^2\theta_i + \cos^2\phi_i \sin^2\theta_i}}\,E_{inc}(t - z\cos\theta_i) \tag{3.3a}$$

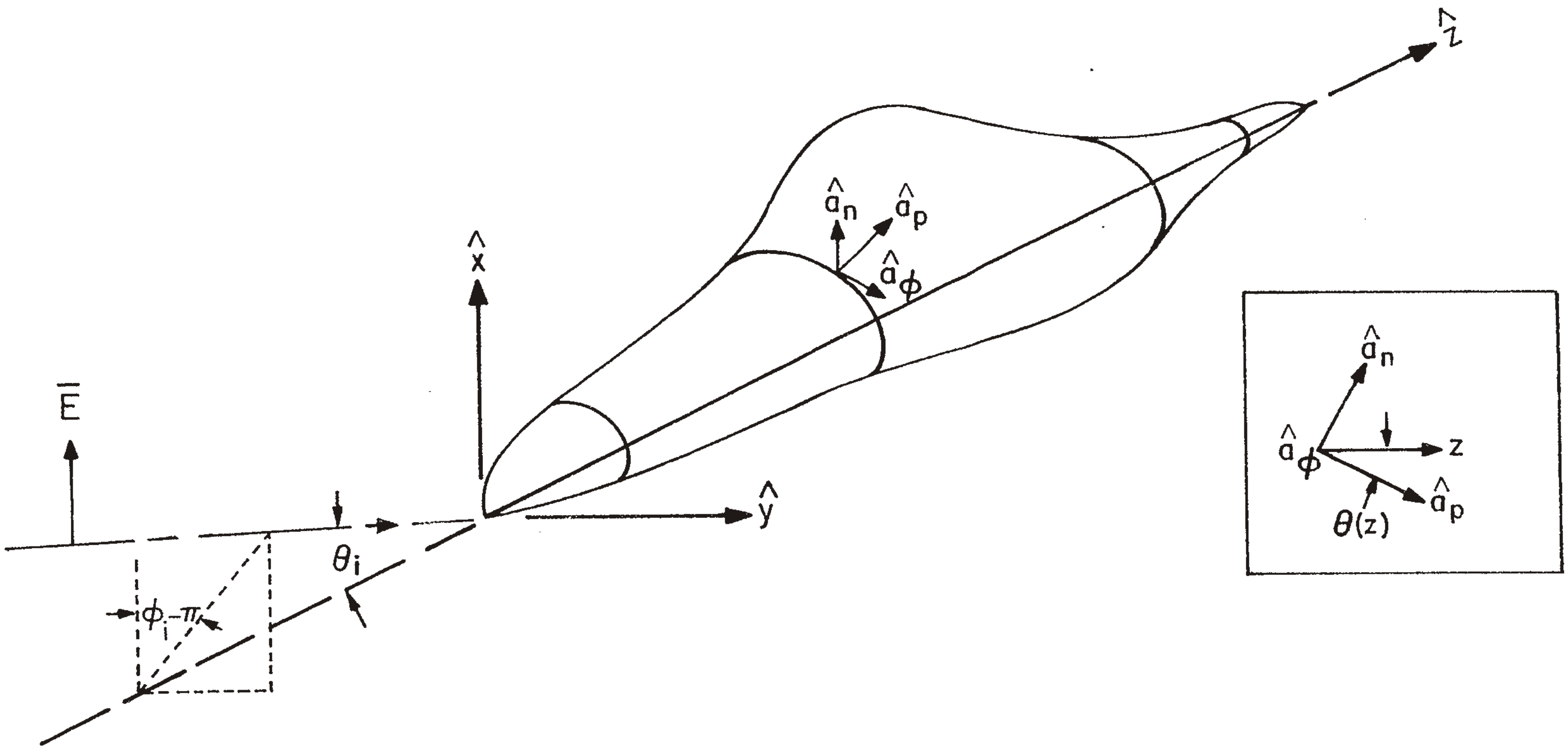

Figure 2. Geometry of the body of rotation and the related coordinate systems.

and

$$\hat{n} \times E_{inc}(\bar{x}, t) \simeq [(\cos\theta_i \sin\theta \cos\phi + \cos\phi_i \sin\theta_i \cos\theta)\,\hat{a}_\phi$$

$$-\cos\theta_i \sin\phi\,\hat{a}_p]\,\frac{E_{inc}(t - z\cos\theta_i)}{\sqrt{\cos^2\theta_i + \cos^2\phi_i \sin^2\theta_i}} \quad . \tag{3.3b}$$

The surface current is now decomposed into two components

$$\bar{J}_{s'} = J_{\phi'}\,\hat{a}_{\phi'} + J_{p'}\,\hat{a}_{p'} \tag{3.4}$$

and

$$\hat{n} \times \bar{J}_{s'} = [J_{\phi'} \sin\theta \sin(\phi - \phi') - J_{p'} (\cos\theta \cos\theta' + \sin\theta$$

$$\cdot \sin\theta' \cos(\phi - \phi'))]\,\hat{a}_\phi + [J_{\phi'} \cos(\phi - \phi')$$

$$+ J_p' \sin\theta' \sin(\phi - \phi')]\,\hat{a}_p \ .$$

Other terms which appear in Equation (3.1) can be written as

$$\nabla' \cdot \bar{J}_{s'} = -\frac{\sin\theta'}{\rho(z')} J_{p'} + \frac{1}{\rho(z')}\frac{\partial J_{\phi'}}{\partial \phi} + \frac{\partial J_{p'}}{\partial p} \tag{3.5a}$$

$$\hat{n} \times (\bar{x} - \bar{x}') = [\rho(z)\sin\theta - \rho(z')\sin\theta\cos(\phi - \phi') - \cos\theta(z - z')]$$

$$\cdot\,\hat{a}_\phi + \rho(z')\sin(\phi - \phi')\,\hat{a}_p \tag{3.5b}$$

and

$$R = |\bar{x} - \bar{x}'| = [(z - z')^2 + \rho(z)^2 + \rho(z')^2 - 2\rho(z)\,\rho(z')\cos(\phi - \phi')]^{1/2}. \tag{3.5c}$$

At this stage, let us substitute (3.4) and (3.5) into Equation (3.1) and separate the resultant equation into components. We now get, for

$\hat{a}_\phi$ and $\hat{a}_p$ components, respectively,

$$\frac{4\pi\, E_{inc}(t - z\cos\theta_i)}{Z_o\sqrt{\cos^2\theta_i + \cos^2\phi_i \sin^2\theta_i}} (\cos\theta_i \sin\theta \cos\phi + \cos\phi_i \sin\theta_i \cos\theta) =$$

$$\int_S \Bigg\{ (\rho(z)\sin\theta - \rho(z')\sin\theta\cos(\phi-\phi') - \cos\theta\,(z-z'))$$

$$\cdot\left(\frac{1}{R^3} + \frac{1}{R^2}\frac{\partial}{\partial\tau}\right) \int_{-\infty}^{\tau} \left(-\frac{\sin\theta'}{\rho(z')} J_{p'} + \frac{1}{\rho(z')}\cdot\frac{\partial J_{\phi'}}{\partial\phi} + \frac{\partial J_{p'}}{\partial p}\right) d\tau'$$

$$+ \frac{1}{R}\left[\sin\theta\sin(\phi-\phi')\frac{\partial J_{\phi'}}{\partial\tau} - (\cos\theta\cos\theta' + \sin\theta\sin\theta'\right.$$

$$\left.\left.\cdot\ \cos(\phi-\phi'))\frac{\partial J_{p'}}{\partial\tau}\right]\Bigg\}\right|_{\tau=t-R} ds' \qquad (3.6a)$$

$$\frac{4\pi\, E_{inc}(t - z\cos\theta_i)}{Z_o\sqrt{\cos^2\theta_i + \cos^2\phi_i \sin^2\theta_i}} \cos\theta_i \sin\phi = -\int_S \Bigg\{\rho(z')\sin(\phi-\phi')$$

$$\cdot\left(\frac{1}{R^3} + \frac{1}{R^2}\frac{\partial}{\partial\tau}\right) \int_{-\infty}^{\tau} \left(-\frac{\sin\theta'}{\rho(z')} J_{p'} + \frac{1}{\rho(z')}\frac{\partial J_{\phi'}}{\partial\phi} + \frac{\partial J_{p'}}{\partial p}\right) d\tau'$$

$$+ \frac{1}{R}\cdot\left[\frac{\partial J_{\phi'}}{\partial\tau}\cos(\phi-\phi') + \frac{\partial J_{p'}}{\partial\tau}\sin\theta'\sin(\phi-\phi')\right]\Bigg\}\Bigg|_{\tau=t-R} ds'. \qquad (3.5b)$$

It is easily seen from Equation (3.3) that the projection of the incident fields onto the surface varies in ϕ as a constant and first-order trigonometric functions. Each component of the surface currents must also vary in the same manner since both the currents and fields rotate identically as the incident field is rotated. Writing the currents in terms of a constant, cos ϕ, and sin ϕ, we have

$$
\begin{aligned}
J_{\phi'} &= J^o_{\phi'} + J^c_{\phi'} \cos \phi' + J^s_{\phi'} \sin \phi' \\
J_{p'} &= J^o_{p'} + J^c_{p'} \cos \phi' + J^s_{p'} \sin \phi' \; .
\end{aligned}
\tag{3.7}
$$

The next step is to substitute Equation (3.7) for $J_{\phi'}$ and $J_{p'}$, make the change of variables $\alpha = \phi' - \phi$, and separate the terms cos ϕ, sin ϕ, and constant in ϕ. Three of the resulting six equations are independent of the incident field and are satisfied by J^c_{ϕ}, J^s_{p}, and J^o_{ϕ} being identically zero. The remaining three non-trivial equations are independent of the first three and may be written

$$
\frac{4\pi\, E_{inc}(t - z\cos\theta_i)}{Z_o \sqrt{\cos^2\theta_i + \cos^2\phi_i \sin^2\theta_i}} \cos\theta_i \sin\theta = \oint_S \Bigg\{ (\rho(z)\sin\theta
$$

$$
-\rho(z')\sin\theta\cos\alpha - (z - z')\cos\theta)\cos\alpha
$$

$$
\cdot \left(\frac{1}{R^3} + \frac{1}{R^2}\frac{\partial}{\partial\tau}\right) \int_{-\infty}^{\tau} \left[\frac{J^s_{\phi'}}{\rho(z')} + \left(-\frac{\sin\theta'}{\rho(z')} + \frac{\partial}{\partial p} \right) J^c_{p'} \right] d\tau'
$$

$$
-\frac{1}{R}\left[\sin\theta\sin^2\alpha\,\frac{\partial J^s_{\phi'}}{\partial\tau} + \left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\alpha\right)\right.
$$

$$
\left.\left. \cdot \cos\alpha\,\frac{\partial J^c_{p'}}{\partial\tau}\right]\Bigg\}_{\tau=t-R} ds' \tag{3.8a}
$$

$$\frac{4\pi\, E_{inc}(t - z\cos\theta_i)}{Z_o\sqrt{\cos^2\theta_i + \cos^2\phi_i \sin^2\theta_i}} \cos\theta_i = \int_S \Bigg\{ -\rho' \sin^2\alpha$$

$$\cdot\left(\frac{1}{R^3} + \frac{1}{R^2}\frac{\partial}{\partial\tau}\right)\int_{-\infty}^{\tau}\left[\frac{J_{\phi'}^{s}}{\rho(z')} + \left(-\frac{\sin\theta'}{\rho(z')} + \frac{\partial}{\partial p}\right) J_{p'}^{c}\right] - \frac{1}{R}\Bigg[\cos^2\alpha$$

$$\cdot\frac{\partial J_{\phi'}^{s}}{\partial\tau} + \sin\theta' \sin^2\alpha \,\frac{\partial J_{p'}^{c}}{\partial\tau}\Bigg]\Bigg\}_{\tau=t-R} ds' \qquad (3.8b)$$

$$\frac{4\pi\, E_{inc}(t-z\cos\theta_i)}{Z_o\sqrt{\cos^2\theta_i + \cos^2\phi_i \sin^2\phi_i}} \cos\theta \sin\theta_i \cos\phi_i = \int_S \Bigg\{ (\rho(z) \sin\theta$$

$$-\rho(z')\sin\theta\cos\alpha - (z - z')\cos\theta)\cdot\left(\frac{1}{R^3} + \frac{1}{R^2}\frac{\partial}{\partial\tau}\right)\int_{-\infty}^{\tau}$$

$$\left(-\frac{\sin\theta'}{\rho(z')} + \frac{\partial}{\partial p}\right) J_{p'}^{o}\, d\tau' - \frac{(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\alpha)}{R}$$

$$\cdot\frac{\partial J_p^o}{\partial\tau}\Bigg\}_{\tau=t-R} ds' \,. \qquad (3.8c)$$

From the solutions of these three equations, (3.8), the total surface current is determined by

$$\overline{J}_s = (J_\phi^s \sin\phi)\,\hat{a}_\phi + (J_p^o + J_p^c \cos\phi)\,\hat{a}_p \,.$$

It should be noted that Equation (3.8c) for J_p^o is uncoupled from (3.8a) and (3.8b) for J_p^c and J_ϕ^s. As a result, J_p^o can be solved for independently; this is useful for the thin-wire problem.

3.2 Semi-Infinite Cylinder and Analytical Extension to the Half-Plane Problem

In this section, the EFIE for the general bodies of rotation derived in the previous section is first specialized to the case in which the scatterer is a perfectly conducting semi-infinite hollow cylinder. Next, the integral equation for the half-plane problem will be derived by taking the limit of the semi-infinite hollow cylinder case in an appropriate manner. The geometry for the hollow cylinder is shown in Figure 3. To obtain the equations for the semi-infinite hollow cylinder, we let θ and θ' be equal to zero and $\rho(z) = \rho$ = constant in Equations (3.8). Rewriting Equations (3.8) with these conditions, we have

$$0 = \int_S \left\{ (z - z') \cos\alpha \left(\frac{1}{R^3} + \frac{1}{R^2} \frac{\partial}{\partial \tau} \right) \int_{-\infty}^{\tau} \left(\frac{J_{\phi'}^{s}}{\rho} + \frac{\partial J_{z'}^{c}}{\partial z} \right) d\tau' + \frac{\cos\alpha}{R} \cdot \frac{\partial J_{z'}^{c}}{\partial \tau} \right\}_{\tau = t-R} ds' \tag{3.9a}$$

$$\frac{4\pi E_{inc}(t - z\cos\theta_i)}{Z_o\sqrt{\cos^2\theta_i + \cos^2\phi_i \sin^2\theta_i}} \cos\theta_i = -\int_S \left\{ \rho \sin^2\alpha \cdot \left(\frac{1}{R^3} + \frac{1}{R^2}\frac{\partial}{\partial\tau} \right) \cdot \int_{-\infty}^{\tau} \left(\frac{J_{\phi'}^{s}}{\rho} + \frac{\partial J_{z'}^{c}}{\partial z} \right) d\tau' + \frac{\cos^2\alpha}{R} \frac{\partial J_{\phi'}^{s}}{\partial \tau} \right\}_{\tau=t-R} \cdot ds' \tag{3.9b}$$

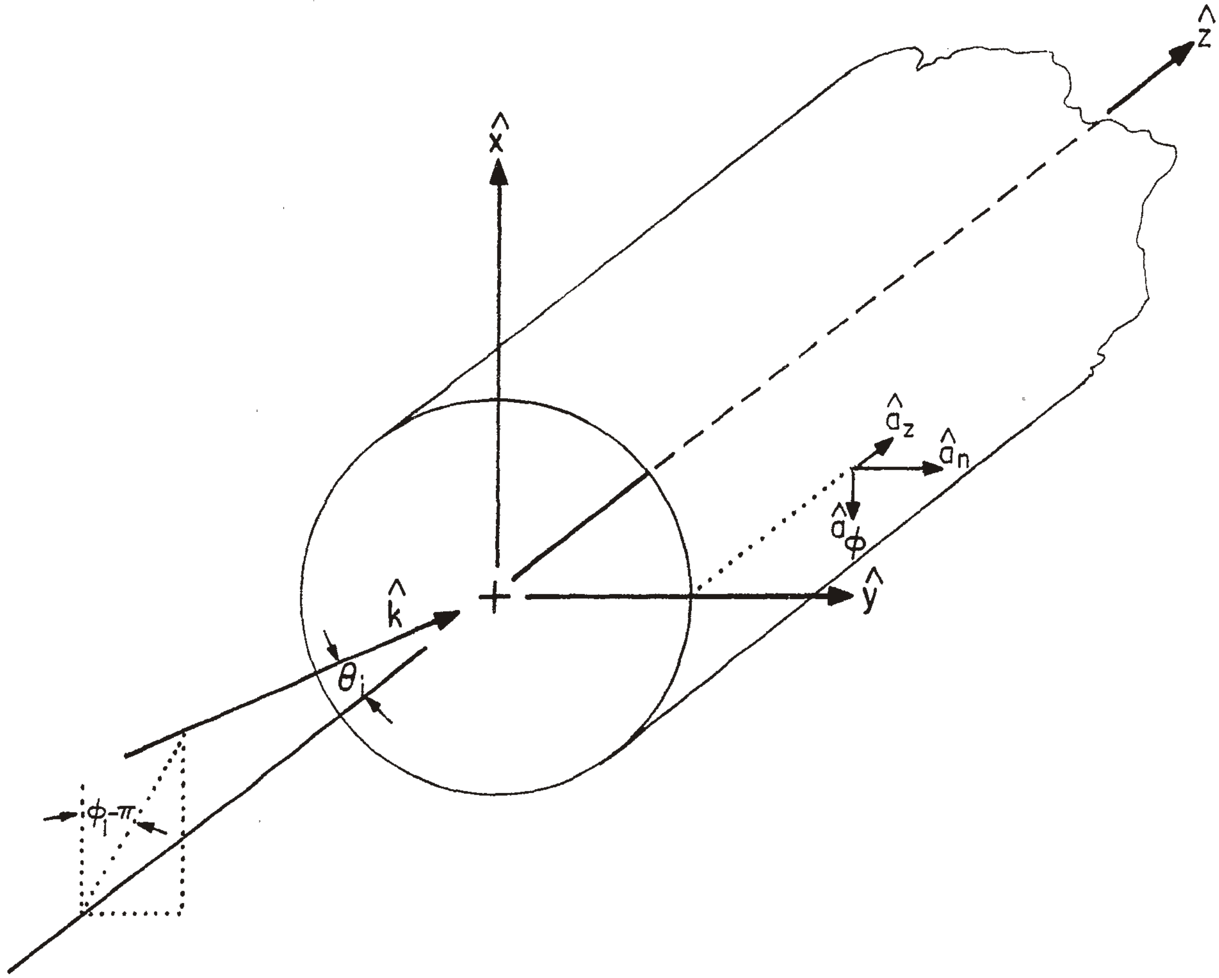

Figure 3. Semi-infinite hollow cylinder geometry.

$$\frac{4\pi\, E_{inc}(t - z\cos\theta_i)}{Z_o\sqrt{\cos^2\theta_i + \cos^2\phi_i \sin^2\theta_i}} \sin\theta_i \cos\phi_i = \int_S \left\{ -(z - z')\left(\frac{1}{R^3} + \frac{1}{R^2}\frac{\partial}{\partial\tau} \right) \int_{-\infty}^{\tau} \frac{\partial J^o_{z'}}{\partial z}\, d\tau' - \frac{1}{R}\frac{\partial J^o_{z'}}{\partial\tau} \right\}_{\tau=t-R} ds'. \qquad (3.9c)$$

The solution to these equations gives the induced surface current on the semi-infinite hollow cylinder.

Now let us go to the half-plane problem. To this end, we must first take a limit of Equations (3.9) as the radius, ρ, of the cylinder goes to infinity. As the limit is taken, the surface of integration becomes a half-plane which is mathematically closed at infinity. This closure provides no contribution to the integral for times less than infinity due to the retarded potential form of the problem. The half-plane geometry is shown in Figure 4.

To obtain the limit, we first expand $\sin\alpha$ and $\cos\alpha$ in Taylor series about $\alpha = 0$. Using one term of each series* and defining $x = \rho\alpha$, we rewrite Equations (3.9)

$$0 = \lim_{\rho\to\infty} \int_S \left\{ (z - z')\left(\frac{1}{R^3} + \frac{1}{R^2}\frac{\partial}{\partial\tau} \right) \int_{-\infty}^{\tau} \left(\frac{J^s_{\phi'}}{\rho} + \frac{\partial J^c_{z'}}{\partial z} \right) d\tau' + \frac{1}{R} \cdot \frac{\partial J^c_{z'}}{\partial\tau} \right\}_{\tau=t-R} dx'dz' \qquad (3.10a)$$

$$\frac{4\pi\, E_{inc}(t - z\cos\theta_i)}{Z_o\sqrt{\cos^2\theta_i + \cos^2\phi_i \sin^2\theta_i}} \cos\theta_i = -\lim_{\rho\to\infty} \int_S \frac{x^2}{\rho}\left(\frac{1}{R^3} + \frac{1}{R^2}\right.$$

* We use two terms for $\cos\alpha$ in R.

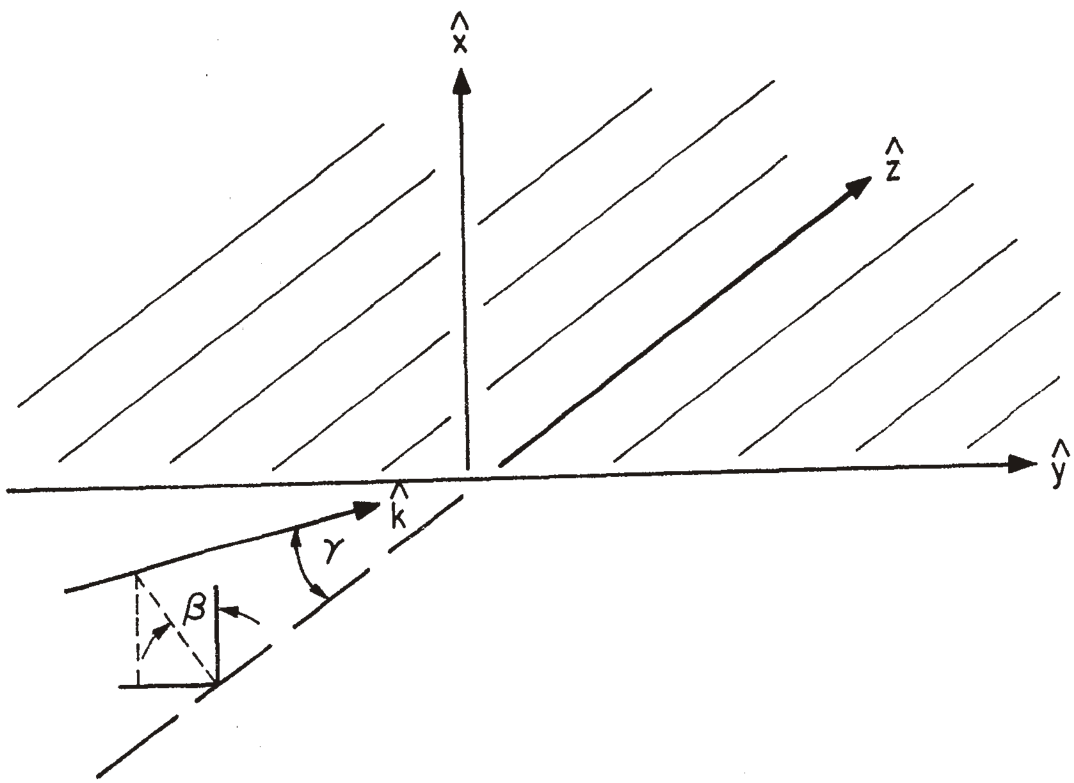

Figure 4. The half-plane geometry with the direction of incidence, $\hat{k}$, defined by (β, α)

$$\cdot \frac{\partial}{\partial \tau}\Bigg) \int_{-\infty}^{\tau} \left(\frac{J_{\phi'}^{s}}{\rho} + \frac{\partial J_{z'}^{c}}{\partial z} \right) d\tau' + \frac{1}{R} \frac{\partial J_{\phi'}^{s}}{\partial \tau} \Bigg\}_{\tau=t-R} dx'\, dz'$$

(3.10b)

$$\frac{4\pi\, E_{inc}(t - z \cos \theta_i)}{Z_o \sqrt{\cos^2 \theta_i + \cos^2 \phi_i \sin^2 \theta_i}} \sin \theta_i \cos \phi_i = - \lim_{\rho \to \infty} ⨍_S \Bigg\{ (z - z')$$

$$\cdot \left(\frac{1}{R^3} + \frac{1}{R^2} \frac{\partial}{\partial \tau} \right) \int_{-\infty}^{\tau} \frac{\partial J_{z'}^{o}}{\partial z} d\tau' + \frac{1}{R} \frac{\partial J_{z'}^{o}}{\partial \tau} \Bigg\}_{\tau=t-R} dx'\, dz'$$

(3.10c)

where $R = [(x - x')^2 + (z - z')^2]^{1/2}$.

In the limit as ρ goes to infinity Equations (3.10a) and (3.10b) become uncoupled, the former requiring $J_{z'}^{c}$ to be identically zero. At the same time, Equations (3.10b) and (3.10c) may be written

$$\frac{4\pi\, E_{inc}(t - z \cos \theta_i)}{Z_o \sqrt{\cos^2 \theta_i + \cos^2 \phi_i \sin^2 \theta_i}} \cos \theta_i = - ⨍_S \left[\frac{1}{R} \frac{\partial J_{\phi'}^{s}}{\partial \tau} \right]_{\tau=t-R} dx'\, dz'$$

(3.11a)

$$\frac{4\pi\, E_{inc}(t - z \cos \theta_i)}{Z_o \sqrt{\cos^2 \theta_i + \cos^2 \phi_i \sin^2 \theta_i}} \sin \theta_i \cos \phi_i = - \int_S \Bigg\{ (z - z')$$

$$\cdot \left(\frac{1}{R^3} + \frac{1}{R^2} \frac{\partial}{\partial \tau} \right) \int_{-\infty}^{\tau} \frac{\partial J_{z'}^{o}}{\partial z} d\tau' + \frac{1}{R} \frac{\partial J_{z'}^{o}}{\partial \tau} \Bigg\}_{\tau=t-R} dx'\, dz'.$$

(3.11b)

It should be noted that Equations (3.11) are not valid for incident angles β not equal to 0 or π (see Figure 4). Furthermore, it is sufficient to restrict ourselves to edge-on incidence in order to compare our half-plane solution to the exact solutions available.[4]

By restricting the half-plane problem to edge-on incidence, $\theta_i = 0$, Equation (3.11b) becomes trivial and requires $J_{z'}^{o}$ to be zero. Equation (3.11a) may be written

$$\frac{4\pi}{Z_o} E_{inc}(t - z) = - \int_S \left[\frac{1}{R} \frac{\partial J_{\phi'}^{s}}{\partial \tau} \right]_{\tau=t-R} dx' \, dz' \tag{3.12}$$

and

$$\overline{J}_s = J_{\phi'}^{s} \sin \phi \, \hat{a}_\phi \, . \tag{3.13}$$

In choosing ϕ, we note that there are two basic types of half-plane problems referred to as the transverse electric (TE) and transverse magnetic (TM) cases. In the TE case the E-field is parallel to the edge. On the other hand, it is perpendicular to the edge in the TM case. From the geometry for the bodies of rotation as shown in Figure 2, it can be seen that the TE case corresponds to $\phi = 90^o$ and the TM case corresponds to $\phi = 0^o$.

From Equation (3.13) we see that the currents on the half-plane are zero for the TM case, as would be expected. For the TE case we may rewrite Equation (3.12) as

$$\frac{4\pi}{Z_o} E_{inc}(t - z) = \int_S \left[\frac{1}{R} \frac{\partial J_y}{\partial \tau} \right]_{\tau=t-R} dy' \, dz' \tag{3.14}$$

where J_y replaces $-J_{\phi'}^{s} \sin(\pi/2)$ and the incident E-field is directed in the $\hat{y}$-direction.

IV. NUMERICAL SOLUTION OF THE ELECTRIC-FIELD INTEGRAL EQUATION

In this chapter we present the numerical results for a half-plane and a large semi-infinite hollow cylinder. Before going into the specific problems, it is necessary to first investigate the general numerical procedure for solving time-domain integral equations. Then, the numerical procedure for the half-plane will be investigated and the numerical solution will be compared to the known exact solution. Finally, we will investigate the large semi-infinite hollow cylinder problem and compare its solution for early time to the exact solution for the half-plane.

4.1 General Numerical Solution

In this section we proceed with the general, electric-field integral equation (EFIE) given by

$$\hat{n} \times \overline{E}_{inc}(\overline{x}, t) = \frac{Z_o}{4\pi} \hat{n} \times \int_S \left\{ \frac{1}{R} \frac{\partial}{\partial \tau} \overline{J}_s(\overline{x}', \tau) + (\overline{x} - \overline{x}') \left(\frac{1}{R^3} + \frac{1}{R^2} \frac{\partial}{\partial \tau} \right) \int_{-\infty}^{\tau} (\nabla \cdot \overline{J}_s (\overline{x}', \tau')) \, d\tau' \right\}_{\tau = t - R} ds'. \tag{4.1}$$

The solution of Equation (4.1) depends on the retardation in time of the fields at a point $\overline{x}$ due to currents at $\overline{x}'$. This retardation requires that the surface currents at $(\overline{x}, t)$ on the scatterer depend only on the incident field at $(\overline{x}, t)$and the induced currents at $\overline{x}'$ prior to the time $\tau = t - |\overline{x} - \overline{x}'|$.* This effect may be understood more fully by considering the two-dimensional space-time cone in Figure 5 where only the currents in the shaded region contribute to the current at $(\overline{x}, t)$. As a result,

*The units of t are in light meters.

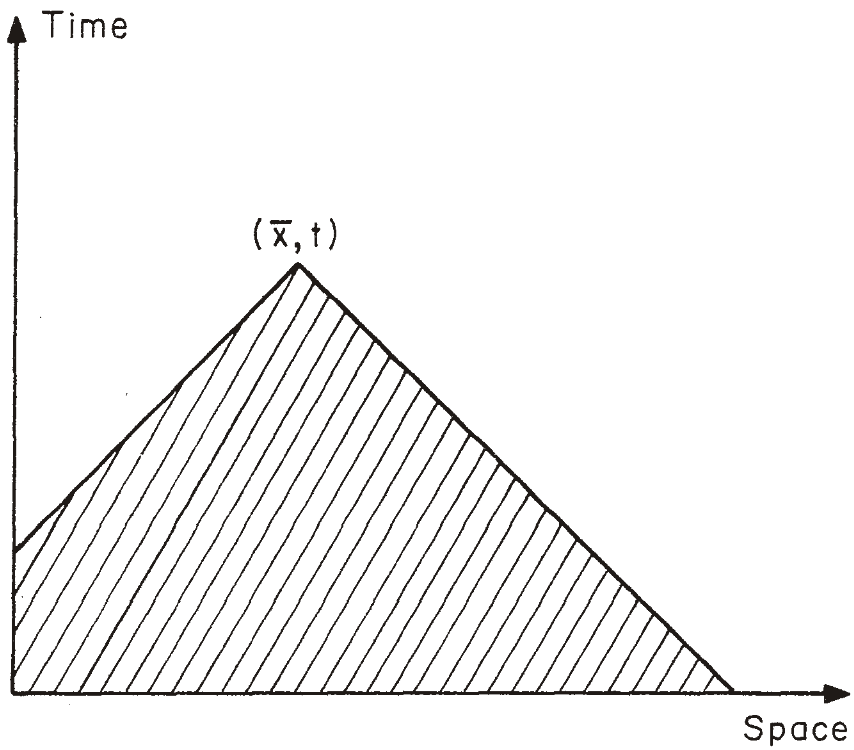

Figure 5. Time-space cone for the currents on a scatterer.

the currents $\overline{J}_s$ on the scatterer may be determined by stepping in time instead of inverting a matrix as is required for the frequency-domain solution.

In choosing the step size in time and space, there are two basic criteria to follow. The first criterion is that the step size in time must be less than or equal to that in space, i.e.,

$$\Delta t \leq \Delta x \tag{4.2}$$

such that the current at a specific time and location on the scatterer surface does not contribute to the current at another location at the same time. The second criterion is that both step sizes be sufficiently small to adequately represent the incident field in a discretized form.

The first step in the solution of Equation (4.1) is the application of the method of moments.* To this end, we choose the Dirac delta weighting function

$$W_{ij} = \delta(\overline{x} - \overline{x}_i)\, \delta(t - t_j) \tag{4.3}$$

and obtain from Equation (4.1)

$$\hat{n} \times \overline{E}_{inc}(\overline{x}_i, t_j) = \frac{Z_o}{4\pi}\, \hat{n} \times \oint_S \left\{ \frac{1}{|\overline{x}_i - \overline{x}'|} \frac{\partial}{\partial \tau} \overline{J}_s(\overline{x}', \tau) + \left(\frac{\overline{x}_i - \overline{x}'}{|\overline{x}_i - \overline{x}'|^3} + \frac{\overline{x}_i - \overline{x}'}{|\overline{x}_i - \overline{x}'|^2} \frac{\partial}{\partial \tau} \right) \int_{-\infty}^{\tau} (\nabla \cdot \overline{J}_s(\overline{x}', \tau'))\, d\tau' \right\}_{\tau = t_j - |\overline{x}_i - \overline{x}'|} ds' \tag{4.4}$$

*A good discussion of the method of moments is found in Poggio and Miller.[1]

where

$$i = 1, 2, \cdot \cdot \cdot, N_s$$

$$j = 1, 2, \cdot \cdot \cdot, N_t \ .$$

Equation (4.4) is well-suited to the application of the method of subsectional collocation. In this method the currents are approximated on each subsection of the space-time coordinates by using a given set of basis functions. Hence, letting Δx_m and Δt_n represent the sub-sections about $\bar{x}_m$ and t_n, respectively, we may write

$$\bar{J}_s(\bar{x}, t) = \sum_{m=1}^{N_s} \sum_{n=1}^{N_t} \bar{J}_{mn} (\bar{x}, t) S_m T_n \tag{4.5}$$

where

$$S_m = \begin{cases} 1, & \bar{x} \text{ in } \Delta x_m \\ 0, & \text{elsewhere} \end{cases} \quad ; \quad T_n = \begin{cases} 1, & t \text{ in } \Delta t_n \\ 0, & \text{elsewhere} \end{cases} .$$

The coefficients $\bar{J}_{mn} (\bar{x}, t)$ can be expressed in terms of the appropriate basis functions with unknown coefficients at the (m, n) subsections. For bodies of rotation it is convenient to use basis functions that are composed of trigonometric functions in ϕ and polynomials in time and the longitudinal direction z.

Having expanded the currents in a set of basis functions, we may solve for the unknown coefficients in $\bar{J}_{mn}$ by substitution of Equation (4.5) into (4.4). In many cases, one may assume that $\bar{J}_s$ and $\nabla \cdot \bar{J}_s$ are constants in space over each subsection with a value determined by the central point of the subsection. To complete the solution of Equation (4.4) in such cases, one only needs to evaluate the currents up to the corresponding time of the central point, integrate the kernels over each subsection, and apply the time-stepping procedure. If we consider the general electric-field integral equation, we see that a solution may theoretically be obtained for any causal incident field. The solution to an incident field with impulse (or Dirac delta) variation is of considerable interest. Such a solution is referred to as an impulse response. Since our problem is linear, it can be shown that the solution due to any other incident field is simply the convolution of the impulse response with the given incident field variation. Therefore, it is desirable to obtain the impulse response.

To this end, one defines an approximate impulse function as the Gaussian impulse which is given by

$$\delta_g(t) = \frac{g}{\sqrt{\pi}} \; e^{-g^2 t^2}$$

where g is referred to as the pulse half-width.* Hence, the incident E-field propagating in the z-direction may be expressed as

$$\overline{E}_{inc}(\overline{x}, t) = \hat{x} \frac{g}{\sqrt{\pi}} \; \exp\left[-g^2 (t - z/c)^2\right] \quad . \tag{4.6}$$

Since the amplitude of the Gaussian pulse decreases rapidly to zero away from its center, one may truncate it to solve the EFIE without a noticeable loss of accuracy. The resultant solution is called the approximate impulse response.

4.2 The Half-Plane

Let us consider the half-plane TE problem with edge-on incidence. This problem is described by Equation (3.14) which we rewrite for convenience

*Time is defined in seconds here.

$$\frac{4\pi}{Z_o} E_{inc}(t - z) = \int_S \left[\frac{1}{R} \frac{\partial J_y}{\partial \tau} \right]_{\tau=t-R} dy' \, dx' \quad . \tag{4.7}$$

At any specific time, the current J_y is independent of y. Hence, we may rewrite the subsectional representation of $\bar{J}_s$ given in Equation (4.5) as

$$J_y(z, t) = \sum_{m=1}^{N_s} \sum_{n=1}^{N_t} J_{mn}(z, t) Z_m T_n \tag{4.8}$$

where Z_m is unity over a surface strip Δz_m about z_m and zero elsewhere.

Applying the method of moments to Equation (4.7) and substituting Equation (4.8) for the currents, one obtains

$$\frac{4\pi}{Z_o} E_{inc}(t_i - z_j) = \int_S \left[\frac{1}{|\bar{x}_j - \bar{x}'|} \sum_{m=1}^{N_s} \sum_{n=1}^{N_t} \left(\frac{\partial}{\partial \tau} J_{mn}(z, \tau) \right) \right.$$

$$\left. \cdot Z_m T_n \right]_{\tau=t_i-|\bar{x}_j-\bar{x}'|} dy' \, dz' \quad . \tag{4.9}$$

It should be noted that τ is a function of both y and z, complicating the space integration. Therefore, to simplify the space integration, one divides the surface into squares S_{mp} with centers at (y_p, z_m) and defines the current to be constant throughout each square with a value corresponding to that at the center of the square. Hence, Equation (4.9) may be rewritten

$$\frac{4\pi}{Z_o} E_{inc}(t_i - z_j) = \sum_{m=1}^{N_s} \sum_{p=-N_t}^{N_t} \left(\sum_{n=1}^{N_t} \frac{\partial}{\partial \tau} J_{mn}(z, \tau_{mp}) T_n \right) \tag{4.10}$$

$$\cdot \int_{S_{mp}} \left[\frac{1}{R} \right] dy' \, dz'.$$

One should note that it is sufficient in most numerical work to evaluate the surface integral by rectangular rule except in the region of the singularity where the integration must be performed analytically.

Let us present some of the numerical results. The approximate impulse response for the half-plane was obtained numerically using Equations (4.10) and (4.6). The step sizes, Δt, Δz, and Δy, were chosen to be equal and such that the incident field of Gaussian impulse was sampled three times in the pulse width. The numerical results are shown in Figure 6 along with the exact results for a Dirac delta incident wave as can be derived from the frequency-domain results of Born and Wolf.[4,*] For comparison of the exact solution to the numerical results, two unknown constants z_o and t_o had to be determined, where z_o is the location of the surface edge and t_o is the additional time delay resulting from the Gaussian impulse approximation. The determination has been done by matching the current variation derived from (A.4) in the Appendix A to the numerical data in two different points in time at the first space sample on the structure.

4.3 Large Semi-Infinite Hollow Cylinder

The primary purpose of this section is to verify the EFIE for bodies of rotation in the specific case of a large semi-infinite hollow cylinder. We have shown in Chapter 3 that in the limit, as the radius goes to infinity, this problem becomes that of a half-plane for which the solution was obtained in the previous section. In this section we present the results for the large cylinder case.

The semi-infinite hollow cylinder problem is specified by Equations (3.9). One follows the procedure of Section (4.1) to cast Equations (3.9) into a discretized form for numerical solution as was done for the half-plane. Several approaches have been used to satisfy the edge conditions at the end

*See Appendix A for derivation.

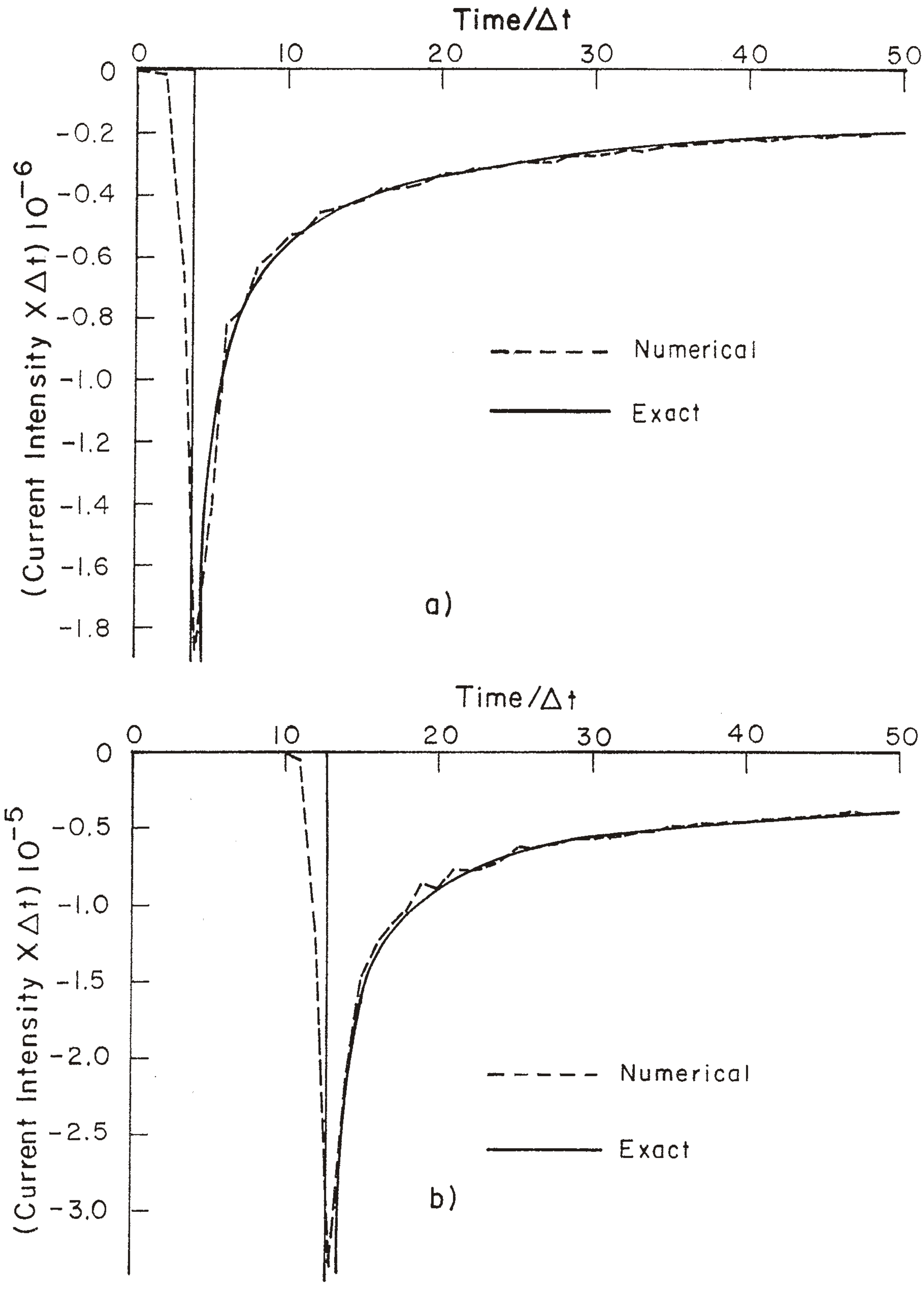

Figure 6. The currents on a half-plane due to an impulse incident field. (a) The current at the first space sample point $(z - z_o)/\Delta z = .284$, $t_o/\Delta t = 3.64$; (b) the currents at the tenth space sample point $(z - z_o)/\Delta z = 9.284$, $t_o/\Delta t = 3.64$.

of the cylinder.* We obtained satisfactory results by allowing z-directed currents to be zero in the edge surface segment and by specifying the corresponding derivative in z to be taken in the forward trapezoidal sense.

The numerical results for the coefficient J_ϕ of the ϕ-directed current are shown in Figure 7 along with the exact half-plane results. The ratio of the radius to the step size used here was 100. The incident field had a Gaussian variation with three samples in the pulse width.

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*Appropriate edge conditions are similar to those for a half-plane for which J_z varies as $\sqrt{z}$ and J_ϕ and $\frac{\partial J_z}{\partial z}$ vary as $1/\sqrt{z}$.

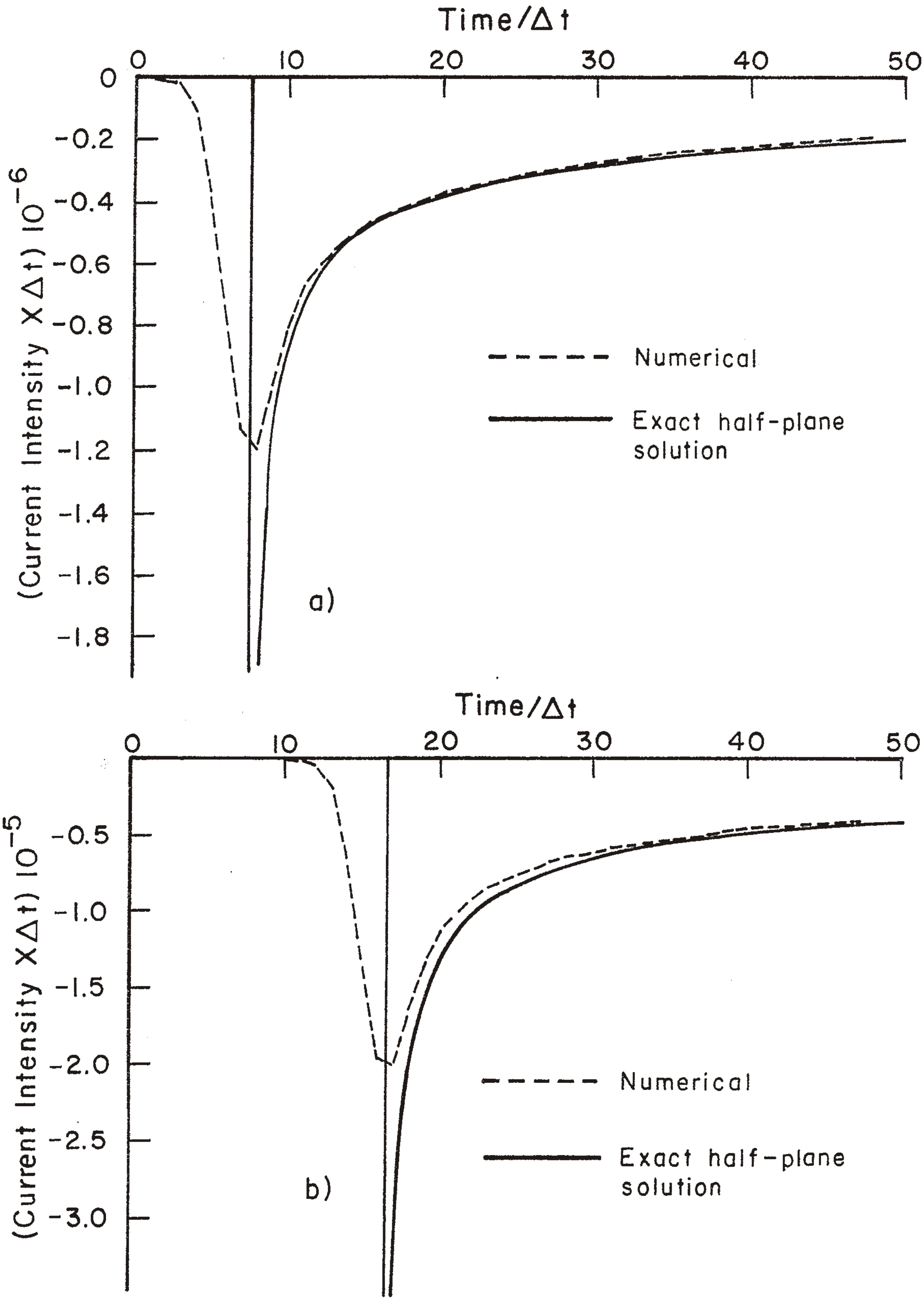

Figure 7. The ϕ-current coefficient on a large, semi-infinite, hollow cylinder due to an impulse incident field. (a) The current coefficient at the first sample point $(z - z_o)/\Delta z = .312$, $t_o/\Delta t = 7.35$; (b) the current coefficient at the tenth sample point $(z - z_o)/\Delta z = 9.312$, $t_o/\Delta t = 7.35$.

V. CONCLUSIONS

In this report we have investigated the integral equation formulation for time-domain scattering by bodies of rotation. Particular attention has been given to the large, semi-infinite, hollow cylinder and half-plane problems.

The integral equation for bodies of rotation was developed using the electric-field integral equation (EFIE) by restricting the incident field to near-axis incidence. Under such a restriction, one may expand the currents in a Fourier series with respect to ϕ having coefficients identically zero for order greater than one.

The time-domain integral equation may be put into digitized form for numerical solution using the method of moments and subsectional collocation. This numerical problem is then solved by stepping in time, thus eliminating the need for a matrix inversion as is required in the corresponding frequency-domain problem.

This development was checked numerically by extending the problem to that of a half-plane by taking an appropriate limit of the semi-infinite, hollow cylinder problem. The results were consistent with the exact solution of the half-plane problem. Early time calculations were also made for a large semi-infinite hollow cylinder and were also consistent with the half-plane solution, as was expected.

Areas of further study include the thin-wire problem and the semi-infinite cylinder with various end caps. There are several other scattering structures that one could investigate using the equations developed in this report. Such structures include the sphere and oblate spheroid.

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3. J. A. Stratton. Electromagnetic Theory. New York: McGraw-Hill, 1941.

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APPENDIX A: DERIVATION OF EXACT HALF-PLANE CURRENTS

Born and Wolf[4] have obtained the exact description of the fields in the frequency domain for scattering by a half-plane* (see Figure 4). The induced current on the half-plane may be obtained from these results by subtracting the tangential magnetic fields on each side of the plane.

For TE polarization with edge-on incidence, the time-harmonic, incident field may be given by

$$\overline{E}_{inc}(\omega) = e^{-i\frac{\omega z}{c}} \hat{y} \quad . \tag{A.1}$$

Thus, the current is given by

$$J_y(z,\ \omega) = H_z(\overline{0},\ z,\ \omega) - H_z(0^+,\ z,\ \omega)$$

$$J_y(z,\ \omega) = \frac{2}{\mu c}\sqrt{\frac{2c}{\pi\omega z}}\ e^{i(\frac{\pi}{4} + \frac{\omega z}{c})} \quad . \tag{A.2}$$

By Fourier transforming the given incident field and the corresponding induced currents, one obtains an incident impulse plane wave and the corresponding induced currents. Using the transform tables such as those of Bateman,[5] one obtains

$$J_y(z,\ t) = \frac{1}{\pi\mu c}\sqrt{\frac{2c}{\pi z}} \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-z/c)}}{\sqrt{-i\omega}}\ d\omega$$

$$= \frac{2}{\pi\mu} \sqrt{\frac{2}{z(ct-z)}}\ \theta(t-z/c) \lim_{\alpha\to\infty} e^{-\alpha(t-z/c)} \quad ,$$

$$\alpha > 0$$

---

*Although Gaussian units are used in Born and Wolf, we use rationalized mks units in this Appendix to be consistent with the body of the report.

$$J_y(z, t) = \frac{2}{\pi\mu} \sqrt{\frac{2}{z(ct-z)}} \quad \theta(t-z/c) \tag{A.3}$$

where $\theta(t)$ is the Heaviside function.

If time is expressed in light meters, the impulse response given in Equation (A.3) becomes

$$J_y(z, t) = \frac{2}{\pi\mu} \sqrt{\frac{2}{z(t-z)}} \quad \theta(t-z) \quad . \tag{A.4}$$

This description of the induced currents on a half-plane is used for comparison with the numerical results in Chapter IV.

APPENDIX B: HALF-PLANE COMPUTER PROGRAM

```
      Program B Plane (Input, Output)
      Dimension AJ(100,100), SE(101)
C          This program was written to determine the currents on a half plane due to an
C     incident field in the plane surface with H perpendicular to the surface.  The
C     solution is in the time-domain using a form of the E-field integral equation.
C
C     This function defines a polynomial approximation to a function derivative with
C     function samples A, B, and C taken at D intervals.
      B2 (A,B,C,D) = (C-A)/2. + (A + C - 2.*B)*D
C     Set the final time sample number.
      NF = 50
C     Set other constants.
      C = 3.0E8
      PI = 3.141592653589
      ZO = 120.0*PI
      ALO = ALOG(1.0 + SQRT(2.0))
C     ALO defines the integral of 1/R over the singular region.
C     Set the currents to zero.
      DO 1 I = 1, NF
      DO 1 J = 1, NF
    1 AJ(I,J) = 0.0
      DO 50 N = 3,NF
      DO 50 M = 3,N
C     Calculate E(N - M). E(N - M) is Gaussian E-field times DT, the time sample spacing.
C     Thus, to obtain the impulse response, one must divide the current values given by
C     DT.  It is assumed that DT = DZ.
C     A is the Gaussian width (in 1/sec) *DT/C.
C     B is the half truncated width.
      A = 1.0
      B = 3.0
      X = A*(N - M)-B
      IF (ABS(X)-B) 2,2,3
    2 E = A/SQRT(PI)*EXP(-X**2)*C
      Go to 4
    3 E = 0.0
    4 Continue
C     Calculate current
      AJ(M,N) = AJ(M,N-1) - PI*E/(ZO*ALO)
      DO 9 M1 = 3,N
C     Define the range of integration in the direction parallel to the edge, +X and -X.
      NPF = N - M1
      IF (M - M1) 5,6,5
    5 N1 = N - IABS(M - M1)
C     Add term at (0,Z)
      AJ(M,N) = AJ(M,N) - (AJ(M1,N1) - AJ(M1, N1 - 1))/(4.0*ALO*IABS(M - M1))
    6 IF (NPF) 9,9,7
    7 DO 85 NP = 1,NPF
C     Add terms at (+X,Z) and (-X,Z) which are equal.  Use the appropriate polynomial
C     approximation.
      A = SQRT(((M - M1)**2 + NP**2)*1.0)
      N1 = N-A+1.0
      IF (N1 - 3) 9,91,91
   91 IF (N - N1)  9,95,93
   95 N1 = N-1
```

```
   93 DTAU = N-A-N1
      AJN1 = (AJ(M1,N1)-AJ(M1,N1-1)*(1.0+DTAU) - (AJ(M1,N1-1)-AJ(M1,N1-2))
     1*DTAU
    8 AJ(M,N) = AJ(M,N) - AJN1/(2.0*ALO*A)
   85 Continue
    9 Continue
   50 Continue
C     Print Current
      Print 10
   10 Format (1H1)
      DO 12 M = 3,NF
   12 Print 11,(AJ(M,N), N = 3,NF)
   11 Format (10X,12(9E12.4/))
      Stop
      End
```