SCATTERING OF ELECTROMAGNETIC RADIATION BY APERTURES; II. OBLIQUE INCIDENCE ON THE SLOTTED PLANE FOR PARALLEL POLARIZATION

BY:

L. F. Libelo
U.S. Naval Ordnance Laboratory

and

J. Bombardt
Harry Diamond Laboratories

ABSTRACT: This report is the second in a series of investigations into the diffraction of electromagnetic radiation by apertures in conducting screens. Herein is presented a technique for obtaining the fields everywhere for plane electromagnetic radiation incident obliquely on a slotted conducting plane. The case studied is that in which the incident electric field is linearly polarized parallel to the slot axis. Numerical solution for the scattering cross-section and the angular distribution for arbitrary incidence direction and for a large range of values of the ratio of slot width to incident wavelength are presented in this work. Comparison with results obtained earlier by others show the newer results to be in excellent agreement.

PHYSICS RESEARCH DEPARTMENT
U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

NOLTR 72-25 15 January 1972

SCATTERING OF ELECTROMAGNETIC RADIATION BY APERTURES;
II. OBLIQUE INCIDENCE ON THE SLOTTED PLANE

The research reported herein was performed jointly at the Naval Ordnance Laboratory, Physics Research Department, and at the USA Harry Diamond Laboratories. This work was partially supported by the Independent Research Program at NOL (Task Number MAT-03L 000/ZR00-001-010) and by the Defense Nuclear Agency at the Harry Diamond Laboratories (HDL Proj: E05E3). This document is for information only.

The authors acknowledge the efforts of Mr. D. Lozier, presently with the National Bureau of Standards in Washington, D. C., for numerical aspects of the work reported herein.

ROBERT WILLIAMSON II

Z. I. SLAWSKY
By direction

TABLE OF CONTENTS

ILLUSTRATIONS

ILLUSTRATIONS (cont'd)

I. THE GENERAL PROBLEM

We shall consider a plane, monochromatic, linearly polarized electromagnetic wave incident at an oblique angle ϕ_0 on an infinite perfectly conducting plane containing an infinite slot. Let the width of this slot be $2\rho_0$. We shall assume the conducting plane screen lies in the XZ-plane and that the remaining space is simply free space. To fix the polarization we assume the incident electric field parallel to the slot axis (i.e. the Z-axis); thus $\vec{E}_i(\vec{r},t) = E_0(\vec{r},t)\,\hat{e}_z$ where $\hat{e}_z$ is a unit vector along the +Z-direction. To denote the direction of incidence of the plane wave, we use the unit vector $\vec{e}_0$. Figure 1 illustrates the geometry and coordinates for the screen and incident radiation.

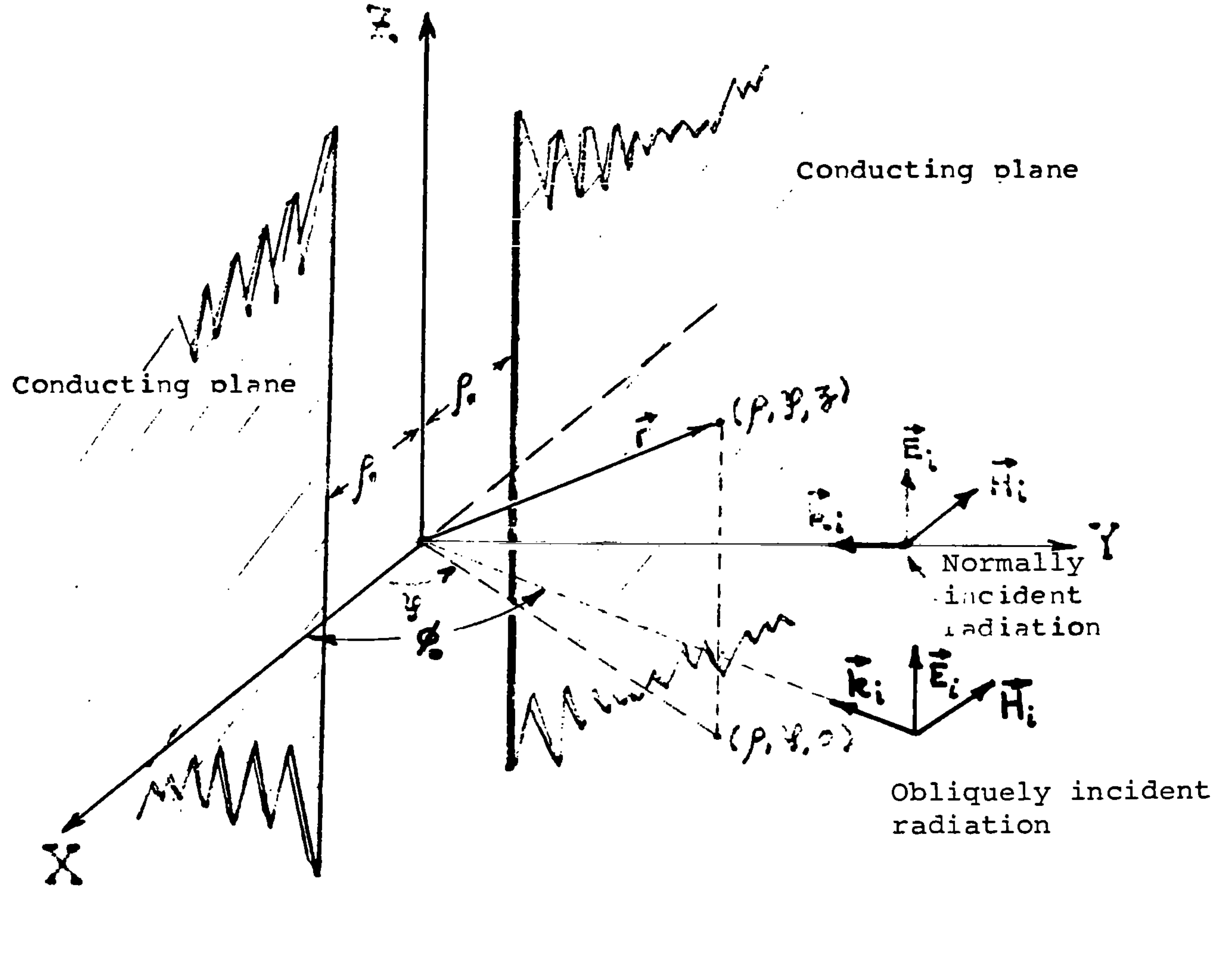

Figure 1. The Slotted Infinite Conducting Plane Showing the Direction of Incidence

Using the constitutive equations

$$\vec{D}\ (\vec{r},t) = \epsilon_o\ \vec{E}\ (\vec{r},t),\ \vec{B}\ (\vec{r},t) = \mu_o\ \vec{H}\ (\vec{r},t) \qquad \text{(I-1)}$$

the Maxwell equations in free space are

$$\vec{\nabla}_r \times \vec{E}\ (\vec{r},t) = 0 \qquad \text{(I-2)}$$

$$\vec{\nabla}_r \cdot \vec{B}\ (\vec{r},t) = 0 \qquad \text{(I-3)}$$

$$\vec{\nabla}_r \times \vec{E}\ (\vec{r},t) = -\frac{\partial \vec{B}\ (\vec{r},t)}{\partial t} \qquad \text{(I-4)}$$

$$\vec{\nabla}_r \times \vec{B}\ (\vec{r},t) = \frac{1}{c^2}\ \frac{\partial \vec{E}\ (\vec{r},t)}{\partial t} \qquad \text{(I-5)}$$

Assuming the time dependence to be of the form $e^{-i\omega t}$, the fields of the incident electromagnetic wave will be

$$\vec{E}_i\ (\vec{r},t) = E_o\ (\vec{r})\ e^{-i\omega t}\ \vec{e}_z \qquad \text{(I-6)}$$

$$\vec{B}_i\ (\vec{r},t) = B_o\ (\vec{r})\ e^{-i\omega t}\ \vec{e}_o \times \vec{e}_z \qquad \text{(I-7)}$$

It will be useful to also have the unit vector $\vec{e}_o$ explicitly in terms of unit vectors along the coordinate axes. This quantity, simplified by the choice of the polarization of the incident radiation, is

$$\vec{e}_o = -(\cos\phi_o)\ \vec{e}_x\ -(\sin\phi_o)\ \vec{e}_y \qquad \text{(I-8)}$$

We then have for the incident wave

$$\vec{k}_i = k\ \vec{e}_o = -k\left\{\cos\phi_o\ \vec{e}_x + \sin\phi_o\ \vec{e}_y\right\} \qquad \text{(I-9)}$$

where $k = \omega/c$ is the free space propagation constant.

From the symmetry of the problem, we have only a Z component of the electric field. Thus, we have to deal only with the differential equations

$$\nabla^2 E_z(\vec{r}) + k^2 E_z(\vec{r}) = 0 \quad \text{(I-10)}$$

$$B_\rho(\vec{r}) = \frac{1}{(i\omega)} \frac{1}{\rho} \frac{\partial E_z(\vec{r})}{\partial \varphi} \quad \text{(I-11)}$$

$$B_\varphi(\vec{r}) = -\frac{1}{(i\omega)} \frac{\partial E_z(\vec{r})}{\partial \rho} \quad \text{(I-12)}$$

Furthermore, we note that the solution of eq (10) will in turn yield the magnetic field solutions of eq (11) and (12).

We shall use the boundary device of the Kaden cylinder [1] which we discussed in an earlier report [2]. The Kaden cylinder divides all of space into the three regions:

(i) region (1): $\rho > \rho_0$, $0 < \varphi < \pi$

(ii) region (2): $\rho < \rho_0$, $0 < \varphi < 2\pi$

(iii) region (3): $\rho > \rho_0$, $\pi < \varphi < 2\pi$

This is illustrated in Figure 2.

Our problem is to determine the fields E_z, B_φ, B_ρ in each of these regions such that the appropriate boundary conditions are satisfied on the conducting plane and on the upper and lower halves of the Kaden cylinder as well. In addition, of course, the radiation condition must be satisfied for $\rho \to \infty$ in regions (2) and (3).

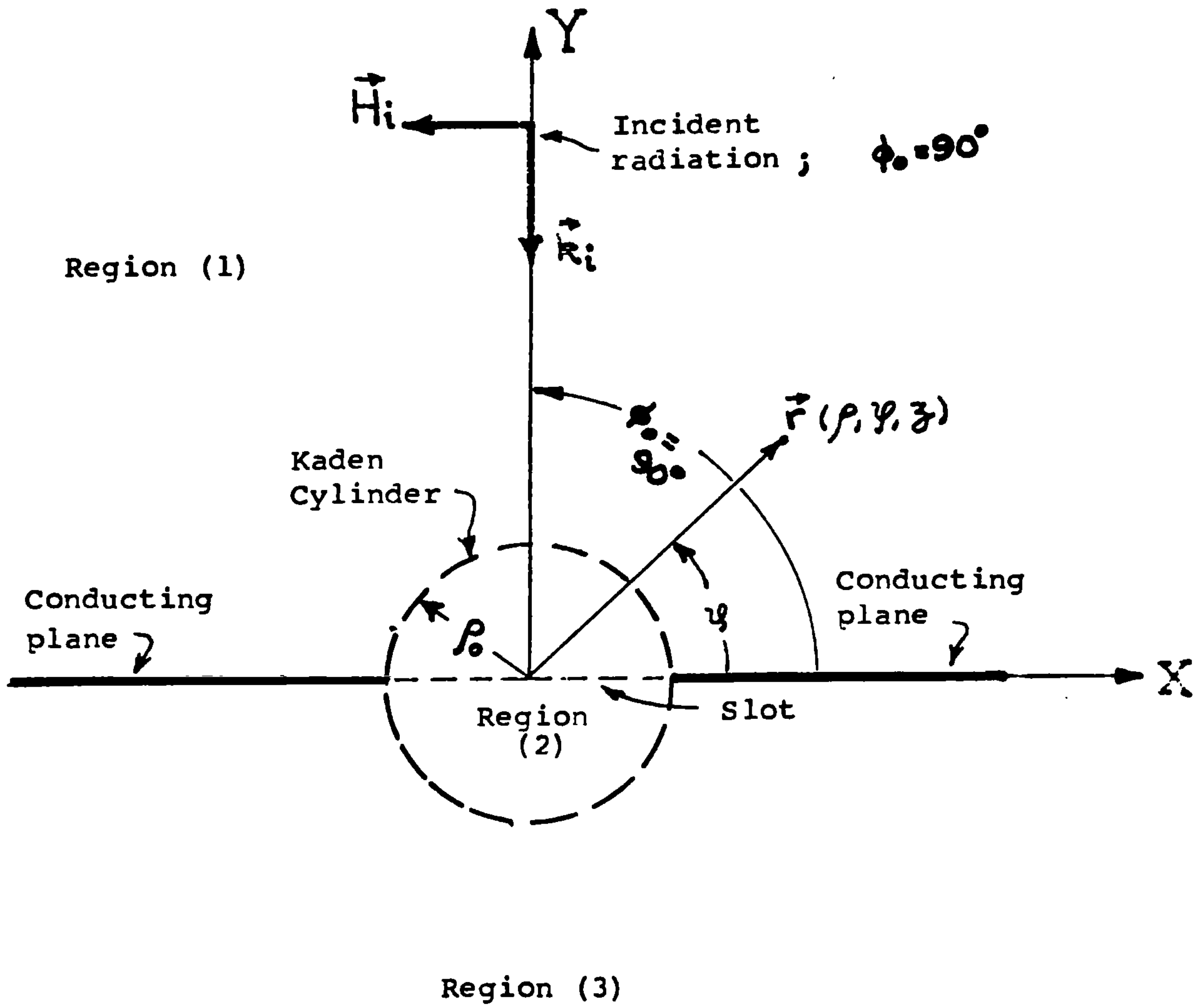

Figure 2. Cross-Section of Three Scattering Regions Used In Formulating The Diffraction Problem.

II. BOUNDARY CONDITIONS; FORMAL SOLUTION FOR THE FIELDS

Above and below the slot region, that is in regions (1) and (3), the fields must satisfy the following boundary conditions

$$
\begin{aligned}
E_z^{(1)}(\rho, o) &= 0 \\
E_z^{(1)}(\rho, \pi) &= 0 \\
H_\varphi^{(1)}(\rho, o) &= 0 \\
H_\varphi^{(1)}(\rho, \pi) &= 0
\end{aligned}
\tag{II-13}
$$

and

$$
\begin{aligned}
E_z^{(3)}(\rho, \pi) &= 0 \\
E_z^{(3)}(\rho, 2\pi) &= 0 \\
H_\varphi^{(3)}(\rho, \pi) &= 0 \\
H_\varphi^{(3)}(\rho, 2\pi) &= 0
\end{aligned}
\tag{II-14}
$$

Consistency requires that the solution of the slot problem reduce properly to the limiting case of zero slot width. We therefore digress briefly to consider the problem of our plane wave incident, obliquely, on the perfectly conducting plane screen without a slot. The geometry is illustrated in cross-section in Figure 3. The incident unit vector is identical with that given in Eq (8). For the reflected radiation, the unit vector in the direction of propagation is simply

$$\vec{e}_r = -(\cos\phi_o)\,\vec{e}_x + (\sin\phi_o)\,\vec{e}_y \tag{II-15}$$

and of course the corresponding propagation constant is given by

$$k_r \equiv k = \frac{\omega}{c} \tag{II-16}$$

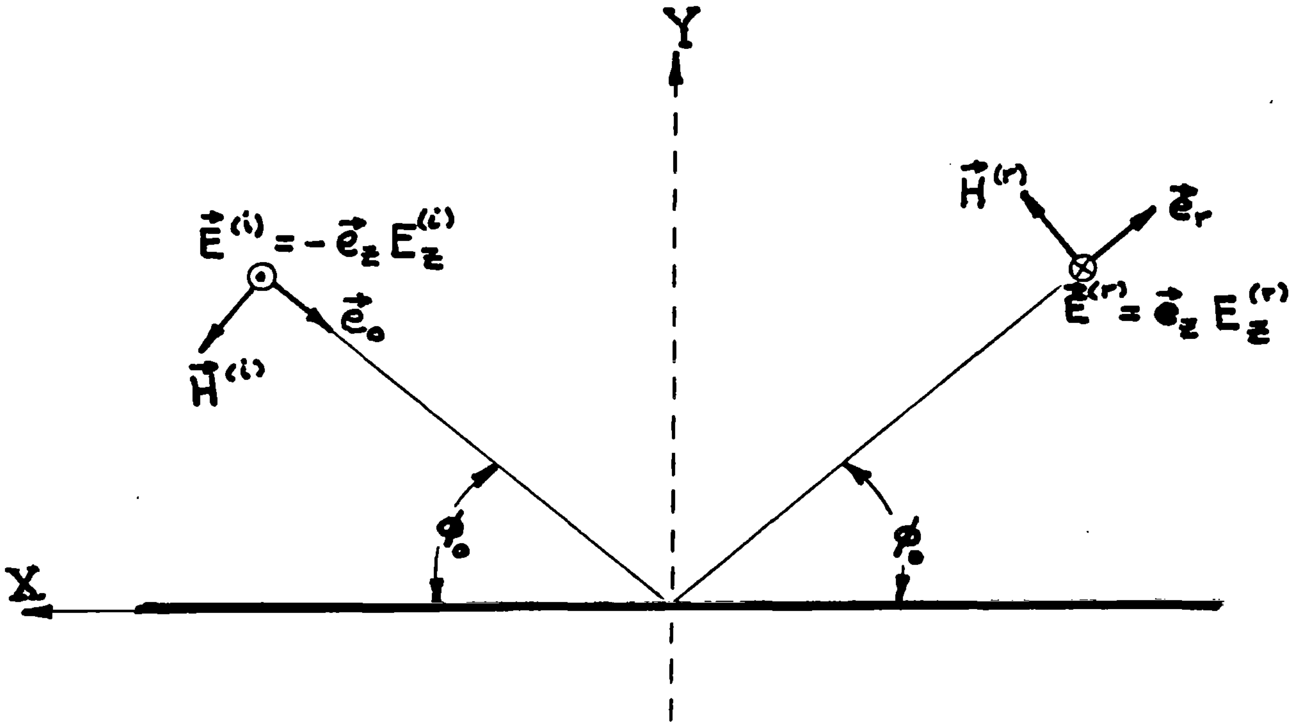

Figure 3. Linearly Polarized Plane Wave Incident Obliquely at Angle ϕ_o on Conducting Plane Screen Containing No Apertures. Polarization of Incident Radiation is Parallel to the Screen.

At any point $\vec{r}$ above the screen, the incident, and reflected fields are respectively

$$E_z^{(i)}(\vec{r}) = -E_o\, e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

and

$$E_z^{(r)}(\vec{r}) = C\, e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

where the unknown coefficient C is to be determined by invoking the boundary condition that $E_z^{(i)} + E_z^{(r)} = 0$ on the screen. From this we find that at $\vec{r}$ the total electric field has the amplitude

$$E_z^{(i)}(\vec{r}) + E_z^{(r)} = 2i\, E_o e^{-i[k\, x \cos\phi_o + \omega t]} \sin(ky \sin\phi_o) \qquad \text{(II-17)}$$

We shall use this result below.

Returning to the slot problem, we consider the general form of the electric field above the slot region namely $E_z^{(1)}(\rho,\varphi)$. In region (1) the total field is considered to be due to the three contributions:

(i) $E_z^{(i)}$ ≡ field of the incident radiation

(ii) $E_z^{(r)}$ ≡ field of the reflected radiation in the absence of the slot

(iii) $E_z^{(slot)}$ ≡ field due to the presence of the slot

Consequently at any point $\vec{r}$ (ρ,φ) in region (1) the electric field has the amplitude (where the time dependence is suppressed)

$$E_z^{(1)}(\rho,\varphi) = E_z^{(i)}(\rho,\varphi) + E_z^{(r)}(\rho,\varphi) + E_z^{(slot)}(\rho,\varphi) \qquad \text{(II-18)}$$

It should be emphasized here that we are representing the total field in region (1) in such a manner that in the limit of no slot eq (18) reduces to eq (17). This requires that $E_z^{(i)} + E_z^{(r)}$ and $E_z^{(slot)}$ satisfy the boundary conditions of eq (13) independently. At large distances from the slot, $E_z^{(slot)}$ must correspond to the field of an outgoing wave. It then follows that the appropriate form of the contribution to the total field arising from the slot will be

$$E_z^{(slot)}(\rho,\varphi) = \sum_{n=1}^{\infty} A_n H_n^{(1)}(k\rho) \sin n\varphi \qquad \text{(II-19)}$$

Combining eqs (18) and (19), we have for the total electric field in region (1)

$$E_z^{(1)}\ (\rho,\varphi) = 2i\ E_o\ e^{-ik\rho\ \cos\varphi\ \cos\phi_o}\ \sin(k\rho\sin\varphi\sin\phi_o) + \qquad \text{(II-20)}$$

$$+ \sum_{n=1}^{\infty} A_n\ H_n^{(1)}\ (k\rho)\ \sin n\varphi$$

If the electric field as given in eq (20) is substituted into eq (12), it is easy to see that the total field $H_\varphi^{(1)}$ satisfies the boundary conditions stated in eq (13) at the conducting screen.

Next, we consider the fields in region (3). This we accomplish through the use of known symmetry conditions relating slot fields above and below the perfectly conducting plane. By considering the current density on the screen, it can be argued[3] by reason of symmetry that the following is true

$$\vec{n} \times \vec{E}^{(slot)}\ (\vec{r}\,') = \vec{n} \times \vec{E}^{(3)}\ (\vec{r}) \qquad \text{(II-21)}$$

where the point $\vec{r}\,'$ is the image in the screen of point $\vec{r}$ and $\vec{n}$ is a unit vector directed normally from the conducting plane into region (1). In our geometry $\vec{n} \equiv \vec{e}_y$. We can then write

$$\vec{e}_y \times \vec{e}_z\ \ E_z^{(slot)}\ (\vec{r}\,') = \vec{e}_y \times \vec{e}_z\ \ E_z^{(3)}\ (\vec{r})$$

or equivalently

$$E_z^{(3)}(\rho,\varphi) = E_z^{(slot)}\ (\rho,-\varphi) \qquad \text{(II-22)}$$

From this result we find more explicitly the relation

$$E_z^{(3)}(\rho,\varphi) = -\sum_{n=1}^{\infty} A_n\ H_n^{(1)}\ (k\rho)\ \sin n\varphi \qquad \text{(II-23)}$$

This solution for the electric field in region (3) satisfies the boundary condition on the screen and also the radiation condition.

The field $B_\varphi^{(3)}$ obtained from eqs (23) and (12) does likewise.

Next, we must obtain the fields within the slot region. We shall carry this out after we have obtained the $\vec{E}$ fields of regions (1) and (3) in a convenient form. After some straightforward manipulation we have the relation

$$2i\, E_o\, e^{-ik\rho\cos\varphi\cos\phi_o} \sin k\rho \sin(\varphi\sin\phi_o) = E_o \left\{ e^{ik\rho\cos(\varphi+\phi_o)} - e^{-ik\rho\cos(\varphi-\phi_o)} \right\} \qquad \text{(II-24)}$$

Using the series representations[4]

$$\cos[k\rho\cos(\varphi\pm\phi_o)] = J_o(k\rho) + 2\sum_{n=2,4,6,\ldots}^{\infty} (-1)^{n/2} J_n(k\rho)\cos n(\varphi\pm\phi_o) \qquad \text{(II-25a)}$$

$$\sin[k\rho\cos(\varphi\pm\phi_o)] = 2\sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} J_n(k\rho)\cos n(\varphi\pm\phi_o) \qquad \text{(II-25b)}$$

we can rewrite eq (24) in the form

$$2i\, E_o\, e^{-ik\rho\cos\varphi\cos\phi_o} \sin(k\rho\sin\varphi\sin\phi_o) = 4E_o \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{n/2} J_n(k\rho)\cdot$$

$$\cdot \sin n\varphi \sin(n\phi_o) - 4E_o \sum_{n=2,4,\ldots}^{\infty} (-1)^{n/2} J_n(k\rho)\sin n\varphi \sin n\phi_o . \qquad \text{(II-26)}$$

Let us define the following function

$$g_n(\varphi) \equiv \begin{cases} \sin n\varphi \ ; & 0 \le \varphi \le \pi \\ -\sin n\varphi \ ; & 0 \le \varphi \le 2\pi \end{cases} \qquad \text{(II-27)}$$

With the aid of this function and eq (26), we can write the Fields $E^{(1)}$ and $E^{(3)}$ (eqs (20) and (23)) as the single expression

$$E_z^{(1,3)}(\rho,\varphi) = 2E_o \sum_{n=1,3,\ldots}^{\infty}(-1)^{n/2} J_n(k\rho)\sin n\varphi \sin n\phi_o - 2E_o \sum_{n=2,4,\ldots}^{\infty}(-1)^{n/2}\Big[$$

$$J_n(k\rho)\sin n\varphi \sin n\phi_o\Big] + 2E_o \sum_{n=1,3,\ldots}^{\infty}(-1)^{n/2} J_n(k\rho)\, g_n(\varphi)\sin n\phi_o - 2E_o \cdot$$

$$\cdot \sum_{n=2,4,\ldots}^{\infty}(-1)^{n/2} J_n(k\rho)\, g_n(\varphi)\sin n\phi_o + \sum_{n=1,3,\ldots}^{\infty} A_n H_n^{(1)}(k\rho)\, g_n(\varphi)$$

$$+ \sum_{n=2,4,\ldots}^{\infty} A_n H_n^{(1)}(k\rho)\, g_n(\varphi) \qquad \text{(II-28)}$$

The function $g_n(\varphi)$ will next be processed so as to produce a representation of it which will help us to find the fields inside the Kaden cylinder in a convenient form. Since $g_n(\varphi)$ is an even function, we can write

$$g_n(\varphi) = \sum_{m=0,2,4,\ldots}^{\infty} Cmn \;\; \mathrm{Cos}\, m\varphi \;; \qquad n = 1,3,5,\ldots \qquad \text{(II-29)}$$

where

$$Cmn = \frac{2}{\pi}\int_0^{\pi} g_n(\varphi)\, d\varphi = \frac{2}{\pi}\left(\frac{1}{n+m} + \frac{1}{n-m}\right); \quad m = 0,2,\ldots \quad n = 1,3,\ldots \qquad \text{(II-30)}$$

Eq (29) enables us to rewrite eq (28) as

$$E_z^{(1,3)}(\rho,\varphi) = 2E_o \left\{ \sum_{n=1,3,\ldots}^{\infty}(-1)^{n/2} J_n(k\rho)\sin n\varphi \sin n\phi_o - \sum_{n=2,4\ldots}(-1)^{n/2} \right.$$

$$\cdot J_n(k\rho)\sin n\varphi \sin n\phi_o +$$

$$+ \sum_{n=1,3,\ldots}^{\infty} \sum_{m=0,2,\ldots}^{\infty}(-1)^{n/2} J_n(k\rho)\, C_m^n \cos m\varphi \sin\phi_o - \sum_{n=2,4,\ldots}^{\infty} \cdot$$

$$\cdot \sum_{m=1,3,\ldots}^{\infty} (-1)^{n/2} J_n(k\rho)\, C_{mn} \cos m\varphi \sin n\phi_o +$$

$$+ \sum_{n=1,3,\ldots}^{\infty} \sum_{m=0,2,\ldots}^{\infty} A_n H_n^{(1)}(k\rho) \cos m\varphi + \sum_{n=2,4,\ldots}^{\infty} \sum_{m=1,3,\ldots}^{\infty} \Big[A_n \times$$

$$\times H_n^{(1)}(k\rho)\, C_{mn} \cos m\varphi \Big] \qquad \text{(II-31)}$$

where we have used the notation

$$C_m^n = \begin{cases} 1/2\; c_{on} & m=0 \;;\; n=1,3,5,\ldots \\ c_{mn} & m=2,4,\ldots \;;\; 1,3,5,\ldots \end{cases} \qquad \text{(II-32)}$$

Using eq (31) for the form of the field outside the slot region, we can deduce the field within the Kaden cylinder. In the plane of the conducting screen, the tangential component of the magnetic field within the slot must be the same as that of the incident field at the same position[5]. Consequently, the first and second series in eq (31) will also be present in the expression for the electric field within the slot region, $E_z^{(2)}$. Also present will be a contribution arising from the slot. Since the latter component must remain finite as $\rho \to o$ we have for the form of $E_z^{(2)}$

$$E_z^{(2)}(\rho,\varphi) = 2E_o \sum_{n=1,3,\ldots}^{\infty} (-1)^{n/2} J_n(k\rho) \sin n\varphi \sin \phi_o - 2E_o \sum_{n=2,4,\ldots}^{\infty} (-1)^{n/2} \times$$

$$\times J_n(k\rho) \sin n\varphi \sin n\phi_o + \sum_{m=o}^{\infty} B_m J_m(k\rho) \cos m\varphi \qquad \text{(II-33)}$$

Eqs (28) and (33) give the formal expressions for the electric field in regions (1), (2) and (3). From these the magnetic fields everywhere can be obtained.

We must complete the set of boundary conditions stated in Eqs (13) and (14), which, we recall, apply on the conducting screen itself. This is effected by adding the conditions at the Kaden cylinder:

$$E_z^{(1)}(\rho_o,\varphi) = E_z^{(2)}(\rho_o,\varphi) \qquad \text{(II-34a)}$$

$$B_\varphi^{(1)}(\rho_o,\varphi) = B_\varphi^{(2)}(\rho_o,\varphi) \qquad 0 < \varphi < \pi \qquad \text{(II-34b)}$$

$$B_\rho^{(1)}(\rho_o,\varphi) = B_\rho^{(2)}(\rho_o,\varphi) \qquad \text{(II-34c)}$$

$$\left.\begin{array}{ll} E_z^{(3)}(\rho_o,\varphi) = E_z^{(2)}(\rho_o,\varphi) & \text{(II-35a)} \\ B_\varphi^{(3)}(\rho_o,\varphi) = B_\varphi^{(2)}(\rho_o,\varphi) & \text{(II-35b)} \\ B_\rho^{(3)}(\rho_o,\varphi) = B_\rho^{(2)}(\rho_o,\varphi) & \text{(II-35c)} \end{array}\right\} \quad \pi < \varphi < 2\pi$$

In principle, the sets of still unknown coefficients A_n and B_n are to be determined via the boundary conditions. Proceeding in this direction, we obtain from the pair of eqs (34a) and (35a) (and also the pair of eqs (34c) and (35c)):

$$2E_o \sum_{n=1,3,...}^{\infty} (-1)^{n/2} J_n(\eta)\, C_m^n \sin n\phi_o + \sum_{n=1,3,...}^{\infty} A_n H_n^{(1)}(\eta) \times$$

$$\times C_m^n = B_m J_m(\eta), \; m = 0,2,4,... \qquad \text{(II-36)}$$

$$2E_o \sum_{n=2,4,...}^{\infty} (-1)^{n/2} J_n(\eta)\, Cnm \sin n\phi_o + \sum_{n=2,4,...}^{\infty} A_n H_n^{(1)}(\eta) \times$$

$$\times Cmn = B_m J_m(\eta), \; m = 1,3,5,... \qquad \text{(II-37)}$$

where we have introduced the helpful notation

$$\eta \equiv k\rho_o = \omega\rho_o/c = 2\pi\rho o/\lambda \qquad \text{(II-38)}$$

Similarly from the pair of eqs (34b) and (35b), we obtain

$$2E_o \sum_{n=1,3,...}^{\infty} (-1)^{n/2} J_n'(\eta)\, C_m^n \sin n\phi_o + \sum_{n=1,3,...}^{\infty} A_n H_n^{(1)\prime}(\eta) \times$$

$$\times C_m^n = B_m J_m'(\eta); \; m=0,2,4,... \qquad \text{(II-39)}$$

$$2E_o \sum_{n=2,4,\ldots}^{\infty} (-1)^{n/2} \; J_n'(\eta)\, C_{mn} \; \sin n\phi_e + \sum_{n=2,4,\ldots}^{\infty} A_n \, H_n^{(1)'}(\eta) \times$$

$$C_{mn} = B_m \, J_m'(\eta) \; ; \; m=1,3,\ldots \qquad \text{(II-40)}$$

where the prime indicates differentiation with respect to the argument $k\rho$ evaluated at $\eta = k\rho_o$.

Simultaneous solution of eqs (36), (37), (39) and (40), if it can be achieved, will give explicitly all the coefficients and hence, in turn, the fields everywhere.

We next consider the solution of the problem in the Rayleigh limit of approximation.

III. THE RAYLEIGH APPROXIMATION

In the Rayleigh region, the wavelength of the incident radiation is very large compared to the characteristic dimensions of the scattering region. For the problem in hand, this latter quantity is simply the slot width. Equivalently the condition on the parameter $\eta \equiv 2\pi\rho_o/\lambda << 1$, implies the Rayleigh approximation. We will solve our scattering problem in this asymptotic limit.

Small argument asymptotic forms of the Bessel and Hankel functions that are needed are

$$J_m(\eta) \xrightarrow[\eta<<1]{} \frac{(\eta/2)^m}{\Gamma(m+1)} \; ; \quad m = 0,1,2,3,\ldots$$

$$H_m^{(1)}(\eta) \xrightarrow[\eta<<1]{} \frac{-i}{\pi}\Gamma(m)\,(2/\eta)^m \; ; \quad m = 1,2,3,\ldots \qquad \text{(III-41)}$$

$$J_m'(\eta) \xrightarrow[\eta<<1]{} \frac{m}{\eta}\,\frac{(\eta/2)^m}{\Gamma(m+1)} \; ; \quad m=1,2,3,\ldots$$

$$J_o'(\eta) \xrightarrow[\eta<<1]{} -\eta/2$$

$$H_m^{(1)\prime}(\eta) \xrightarrow[\eta<<1]{} \frac{im}{\pi\eta}\,\Gamma(m)\,(2/\eta)^m \qquad m=1,2,3,\ldots$$

Using these asymptotic forms eqs (36) and (39) become, respectively

$$2E_o \sum_{n=1,3,\ldots}^{\infty} (-1)^{n/2}\, C_m^n \left[\frac{(\eta/2)^n}{\Gamma(n+1)}\right] \sin n\phi_o + \sum_{n=1,3,\ldots}^{\infty} A_n\, C_m^n \times$$

$$\times \left[\frac{-i}{\pi}\Gamma(n)\left(\frac{2}{\eta}\right)^n\right] = B_m \frac{(\eta/2)^m}{\Gamma(m+1)} \quad \text{for } m = 2,4,\ldots \; ; \qquad \text{(III-42)}$$

$$2E_o \sum_{n=1,3,\ldots}^{\infty} (-1)^{n/2}\, C_m^n \left[\frac{n}{\eta}\,\frac{(\eta/2)^n}{\Gamma(n+1)}\right] \sin n\phi_o + \sum_{n=1,3,\ldots}^{\infty} A_n\, C_m^n \times$$

$$\times \left[\frac{in}{\pi\eta}\Gamma(n)\left(\frac{2}{\eta}\right)^n\right] = \frac{mB_m}{\eta}\,\frac{(\eta/2)^m}{\Gamma(m+1)} \; ; \quad m=2,4,\ldots \qquad \text{(III-43)}$$

We shall consider the m = 0 case separately. If we multiply eq (42) through by m and eq (43) by η and subtract the latter from the former we obtain

$$2E_o \sum_{n=1,3,..}^{\infty} (-1)^{n/2} (m-n) \frac{C_m^n (\eta/2)^n}{\Gamma(n+1)} \sin n\phi_o = \frac{i}{\pi} \sum_{n=1,3,..}^{\infty} A_n (m+n) \times$$

$$\times C_m^n \Gamma(n) \left(\frac{2}{\eta}\right)^n \quad \text{for } m=2,4,.. \qquad \text{(III-44)}$$

Now for the case m = 0. If we multiply the asymptotic form of eq (36) by $\eta/2$ and add it to the asymptotic form of eq (39), with m = 0, we obtain

$$2E_o \sum_{n=1,3,...}^{\infty} (-1)^{n/2} n C_o^n \frac{(\eta/2)^n}{\Gamma(n+1)} \sin n\phi_o = -\frac{i}{\pi} \sum_{n=1,3,..}^{\infty} A_n C_o^n n\Gamma(n) (2/\eta)^n$$

(III-45)

where we have used the long wavelength condition

$$n/\eta \gg \eta/2 .$$

Eq (44) with m formally set equal to zero exactly coincides with eq (45). Thus, we can write the set of equations to be solved for the coefficients A_n, for n = 1,3,5,..., as

$$2E_o \sum_{n=1,3,..}^{\infty} (-1)^{n/2} (m-n) C_m^n \frac{(\eta/2)^n}{\Gamma(n+1)} \sin n\phi_o = \frac{i}{\pi} \sum_{n=1,3,5,...}^{\infty} A_n (m+n) \times$$

$$\times C_m^n \Gamma(n) (2/\eta)^n \quad \text{for } m = 0,2,4,... \qquad \text{(III-46)}$$

In precisely the same manner given in BL-I[6], we obtain for the coefficients outside the slot region

$$A_n = -\frac{\pi E_o \sin\phi_o}{(n+1)} \times \frac{(\eta/2)^{n+1}}{2^{(n-2)}\left[\left(\frac{n-1}{2}\right)!\right]^2} \quad n=1,3,.. \qquad \text{(III-47)}$$

The next immediate task is to determine the even subscripted series expansion coefficients A_n. Equations (37) and (40) become with the aid of eqs (41)

$$2E_o \sum_{n=2,4,\ldots}^{\infty} (-1)^{n/2} \; C_{mn} \left[\frac{(\eta/2)^n}{\Gamma(n+1)}\right] \sin n\phi_o + \sum_{n=2,4,\ldots}^{\infty} A_n \; C_{mn} \times$$

$$\times \left[-\frac{i}{\pi}\,\Gamma(n)\left(\frac{2}{\eta}\right)^n\right] = \frac{B_m \; (\eta/2)^m}{\Gamma\;(m+1)} \;; \quad m=1,3,5,\ldots \tag{III-48}$$

and

$$2E_o \sum_{n=2,4,\ldots}^{\infty} (-1)^{n/2} \; C_{mn} \; \frac{n}{} \left[\frac{(\eta/2)^n}{\Gamma\;(n+1)}\right] \sin n\phi_o + \sum_{n=2,4,\ldots}^{\infty} A_n \; C_{mn} \times$$

$$\times \left[\frac{in}{\pi\eta}\;\Gamma(n)\left(\frac{2}{\eta}\right)^n\right] = \frac{m B_m \; (\eta/2)^m}{\eta\,\Gamma\;(m+1)} \;; \quad m=1,3,5,\ldots \tag{III-49}$$

Processing this in exactly the manner we handled eqs (42) and (43), we obtain finally the solution

$$A_n = \frac{i\pi\; E_o \; \sin 2\phi_o}{(n+2)} \quad \frac{n}{2^n\left[\left(\frac{n}{2}\right)!\right]^2} \left(\frac{\eta}{2}\right)^{n+2} ; \qquad n=2,4,\ldots \tag{III-50}$$

Now we shall find the Rayleigh region series coefficients within the Kaden cylinder namely the B_m. In the long wavelength limit eqs (42) and (43) are respectively just

$$iE_o \; C_m^1 \,\eta \sin\phi_o + \sum_{n=1,3,\ldots}^{\infty} A_n \; C_m^n \left[\frac{-i}{\pi}\;\Gamma\;(n)\;\left(\frac{2}{\eta}\right)^n\right] = \frac{B_m \;(\eta/2)^m}{\Gamma\;(m+1)} \;; \quad m = 2,4,\ldots \tag{III-51}$$

and

$$iE_o \; C_m^1 \,\eta \sin\phi_o + \sum_{n=1,3,\ldots}^{\infty} A_n \; C_m^n \left[\frac{i}{\pi}\;\Gamma\;(n)\;\left(\frac{2}{\eta}\right)^n\right] = \frac{m B_m \;(\eta/2)^m}{\Gamma\;(m+1)} \;; \quad m=2,4,\ldots \tag{III-52}$$

Subtracting these and using the explicit form of A_n as given in eq (47), we obtain quite readily

$$\left(\frac{\rho_0 E_0 \sin\phi_0}{c} \sum_{n=1,3,\ldots}^{\infty} C_m^n \frac{(n-1)!}{2^{(n-1)}\left[\left(\frac{n-1}{2}\right)!\right]^2} = \frac{(1-m)\ (\eta/2)^m B_m}{\Gamma\ (m+1)} ; \quad m=2,4,6\ldots \right)$$

(III-53)

Following the same procedure used above to find the A_n, we obtain the coefficients in question:

$$B_m = \frac{\rho_0 E_0 \sin\phi_0}{\sqrt{\pi}\ c} \frac{m!}{(1-m)} \frac{\Gamma\left(\frac{m+1}{2}\right)}{(m/2)!} \left(\frac{2}{\eta}\right)^m \text{ for } m = 2,4,\ldots \qquad \text{(III-54)}$$

We next fill in for the missing coefficient B_o. From eqs (36) and (39), the long wavelength approximation gives for $m = 0$

$$2E_o C_o^1 (\eta/2) \sin\phi_o + \sum_{n=1,3,\ldots}^{\infty} A_n C_o^n \left[- \frac{i}{\pi}\Gamma(n) \ (2/\eta)^n\right] = B_o$$

and

$$\eta E_o C_o^1 \sin\phi_o + \sum_{n=1,3,\ldots}^{\infty} A_n C_o^n \frac{n}{\pi} \Gamma(n) \ (2/\eta)^n = -B_o \ \eta^2/2$$

Solving for B_o (where we recall $\eta \ll 1$) we have

$$B_o = \frac{i}{\pi} \sum_{n=1,3,\ldots}^{\infty} A_n C_o^n (n+1)\ \Gamma(n) \ (2/\eta)^n \qquad \text{(III-55)}$$

Comparing this with eq (54), we find that it coincides in form if we set $m = 0$. Thus eq (54) can be modified to include $m = 0$:

$$B_m = \frac{\rho_0 E_0 \sin\phi_0}{\sqrt{\pi}\ c} \frac{m!}{(1-m)} \frac{\Gamma\left(\frac{m+1}{2}\right)}{(m/2)!} \left(\frac{2}{\eta}\right)^m ; \quad m=0,2,4\ldots \qquad \text{(III-56)}$$

We next complete this solution by obtaining the odd subscripted coefficients B_m. From eqs (48) and (49), we have Rayleigh limit forms:

$$2E_o (-1)^1 \ C_{m2} \frac{(\eta/2)^2}{\Gamma\ (3)} \sin 2\phi_o + \sum_{n=2,4,\ldots}^{\infty} A_n C_{mn} \left[\frac{-i}{\pi} \Gamma(n) \ (2/\eta)^n\right]$$

$$= \frac{B_m \ (\eta/2)^m}{\Gamma\ (m+1)} ; \qquad m=1,3,5,\ldots$$

$$2E_o\ (-1)^1 C_{m2} \left(\frac{2}{\eta}\right) \frac{(\eta/2)^2}{\Gamma(3)} \sin 2\phi_o + \sum_{n=2,4,\ldots}^{\infty} A_n C_{mn} \left[\frac{in}{\pi\eta}\Gamma(n)\left(\frac{2}{\eta}\right)^n\right]$$

$$= \frac{m}{\eta} \frac{B_m (\eta/2)^m}{\Gamma(m+1)} \quad ; \quad m=1,3,5,\ldots \qquad \text{(III-58)}$$

Multiplying eq (57) by 2 and eq (58) by η and then subtracting yields

$$-\frac{i}{\pi} \sum_{n=2,4,\ldots}^{\infty} A_n\ C_{mn}(n+2)\ \Gamma(n)\ (2/\eta)^n = \frac{(2-m)\ B_m\ (\eta/2)^m}{\Gamma(m+1)} ;$$

$$m=1,3,5,\ldots \qquad \text{(III-59)}$$

Subtracting for A_n as given by eq (50) and solving for the B_m we obtain finally

$$B_m = \frac{\eta E_o \sin 2\phi_o}{2\sqrt{\pi}} \frac{m!}{(2-m)} \frac{\Gamma(m/2)}{\left(\frac{m-1}{2}\right)!} \left(\frac{2}{\eta}\right)^{m-1} ; \quad m=1,3,5,\ldots$$

(III-60)

Using eqs (47) and (50), we have the electric field in region (1) in the Rayleigh approximation

$$E_z^{(1)}(\rho,\varphi) \xrightarrow[\eta<<1]{} 2iE_o \exp(-ik\rho\cos\varphi\cos\phi_o)\ \sin(k\rho\sin\varphi\sin\phi_o) -$$

$$- \pi E_o \sin\phi_o \sum_{n=1,3,\ldots}^{\infty} \frac{H_n^{(1)}(k\rho)}{(n+1)} \frac{\sin n\varphi}{\left[2^{n-2}\right]\left[\left(\frac{n-1}{2}\right)!\right]^2} \left(\frac{\eta}{2}\right)^{n+1} +$$

$$+ i\pi E_o \sin 2\phi_o \sum_{n=2,4,\ldots}^{\infty} \frac{H_n^{(1)}(k\rho)}{(n+2)} \frac{\sin n\varphi}{2^n\left[\left(\frac{n}{2}\right)!\right]^2} \left(\frac{\eta}{2}\right)^{n+2}$$

(III-61)

In region (3), the electric field is similarly

$$E_z^{(3)}(\rho,\varphi)\xrightarrow[\eta<<1]{}\pi E_o \sin\phi_o \sum_{n=1,3,..}^{\infty} \frac{H_n^{(1)}(k\rho)}{(n+1)} \frac{\sin n\varphi}{2^{(n-2)}\left[\left(\frac{n-1}{2}\right)!\right]^2}\left(\frac{\eta}{2}\right)^{n+1}$$

$$- i\pi E_o \sin 2\phi_o \sum_{n=2,4,..}^{\infty} \frac{H_n^{(1)}(k\rho)}{(n+2)} \frac{\sin n\varphi}{2^n\left[\left(\frac{n}{2}\right)!\right]^2}\left(\frac{\eta}{2}\right)^{n+2}$$

(III-62)

Using the identity of eq (26) and the results for B_m in eqs (56) and (60), we can write the long wavelength electric field in region (2):

$$E_z^{(2)}(\rho,\varphi)\xrightarrow[\eta<<1]{} iE_o \exp(-ik\rho\cos\varphi\cos\phi_o)\sin(k\rho\sin\varphi\sin\phi_o) +$$

$$+ \frac{i\eta E_o \sin\phi_o}{\sqrt{\pi}} \sum_{m=0,2,4,..}^{\infty} \frac{\Gamma\left(\frac{(m+1)}{2}\right)}{(1-m)\left(\frac{m}{2}\right)!}\left(\frac{\rho}{\rho_o}\right)^m \cos m\varphi +$$

$$+ \left(\frac{\eta}{2}\right)^2 \frac{E_o \sin 2\phi_o}{\sqrt{\pi}} \sum_{m=1,3,5,..}^{\infty} \frac{\Gamma\left(\frac{m}{2}\right)}{(2-m)\left(\frac{m-1}{2}\right)!}\left(\frac{\rho}{\rho_o}\right)^m \cos m\varphi$$

(III-63)

An extremely important point to elaborate is that of fulfillment of the boundary condition at the slot edges. Consider then the edge defined by $\rho = \rho_o$ and $\varphi = 0$. Here eq (63) takes the form, after setting $m \equiv 2M$ in the first sum and $m \equiv 2M+1$ in the second sum:

$$E_z^{(2)}(\rho_o,0)\xrightarrow[\eta<<1]{} \frac{i\eta E_o \sin\phi_o}{\sqrt{\pi}} \sum_{M=0}^{\infty} \frac{\Gamma(M+1/2)}{M!\,(1-2M)} +$$

$$+ \left(\frac{\eta}{2}\right)^2 \frac{E_o \sin 2\phi_o}{\sqrt{\pi}} \sum_{M=0}^{\infty} \frac{\Gamma(M+1/2)}{M!\,(1-2M)}$$

Using the identity

$$\Gamma(M+1/2) = \frac{\sqrt{\pi}\,(2M)!}{2^{2M}\,M!}$$

and in turn the series definition of the beta-function

$$B(\xi,1/2) = \frac{\Gamma(\xi)\Gamma(1/2)}{\Gamma(\xi + 1/2)} = \sum_{M=0}^{\infty} \frac{(2M)!}{2^{2M}(M!)} \frac{1}{(M+\xi)} , \qquad \text{(III-64)}$$

and recalling that $\Gamma(0) = \infty$, we find that

$$E_z^{(2)} (\rho_o, 0) \xrightarrow[\eta \ll 1]{} 0 \qquad \text{(III-65a)}$$

In precisely the same manner, we can demonstrate that at the other slot edge the field also vanishes

$$E_z^{(2)} (\rho_o, \pi) \xrightarrow[\eta \ll 1]{} 0 \qquad \text{(III-65b)}$$

We can conclude then that our solution in the Rayleigh limit satisfies the boundary conditions at the edges of the slot.

IV. FORMULATION OF THE NUMERICAL SOLUTION FOR ARBITRARY SLOT-WIDTH AND INCIDENT WAVELENGTH

To determine the unknown series coefficients A_n, B_m required to explicitly find the fields in the general situation for arbitrary $\eta = k\rho o = 2\pi\rho o/\lambda$, we adopt a numerical technique developed by one of the authors.[8] Essentially the method of approximation is to require fulfillment of the boundary conditions at only restricted sets of points on the boundary rather than over its entirety and to truncate the infinite series in the formal field expressions at a corresponding finite number of terms. This leads to sets of simultaneous linear equations which are readily solved (at least in principle).

Following this procedure, we define angles φ such that

$$\pi \leq \varphi e \leq 2\pi \qquad \text{(IV-66)}$$

and at these azimuths we have from the boundary condition requirements at the Kaden cylinder.

$$E_z^{(2)}(\rho_o, \varphi_\ell) = E_z^{(3)}(\rho_o, \varphi_\ell) \qquad \text{(IV-67)}$$

Also we have

$$H_\rho^{(2)}(\rho o, \varphi_\ell) = H_\rho^{(3)}(\rho o, \varphi_\ell) \qquad \text{(IV-68a)}$$

or equivalently

$$\left[\left(\frac{\partial E_z}{\partial \varphi}^{(2)}\right)_{\rho=\rho o}\right]_{\varphi=\varphi_\ell} = \left[\left(\frac{\partial E_z}{\partial \varphi}^{(3)}\right)_{\rho=\rho o}\right]_{\varphi=\varphi_\ell} \qquad \text{(IV-68b)}$$

However, if eqs (67) and (68) are written out in detail in terms of the summations, it will be observed that whenever eq (67) holds eq (68) is automatically satisfied. The boundary condition on H_φ at the Kaden surface gives us

$$H_\varphi^{(2)}(\rho o, \varphi_\ell) = H_\varphi^{(3)}(\rho o, \varphi_\ell) \qquad \text{(IV-69)}$$

or equivalently

$$\left[\left(\frac{\partial E_z}{\partial (k\rho)}^{(2)}\right)_{\rho=\rho' o}\right]_{\varphi=\varphi_\ell} = \left[\left(\frac{\partial E_z}{\partial (k\rho)}^{(3)}\right)_{\rho=\rho o}\right]_{\varphi=\varphi_\ell} \qquad \text{(IV-69a)}$$

This, together with eq (67), constitute for each φ_ℓ two independent relations connecting the sets of expansion coefficients A_n and B_m. For a given slot width and a given wavelength for the incident radiation, the A_n's and B_m's that contribute significantly to the composition of the fields can be numerically obtained approximately but, nevertheless, quite accurately if one ignores all the remaining coefficients further out in the series representations. To effect this we truncate the series so that the indices n and m run respectively from 1 thru N and from 0 through M = N-1. We now have 2N unknowns to evaluate. By suitably selecting N values of φ_ℓ we will generate from eqs (67) and (69) the two sets, each of N simultaneous linear equations in 2N unknowns

$$\sum_{n=1}^{N} a_n H_n^{(1)}(\eta) \sin n\varphi_\ell + \sum_{m=0}^{N-1} b_m J_m(\eta) \cos m\varphi_\ell =$$

$$= \frac{e^{-i\eta \cos\varphi_\ell \cos\phi_o}}{\eta} \sin(\eta \sin\varphi_\ell \sin\phi_o) \; ;$$

$$\ell = 1,2,\ldots,N \qquad \text{(IV-70)}$$

$$\sum_{n=1}^{N} a_n H_n^{(1)\prime}(\eta) \sin n\varphi_\ell + \sum_{m=0}^{N-1} b_m J_m^{\prime}(\eta) \cos m\varphi_\ell =$$

$$= \frac{1}{\eta} \sin\varphi_\ell \sin\phi_o \, e^{-i\eta \cos\varphi_\ell \cos\phi_o} \cos(\eta \sin\varphi_\ell \sin\phi_o)$$

$$- \frac{i}{\eta} \cos\varphi_\ell \cos\phi_o \, e^{-i\eta \cos\varphi_\ell \cos\phi_o} \sin(\eta \sin\varphi_\ell \sin\phi_o) ;$$

$$\ell = 1,2,\ldots,N \qquad \text{(IV-71)}$$

where for further convenience we have introduced the additional notation

$$a_n \equiv \frac{iA_n}{\eta E_o} \quad , \qquad b_m \equiv \frac{iB_m}{\eta E_o} \qquad \text{(IV-72)}$$

Before considering the calculated results, we shall briefly derive the transmission cross-section expression in terms of the fields.

V. ANGULAR DISTRIBUTION OF SCATTERED INTENSITY AND TRANSMISSION CROSS-SECTION

If we introduce the following notation

p_t ≡ total power transmitted through the slot per unit length of the slot

S_i ≡ magnitude of the real part of $\vec{S}_i$ the incident complex Poynting vector at the scatterer = $|\mathrm{Re}\ \vec{S}\ |$

we define the transmission cross-section per unit length of the slot as

$$\sigma_t = \frac{\rho_t}{|\mathrm{Re}\ \vec{S}_i\ |} \qquad \text{(V-73)}$$

which in turn can be expressed in terms of the expansion coefficients A_n

Since $$|\mathrm{Re}\ \vec{S}_i| = |1/2\ \mathrm{Re}\ \vec{E}_i \times \vec{H}_i^*| \qquad \text{(V-74)}$$

we can use eqs. (4) and (6) to obtain S_i as follows

$$|\mathrm{Re}\ \vec{S}_i| = 1/2\ \Big|\mathrm{Re}\ \Big\{E_o\ \vec{e}_z \times \Big[\frac{E_o}{i\omega\mu_o}\ \vec{\nabla} \times \vec{e}_z\ e^{i\vec{k}_i\cdot\vec{r}}\Big]^*\Big\} \quad \text{at slot}$$

$$= \frac{E_o^{\,2}}{2\omega\mu_o}\ \Big|\ \mathrm{Re}\ \Big\{\vec{e}_z \times \Big[(i)\ \Big(-ie^{-i\vec{k}_i\cdot\vec{r}}\ \vec{e}_z \times \vec{k}\Big)\Big]\Big\} \quad \text{at slot}$$

which becomes simply

$$S_i = E_o^{\,2}/\ 2\mu_o c \qquad \text{(V-75)}$$

The total power transmitted through a unit length of the slot is

$$p_t = 1/2 \int_{\pi}^{2\pi} d\varphi\ \rho\cdot\vec{e}_\rho \cdot \mathrm{Re}\ \Big[\lim_{\rho\to\infty} (\vec{E}^{(3)} \times \vec{H}^{(3)*})\Big] \qquad \text{(V-76)}$$

or more simply

$$P_t = 1/2 \int_{\pi}^{2\pi} d\varphi \; \rho \, \vec{e}_\rho \cdot \mathrm{Re}\left[\lim_{\rho\to\infty} E_z^{(3)} \; H_\varphi^{(3)*}\right]$$

We repeat Eq (23) for the field $E_z^{(3)}$

$$E_z^{(3)} \quad (\rho, \varphi) = - \sum_{n=1}^{\infty} A_n \; H_n^{(1)} \; (k\rho) \; \sin n\varphi$$

Using the relation

$$H_\varphi^{(3)} = - \frac{1}{i\omega\mu_o} \; \frac{\partial Ez^{(3)}}{\partial \rho}$$

we obtain

$$E_z^{(3)} \; H_\varphi^{(3)*} = \frac{k}{i\omega\mu_o} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} A_n \; A_\ell^* \; H_n^{(1)} \; (k\rho) \; H_\ell^{(2)\prime} \; (k\rho) \times$$

$$\times \sin n\varphi \quad \sin \ell\varphi .$$

In the far field as $\rho \to \infty$, we use the asymptotic forms of the Hankel functions and obtain

$$\rho \lim_{\rho\to\infty} E_z^{(3)} \; H_\varphi^{(3)*} = - \frac{2}{\pi\omega\mu o} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} A_n \; A_\ell^* \; e^{\, i(\ell-n)\pi/2} \sin n\varphi \sin \ell\varphi .$$

The transmitted power is then

$$P_t = 1/2 \; \frac{1}{\omega\mu_o} \sum_{n=1}^{\infty} \left| A_n \right|^2 \qquad \text{(V-77)}$$

and consequently the transmission cross-section per unit slot length is

$$\sigma_t = (kE_o^{\,2})^{-1} \sum_{n=1}^{\infty} \left| A_n \right|^2 \qquad \text{(V-78)}$$

In geometric optics this transmission cross-section is just

$$\sigma_t^{\,optics} = 2\rho_o \qquad \text{(V-79)}$$

which will serve as a convenient normalizing parameter. Thus, we introduce the normalized transmission cross-section, which is hereafter referred to as the <u>transmission coefficient</u>:

$$T_E(\varphi_o,\eta) \equiv \sigma_t(\varphi_o,\eta)/\sigma_t^{\text{Optics}} \qquad \text{(V-80)}$$

which in terms of the scattering coefficient is

$$T_E(\varphi_o,\eta) = \frac{1}{2\eta E_o^2}\sum_{n=1}^{\infty}|A_n|^2 = \frac{\eta}{2}\sum_{n=1}^{\infty}|a_n(\varphi_o,\eta)|^2 \qquad \text{(V-81)}$$

This is truncated in our approximation technique to the sum

$$T_E^{(N)}(\varphi_o,\eta) = \frac{\eta}{2}\sum_{n=1}^{N}|a_n^{(N)}(\varphi_o,\eta)|^2 \qquad \text{(V-82)}$$

where the superscript (N) has been appended to the appropriate symbols to indicate the order of approximation to which they are to be calculated.

In addition to the total radiation transmitted through the slot, another physical parameter is of interest in scattering problems. This is the angular distribution of the intensity of the scattered radiation in the far field. Or in other words, the differential power transmitted per unit length of slot, in the direction $\vec{e}_\rho(\varphi)$. If we denote this quantity by $I_E(\varphi)$, we have

$$I_E(\varphi) = 1/2\ \vec{e}_\rho(\varphi)\cdot \mathrm{Re}\lim_{\rho\to\infty}\rho\left[\vec{E}^{(3)}\times\vec{H}^{(3)*}\right] \qquad \text{(V-83)}$$

which following the same arguments leading to eq (77), becomes in terms of the scattering coefficients, a_n :

$$I_E(\varphi) = \frac{\eta}{\pi}\left|\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} a_\ell^* a_n e^{i(\ell-n)\pi/2}\sin\ell\varphi\sin n\varphi\right| \qquad \text{(V-84)}$$

Truncating this consistem with our method of approximation, this becomes in turn

$$I_E^{(N)}(\varphi) = \frac{\eta}{\pi}\left|\sum_{n=1}^{N}\sum_{\ell=1}^{N} a_n^{(N)} a_\ell^{(N)*} e^{i(\ell-n)\pi/2}\sin\ell\varphi\sin n\varphi\right| \qquad \text{(V-85)}$$

Having completed the formal development, we next present the numerical results obtained for oblique incidence.

VI. NUMERICAL RESULTS AND DISCUSSION

We restrict the specific results presented in this report merely to (1) the calculated values of the transmission coefficient $T_E(\phi_o,\eta)$ given to several orders of approximation; (2) and the angular distribution of the far field intensity for a single but rather high order of approximation. Although we do not include the numerical results obtained for the scattering expansion coefficients; namely, the a_n, we shall nevertheless briefly discuss their behavior. For a given ϕ_o the expansion coefficients exhibit the same behavior obtained and discussed in an earlier report (BLI) as the order of approximation N increases. Here, as in the report just cited, the angles ψ_ℓ, $\ell = 1,2,\ldots,N$ were selected by evenly spacing the range of ψ. These values were chosen via the relation

$$\psi_\ell = \pi + \frac{\ell\pi}{N+1} \qquad \ell = 1,2,\ldots,N \tag{VI-83}$$

The results obtained for the expansion coefficients a_n for a sufficiently large but fixed order of approximation N vary with changes in the choice of the set of angles ψ_ℓ. For given ratio of slot width to incident wavelength, *i.e.* η, the predominant a_n's show quite small dependence on the choice of the ψ_ℓ. This dependence on the ψ_ℓ is more marked for the less significant a_n's. The net result of this behavior is to produce a numerical value for the transmission coefficient which is practically independent of the ψ_ℓ's provided that they are more or less uniformly distributed over the range of ψ. By decreasing N, the order of approximation, this salutary behavior was found to be even more pronounced.

Many calculations were carried out for a large range of order of approximation and for angles of incidence from $\phi_o = 10°$ to $\phi_o = 90°$ in steps of 10°. However, we present in this report only a part of the numerical results obtained. These are in every way representative of all the calculated results and embody all the essentials of the extension of our study of the approximation technique.

Table I contains the numerical value of the transmission coefficient $T_E^{(N)}$ for the ratio of slot-width to wavelength parameter over the range $0.5 \leq \eta \leq 2.2$. The values are listed for approximations obtained by matching at N = 23,25, and 27 boundary angles chosen in accordance with eq (83). These listings are for angles ranging from $\phi_o = 10°$, that is, almost grazing incident, to $\phi_o = 90°$, normal incidence on the slotted plane. We see in Table I(a) for $\phi_o = 10°$ that the results differ at most in the fourth decimal place as N goes from 23 to 27. The entries in Table I were actually only calculated to the fourth decimal place. A more careful inspection of these values strongly suggests that the discrepancies between the results for a given value of η really occurs in the fifth decimal place. This is borne out by the results shown in Table II which gives $T_E^{(N)}$

for $\phi_o = 10°$ for $N = 43, 45$ and 47. Table II, however, covers the larger range $0.5 \leq \eta \leq 6.0$. Over even the extended range of η the values of $T_E^{(N)}$ for fixed η only vary in the fifth decimal place. This indicates that for the range of η covered T_E (10°) has in all probability been accurately determined by choosing $N = 43$. Clearly for $\eta \gtrsim 2.0$, $T_E(10°)$ can be adequately calculated using a lower order of approximation $N \approx 25$. Pretty much the same situation is the case for incidence at $\phi_o = 30°$. One can see in Table I(b) that the transmission coefficient deviates for fixed η in the fourth decimal position as the order of approximation increases from $N = 23$ to $N = 27$. For $\eta \gtrsim 1.5$, $T_E^{(N)}(30°)$ is as given in Table I(b) for $N \approx 25$. At intermediate angles of incidence such as $\phi_o = 50°$, we note that in Table I(c) where N runs from 23 to 27 that the deviations in $T_E(\phi_o)$, fixed η, are in the fourth decimal place for smaller η values and in the third decimal place at the larger values. Examination of Table III, however, reveals that at nearly double the order of approximation in Table I(c) $T_E(\phi_o)$ values obtained vary at fixed η in the fifth decimal place for small η, in the fourth place for all the larger values of η. For the common range of ratio of slot-width to wavelength in Tables I(c) and III the improvement obtained by increasing the order of approximation to twice the number of matching boundary angles is essentially only on the order of a very few percent actually from about 4% to less than 1%. Nevertheless, it appears that at the intermediate range of angles of incidence the calculated results are more reliable for $N \approx 45$. The overall trend continues as ϕ_o increases. Thus in Table I(d), we have the results for incidence just a bit off normal at $\phi_o = 80°$. For the smaller η values the T_E (80°) values vary in the fourth decimal place. For all larger η this deviation occurs in the third decimal place. However, it should be noted that this deviation, for $0.8 < \eta \leq 2.2$, is less than 1% as N goes from 23 to 27. Improvement of this calculation occurs, of course, for higher order of approximation. Finally, we can study the results for normal incidence $\phi_o = 90°$ given in Table I(e) for $0.5 \leq \eta \leq 2.2$ and $N = 23$, 25 and 27 and in Table IV for $0.5 \leq \eta \leq 6.0$ and $N = 45$, 47, and 49. Examination of Table I(e) shows that as N changes from 23 to 27, for fixed η, $T_E(90°)$ varies in the third decimal position although this variation ranges from 2% for $\eta = 0.5$ to about 0.3% for $\eta = 2.2$. Table IV on the other hand exhibits a variation in the fourth decimal position for $\eta < 3.3$ and at most in the third place for η beyond this value , resulting in a smaller spread than for the lower orders of approximation in Table I(e). For the common range in η of the Tables, the improvement at double the order of approximation displayed in Table I goes from .1% at $\eta = 2.2$ to about 4% for $\eta = 0.5$.

One conclusion we draw is that for more accurate results a much higher order of approximation is required the closer we come to normal incidence on the slotted plane. Thus we need $N \approx 45$ for $\phi_o \approx 90°$ whereas $N \approx 25$ is quite adequate for $\phi_o \approx 10°$. The reason underlying this is not too evident as yet and is presently undergoing investigation. We display in Figure 4 the transmission coefficient $T_E^{(49)}(\phi_o, \eta)$ for ϕ_o from 10° to 90°. The results calculated for the other orders of approximation discussed in this report are nearly coincident with the curves shown and hence only one set of curves is presented. Figure 4 also shows the earlier results of Morse and Rubenstein[9] for the transmission coefficient. Our results are in very good agreement with theirs. However, we probably have better numerical accuracy than the earlier tables of Mathieu functions used by Morse and Rubenstein.

In Figures 5, 6, and 7 we show respectively relative angular distributions of the diffracted radiation for angles of incidence of $\phi_o = 30°$, $60°$, and $90°$. The results in each case were obtained, using 49 matching boundary circles on the Kaden cylinder for $\lambda = 2\pi\rho o$, $\lambda = \pi\rho o$, and $\lambda = 2\pi\rho o/3$. Again these results agree quite well with those obtained by Morse and Rubenstein.

In conclusion, we observe that the numerical method of approximation yields accurate results for the obliquely incident slotted plane problem. The technique is being applied to more complex scattering problems and the results will be reported in the near future.

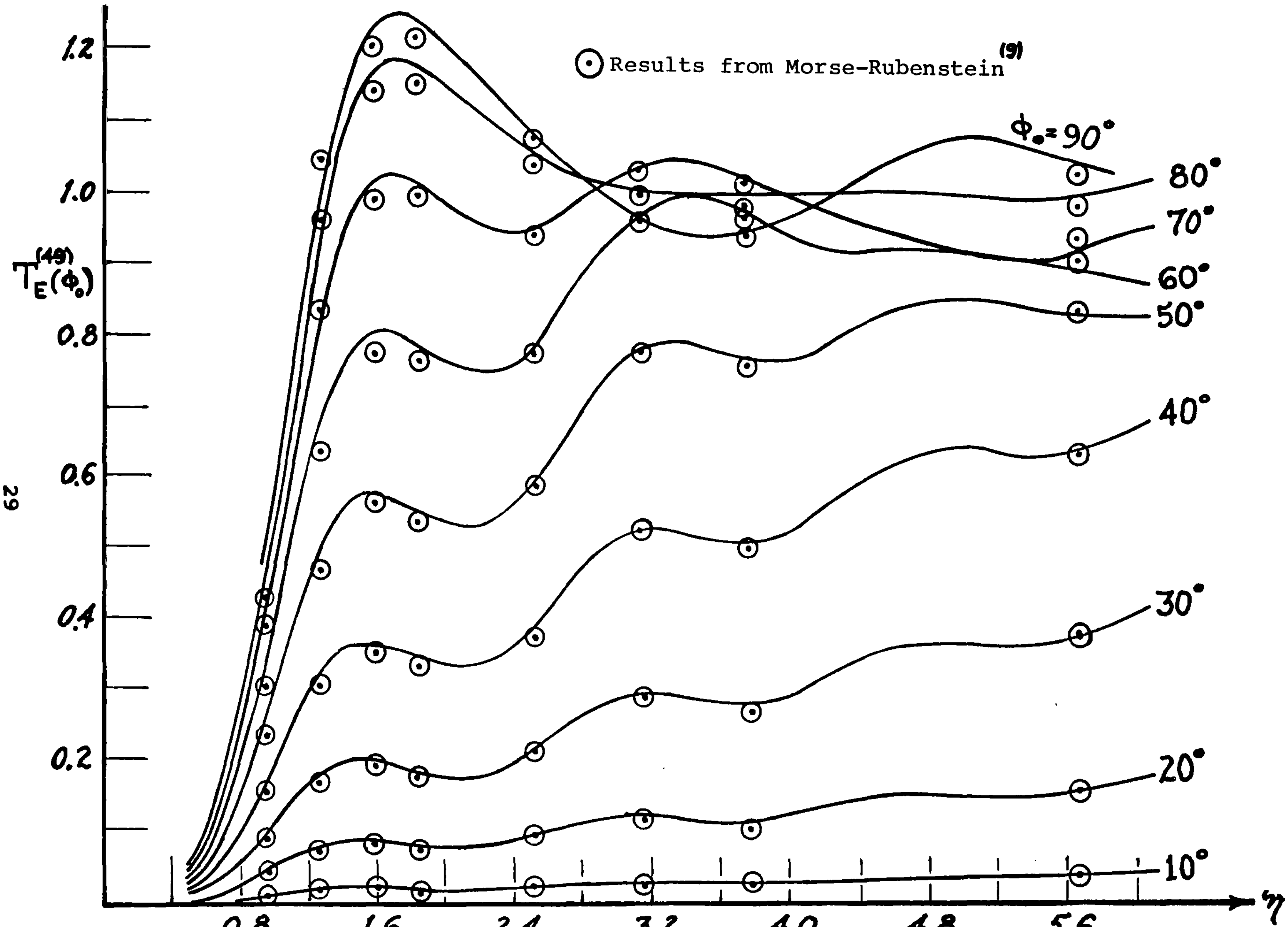

Figure 4. Transmission Coefficient Versus Slot-Width to Wavelength Ratio, for Oblique Incidence

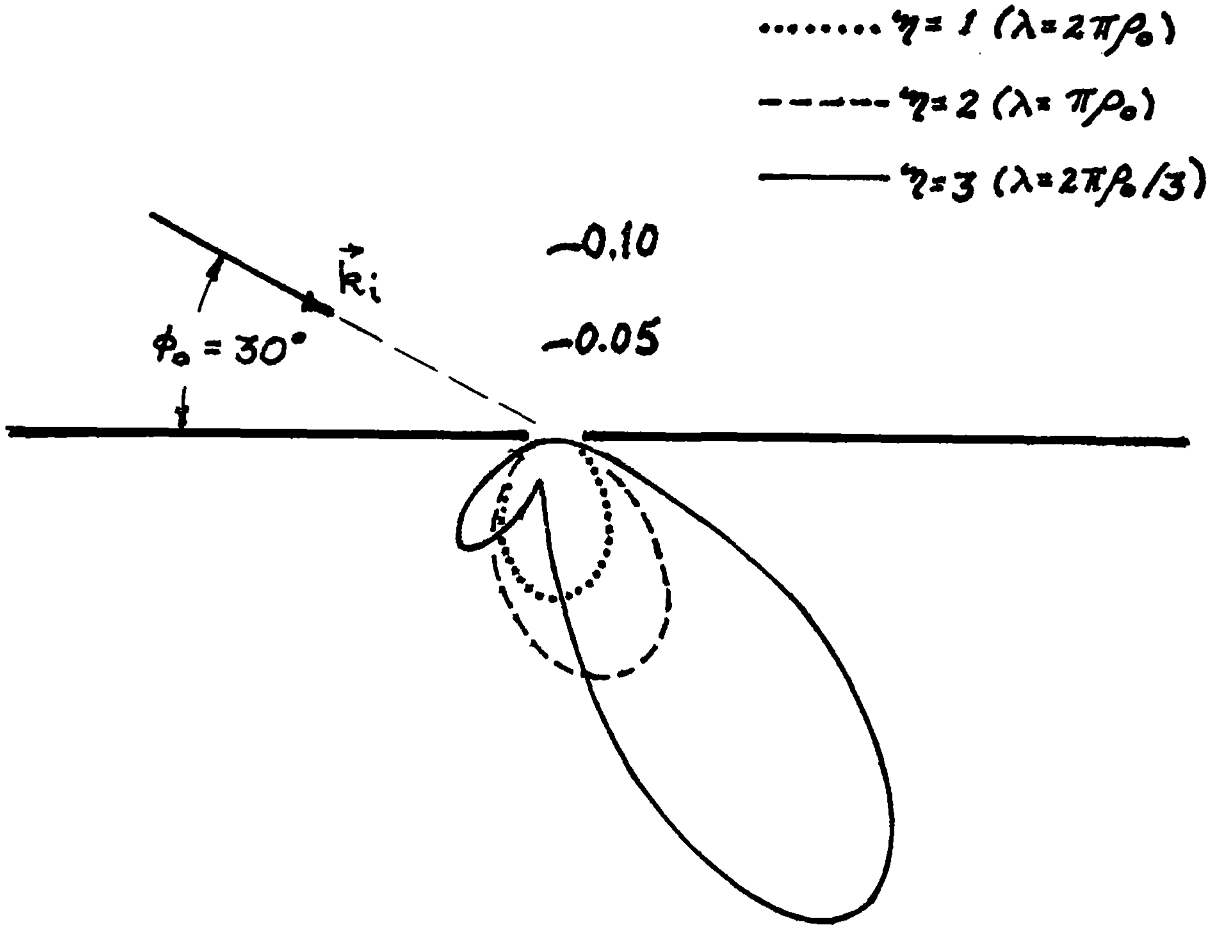

Figure 5. Angular Distribution of the Transmitted, Relative Far-Field Intensity, I_E (ψ), for Several Wavelengths at Angle of Incidence $\phi_o = 30^o$.

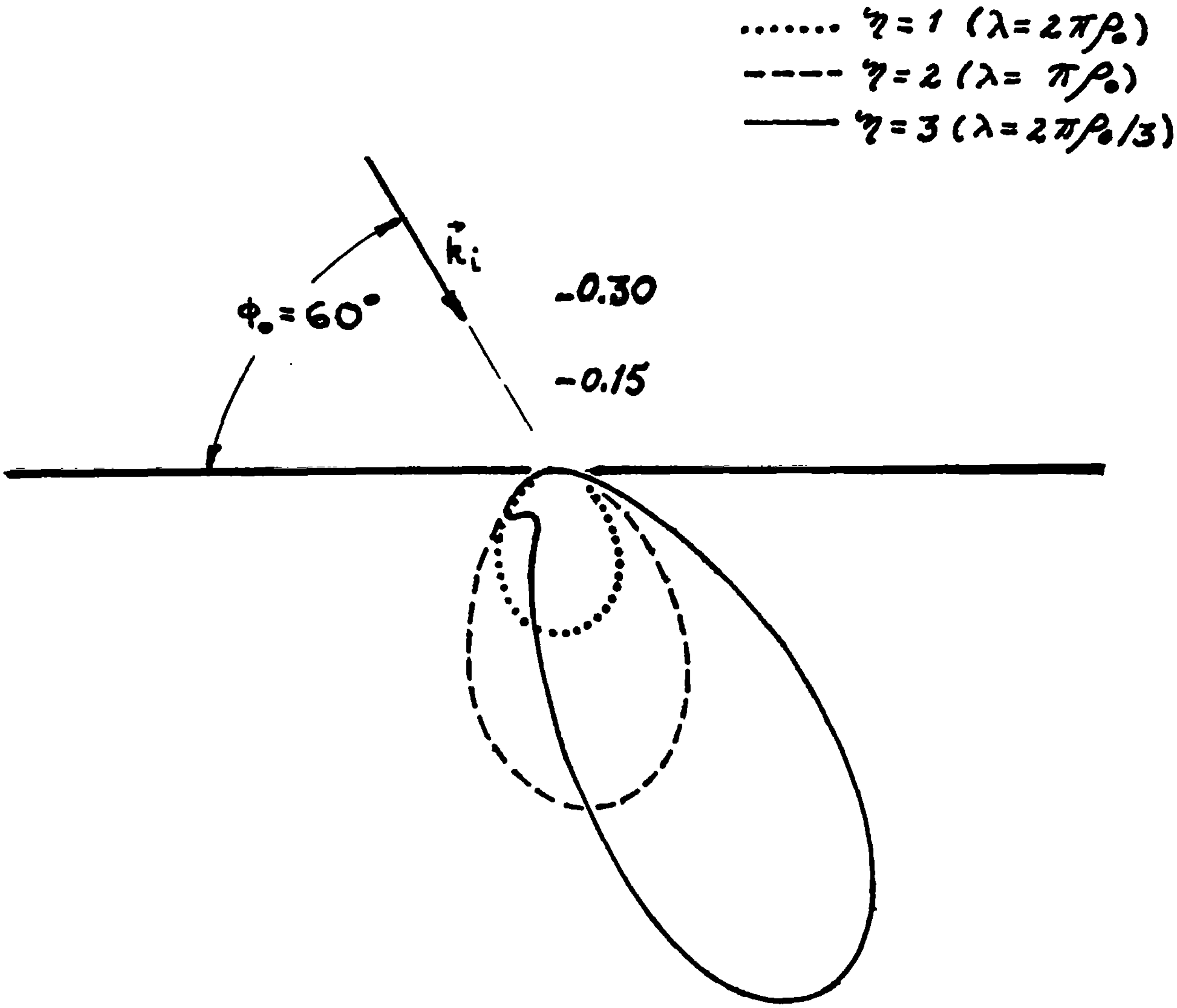

Figure 6. Angular Distribution of the Transmitted, Relative Far-Field Intensity, I_E (ψ), for Several Wavelengths at Angle of Incidence $\phi_0 = 60^\circ$.

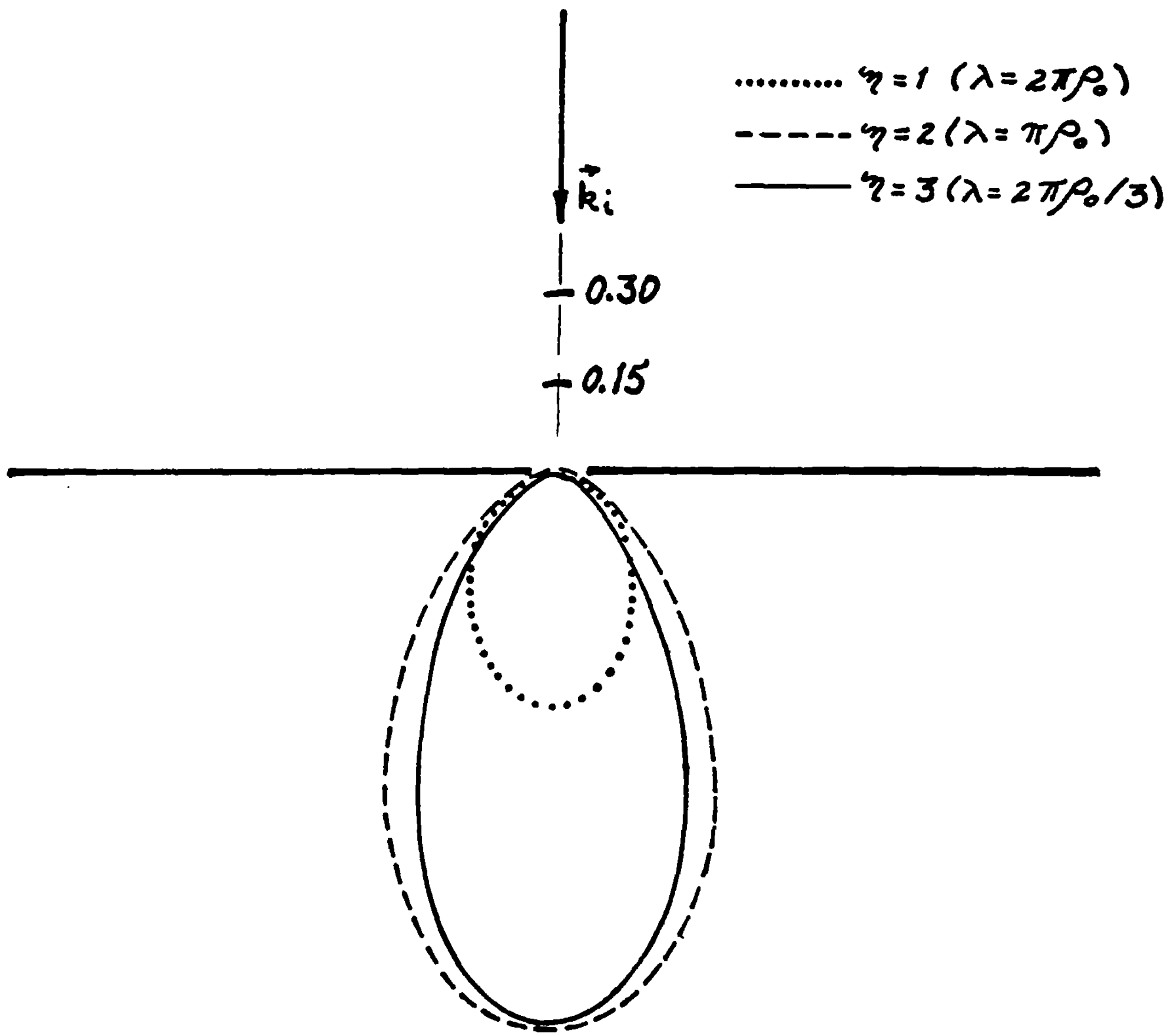

Figure 7. Angular Distribution of the Transmitted, Relative Far-Field Intensity, I_E (ψ), for Several Wavelengths at Angle of Incidence $\phi_0 = 90°$.

Table 1. Slotted Plane Transmission Coefficient, $T_E^{(N)}(\phi_o, \eta)$; E-Polarization

a). Angle of Incidence $\phi_o = 10^o$

η	N = 23	N = 25	N = 27
0.5	0.0016	0.0016	0.0016
0.6	0.0029	0.0029	0.0029
0.7	0.0049	0.0048	0.0048
0.8	0.0075	0.0074	0.0074
0.9	0.0107	0.0106	0.0105
1.0	0.0141	0.0140	0.0140
1.1	0.0174	0.0173	0.0172
1.2	0.0199	0.0198	0.0198
1.3	0.0215	0.0214	0.0214
1.4	0.0220	0.0220	0.0219
1.5	0.0217	0.0217	0.0217
1.6	0.0209	0.0210	0.0210
1.7	0.0200	0.0200	0.0200
1.8	0.0191	0.0191	0.0192
1.9	0.0185	0.0185	0.0185
2.0	0.0183	0.0183	0.0183
2.1	0.0186	0.0185	0.0185
2.2	0.0194	0.0193	0.0193

b). Angle of Incidence $\phi_o = 30^o$

η	N = 23	N = 25	N = 27
0.5	0.0134	0.0133	0.0132
0.6	0.0247	0.0245	0.0243
0.7	0.0414	0.0411	0.0408
0.8	0.0643	0.0637	0.0633
0.9	0.0926	0.0919	0.0913
1.0	0.1240	0.1232	0.1225
1.1	0.1545	0.1536	0.1529
1.2	0.1794	0.1787	0.1781
1.3	0.1960	0.1955	0.1950
1.4	0.2037	0.2033	0.2030
1.5	0.2040	0.2038	0.2037
1.6	0.1995	0.1995	0.1994
1.7	0.1927	0.1927	0.1928
1.8	0.1858	0.1859	0.1859
1.9	0.1804	0.1805	0.1805
2.0	0.1781	0.1780	0.1779
2.1	0.1796	0.1794	0.1792
2.2	0.1857	0.1853	0.1849

Table 1. Slotted Plane Transmission Coefficient, $T_E^{(N)}(\phi_0, \eta)$; E-Polarization (contd)

c). Angle of Incidence $\phi_0 = 50^\circ$

η	N = 23	N = 25	N = 27
0.5	0.0322	0.0320	0.0317
0.6	0.0599	0.0594	0.0590
0.7	0.1016	0.1008	0.1000
0.8	0.1597	0.1584	0.1572
0.9	0.2335	0.2317	0.2302
1.0	0.3180	0.3158	0.3139
1.1	0.4033	0.4008	0.3987
1.2	0.4776	0.4753	0.4733
1.3	0.5325	0.5305	0.5288
1.4	0.5650	0.5635	0.5622
1.5	0.5778	0.5768	0.5758
1.6	0.5762	0.5756	0.5749
1.7	0.5662	0.5657	0.5653
1.8	0.5528	0.5525	0.5522
1.9	0.5403	0.5399	0.5396
2.0	0.5319	0.5313	0.5308
2.1	0.5302	0.5294	0.5286
2.2	0.5372	0.5360	0.5349

d). Angle of Incidence $\phi_0 = 80^\circ$

η	N = 23	N = 25	N = 27
0.5	0.0547	0.0542	0.0538
0.6	0.1027	0.1018	0.1010
0.7	0.1764	0.1749	0.1736
0.8	0.2814	0.2790	0.2769
0.9	0.4185	0.4150	0.4120
1.0	0.5804	0.5759	0.5721
1.1	0.7507	0.7456	0.7412
1.2	0.9082	0.9030	0.8985
1.3	1.0353	1.0304	1.0262
1.4	1.1239	1.1196	1.1159
1.5	1.1757	1.1720	1.1689
1.6	1.1979	1.1949	1.1923
1.7	1.1993	1.1968	1.1946
1.8	1.1876	1.1854	1.1835
1.9	1.1685	1.1665	1.1649
2.0	1.1462	1.1444	1.1428
2.1	1.1235	1.1216	1.1201
2.2	1.1021	1.1002	1.0986

Table 1. (Continued)

e). Angle of Incidence = 90^{o}

η	N = 23	N = 25	N = 27
0.5	0.0565	0.0560	0.0556
0.6	0.1062	0.1052	0.1045
0.7	0.1826	0.1810	0.1797
0.8	0.2916	0.2891	0.2569
0.9	0.4342	0.4306	0.4275
1.0	0.6031	0.5984	0.5945
1.1	0.7813	0.7759	0.7713
1.2	0.9467	0.9412	0.9365
1.3	1.0811	1.0758	1.0713
1.4	1.1756	1.1710	1.1671
1.5	1.2319	1.2280	1.2247
1.6	1.2573	1.2540	1.2512
1.7	1.2606	1.2578	1.2555
1.8	1.2497	1.2472	1.2452
1.9	1.2303	1.2280	1.2262
2.0	1.2064	1.2044	1.2027
2.1	1.1807	1.1787	1.1771
2.2	1.1547	1.1528	1.1512

Table II: Slotted Plane Transmission Coefficient, $T_E^{(N)}$, (ϕ_o, η); E-Polarization, Angle of Incidence, ϕ_o = 10° and $0.5 \leq \eta \leq 6.0$

η	N = 45	N = 47	N = 49
0.5	0.00152	0.00151	0.00151
0.6	0.00277	0.00277	0.00276
0.7	0.00462	0.00461	0.00460
0.8	0.00711	0.00710	0.00708
0.9	0.01019	0.01017	0.01015
1.0	0.01357	0.01355	0.01352
1.1	0.01682	0.01679	0.01677
1.2	0.01944	0.01942	0.01940
1.3	0.02112	0.02111	0.02109
1.4	0.02180	0.02179	0.02178
1.5	0.02167	0.02166	0.02166
1.6	0.02100	0.02100	0.02100
1.7	0.02010	0.02010	0.02010
1.8	0.01922	0.01922	0.01922
1.9	0.01855	0.01855	0.01855
2.0	0.01824	0.01824	0.01823
2.1	0.01839	0.01838	0.01837
2.2	0.01906	0.01905	0.01903
2.3	0.02026	0.02023	0.02021
2.4	0.02190	0.02187	0.02184
2.5	0.02383	0.02380	0.02377
2.6	0.02583	0.02580	0.02577
2.7	0.02764	0.02761	0.02758
2.8	0.02901	0.02900	0.02898
2.9	0.02984	0.02983	0.02982
3.0	0.03009	0.03009	0.03008
3.1	0.02986	0.02986	0.02987
3.2	0.02931	0.02932	0.02933
3.3	0.02862	0.02864	0.02865
3.4	0.02798	0.02800	0.02801
3.5	0.02754	0.02755	0.02756
3.6	0.02742	0.02742	0.02742
3.7	0.02769	0.02769	0.02768
3.8	0.02839	0.02838	0.02837
3.9	0.02949	0.02947	0.02945
4.0	0.03088	0.03086	0.03083
4.2	0.03392	0.03390	0.03388
4.4	0.03609	0.03609	0.03608
4.6	0.03658	0.03660	0.03662
4.8	0.03576	0.03580	0.03582
5.0	0.03470	0.03474	0.03477
5.2	0.03447	0.03449	0.03450
5.4	0.03561	0.03561	0.03561
5.6	0.03790	0.03790	0.03789
5.8	0.04039	0.04039	0.04039
6.0	0.04192	0.04195	0.04198

Table III: Slotted Plane Transmission Coefficient, $T_E^{(N)}(\phi_o, \eta)$; E-Polarization, Angle of Incidence, $\phi_o = 50^o$ and $0.5 \leq \eta \leq 6.0$

η	N = 45	N = 47	N = 49
0.5	0.0306	0.0305	0.0305
0.6	0.0568	0.0567	0.0566
0.7	0.0964	0.0962	0.0960
0.8	0.1517	0.1513	0.1510
0.9	0.2224	0.2219	0.2215
1.0	0.3043	0.3036	0.3031
1.1	0.3881	0.3875	0.3868
1.2	0.4630	0.4623	0.4617
1.3	0.5198	0.5193	0.5187
1.4	0.5552	0.5547	0.5543
1.5	0.5708	0.5705	0.5701
1.6	0.5715	0.5713	0.5711
1.7	0.5630	0.5628	0.5627
1.8	0.5503	0.5502	0.5500
1.9	0.5376	0.5375	0.5374
2.0	0.5283	0.5281	0.5279
2.1	0.5248	0.5246	0.5244
2.2	0.5294	0.5290	0.5287
2.3	0.5432	0.5427	0.5422
2.4	0.5665	0.5659	0.5653
2.5	0.5983	0.5975	0.5968
2.6	0.6358	0.6350	0.6342
2.7	0.6752	0.6743	0.6735
2.8	0.7121	0.7113	0.7105
2.9	0.7430	0.7422	0.7415
3.0	0.7656	0.7649	0.7643
3.1	0.7794	0.7788	0.7784
3.2	0.7852	0.7848	0.7844
3.3	0.7848	0.7845	0.7842
3.4	0.7801	0.7799	0.7797
3.5	0.7734	0.7732	0.7730
3.6	0.7666	0.7664	0.7662
3.7	0.7613	0.7610	0.7608
3.8	0.7589	0.7586	0.7583
3.9	0.7605	0.7601	0.7597
4.0	0.7663	0.7658	0.7653
4.2	0.7891	0.7884	0.7878
4.4	0.8182	0.8174	0.8167
4.6	0.8411	0.8405	0.8399
4.8	0.8511	0.8507	0.8502
5.0	0.8489	0.8485	0.8482
5.2	0.8393	0.8390	0.8388
5.4	0.8281	0.8278	0.8276
5.6	0.8200	0.8197	0.8193
5.8	0.8172	0.8167	0.8162
6.0	0.8184	0.8179	0.8173

Table IV: Slotted Plane Transmission Coefficient, $T_E^{(N)}(\phi_o, \eta)$; E-Polarization, Angle of Incidence, $\phi_o = 90^o$ and $0.5 \leq \eta \leq 6.0$

η	N = 45	N = 47	N = 49
0.5	0.0535	0.0534	0.0533
0.6	0.1006	0.1003	0.1001
0.7	0.1729	0.1725	0.1721
0.8	0.2763	0.2756	0.2750
0.9	0.4122	0.4113	0.4100
1.0	0.5747	0.5735	0.5723
1.1	0.7485	0.7470	0.7457
1.2	0.9127	0.9112	0.9098
1.3	1.0489	1.0474	1.0461
1.4	1.1473	1.1460	1.1448
1.5	1.2079	1.2069	1.2059
1.6	1.2373	1.2364	1.2356
1.7	1.2438	1.2431	1.2424
1.8	1.2352	1.2346	1.2341
1.9	1.2175	1.2170	1.2164
2.0	1.1947	1.1942	1.1938
2.1	1.1697	1.1692	1.1688
2.2	1.1441	1.1436	1.1432
2.3	1.1189	1.1185	1.1181
2.4	1.0947	1.0943	1.0940
2.5	1.0719	1.0715	1.0712
2.6	1.0506	1.0502	1.0498
2.7	1.0308	1.0304	1.0300
2.8	1.0125	1.0121	1.0117
2.9	0.9958	0.9954	0.9950
3.0	0.9808	0.9803	0.9799
3.1	0.9674	0.9669	0.9665
3.2	0.9559	0.9554	0.9550
3.3	0.9465	0.9460	0.9455
3.4	0.9394	0.9388	0.9383
3.5	0.9350	0.9344	0.9338
3.6	0.9337	0.9330	0.9324
3.7	0.9358	0.9350	0.9343
3.8	0.9415	0.9407	0.9399
3.9	0.9510	0.9501	0.9493
4.0	0.9640	0.9630	0.9621
4.2	0.9969	0.9957	0.9947
4.4	1.0304	1.0293	1.0283
4.6	1.0548	1.0538	1.0529
4.8	1.0661	1.0652	1.0644
5.0	1.0659	1.0651	1.0644
5.2	1.0582	1.0575	1.0569
5.4	1.0466	1.0459	1.0453
5.6	1.0333	1.0327	1.0321
5.8	1.0200	1.0194	1.0188
6.0	1.0076	1.0069	1.0063

REFERENCES

1. H. Kaden, "Wirbelstrome und Schirmung in der Nachrichtentechnik" Technische Physik in Einzeldarstellungen 10, Springer-Verlag (1959).

2. L. F. Libelo and J. Bombardt, NOLTR 70-58, April 1970. (This report will hereafter be referred to as B.L.-I.)

3. J. D. Jackson, page 289, "Classical Electrodynamics". J. Wiley & Sons Inc.

4. J. A. Stratton, (eq.63) page 372), Electromagnetic Theory", McGraw-Hill Book Co. Inc., N.Y. (1941)

5. J. VanBladel, page 405, "Electromagnetic Fields", McGraw-Hill Co., N.Y. (1964).

6. B.L.-I, pages 18-20.

7. I. S. Gradshten and I. W. Rhyzik, Formula #3, Sect. 8.382, page 950, "Table of Integrals, Series and Products". Academic Press, N.Y. (1965)

8. J. M. Greenberg, A. C. Lind, R. T. Wang and L. F. Libelo, "Electromagnetic Scattering" Eds. R. L. Rowell, R. S. Stein, Gordon & Breach, (Science Publishers), N.Y. (1967).

9. P. M. Morse and P. J. Rubenstein, Phys. Rev. 54, 895 (1938).