WL-EMP-IN-115

Interaction Notes

Note 115

July 1972

Balanced Transmission Lines in External Fields

K. S. H. Lee*
Northrop Corporate Laboratories
Pasadena, California

transmission lines, effects of radiation

Abstract

Equations for voltage and current of a balanced transmission line are derived from the viewpoint of wave scattering. It is shown that when such a transmission line is immersed in a time varying external field, two source terms appear in the equations: a longitudinal (series) distributed voltage source and a transverse (shunt) distributed current source. The voltage source is equal to the time rate of change of the total magnetic flux minus the flux of the TEM mode resulting from the terminating impedances of the line. The current source is proportional to the time rate of change of the charge induced in one conductor of the line by the incident and scattered electric fields. Alternatively, this induced charge can be thought of as being proportional to the open-circuited voltage between the conductors caused by the incident electric field. For uniform cylindrical lines, such as a two-conductor line or a coaxial cable with long slits in its sheath, only one two-dimensional static problem need be solved for adequately determining all the coefficients and source terms in the transmission-line equations. Detailed calculations are given for a two-cylinder line. An extension of the theory is also considered to cables that do not support a TEM mode, and to cables that have small apertures in their sheaths.

---

*Now with the Dikewood Corporation, Westwood Research Branch, California.

Table of Contents

Illustrations

I. Introduction

The purpose of this note is threefold: (1) to understand an old subject -- the transmission-line theory -- from the viewpoint of a field theorist; (2) to derive the transmission-line equations as rigorously as possible from Maxwell's equations with all underlying assumptions explicitly stated when the line is exposed to a plane wave; (3) to obtain some general and useful results for certain commonly used transmission lines.

When a plane wave (for example, a nuclear EMP) is incident upon a transmission line terminated at both ends by some impedances (Fig. 1), the usual question arises as to what currents would be induced in these impedances. According to the conventional transmission-line theory, one simply proceeds with the usual set of first-order differential equations for the line voltage and current with a distributed voltage source, which is proportional to the longitudinal component of the electric field of the incident wave (that is, the component parallel to the direction of energy propagation along the transmission line). If this distributed voltage source is looked at from Faraday's law of induction, one would say that this source is due, rather, to the time rate of change of magnetic flux linking the conductors of the line. Then, naturally one would ask what role the transverse electric field plays. (By transverse field we mean the component of the field perpendicular to the transmission line.) This transverse electric field will cause a potential difference between the conductors, which is often referred to as the open-circuited voltage in antenna theory. From the equivalence of Thévenin's and Norton's theorems this voltage source can, of course, be thought of as a short-circuited current source applied across the conductors of the line. This current source is absent in the "old" transmission-line theory and, only recently, its existence has been conjectured and proved.[1]

Let us go back to Figure 1 and digest a little more what is really happening in terms of wave scattering. In Figure 1 we have a two-parallel-conductor transmission line terminated at both ends by some impedances and exposed to a plane wave. We ask what kind of transmission-line equations will describe the voltage and current on the line and, in particular, the voltage

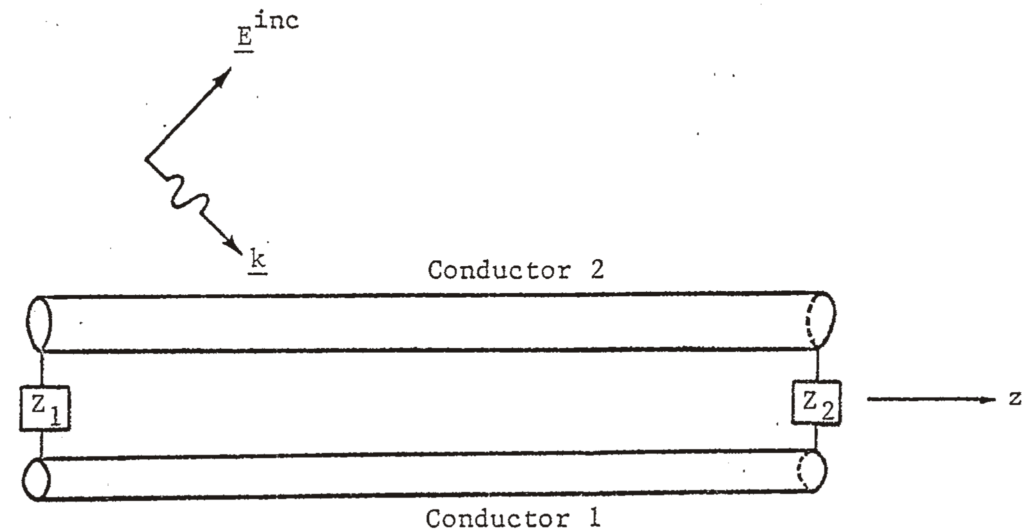

Figure 1a. A terminated transmission line in a plane wave.

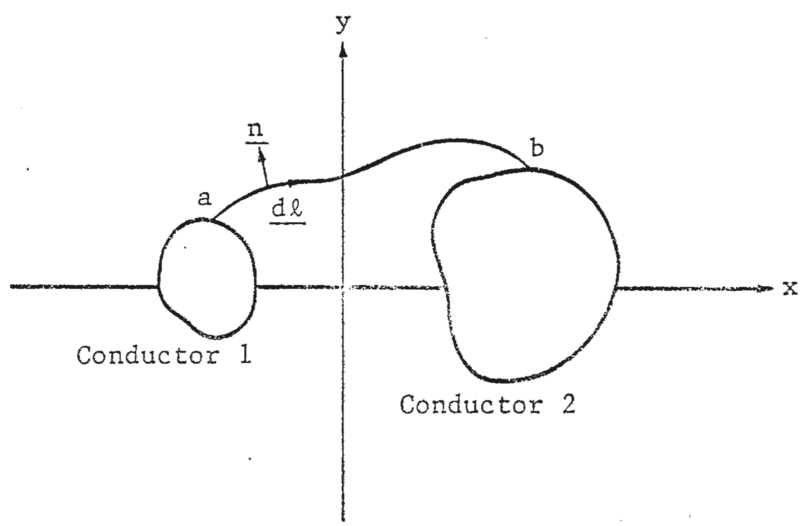

Figure 1b. Cross section of the line shown in Figure 1a (the z-axis points into the page).

or current induced in the terminating impedances. Our following considerations are not limited to a two-conductor line and can be applied to other kind of transmission lines (for example, a coaxial cable with apertures in its sheath). Considering the line in Figure 1 as a wave guiding structure one can easily see that the total field consists of three parts: (1) the incident field, (2) the scattered field from an infinitely long line, and (3) the TEM, TE and TM modes resulting from the inhomogeneities in the line, such as the terminating impedances Z_1 and Z_2 in Figure 1. For wavelengths larger than the cross-sectional dimensions of the line (which is a necessary assumption for the transmission-line theory to hold), these modes, except the TEM mode, decay rapidly away from the inhomogeneities and become negligible at a distance of the order of the separation of the conductors. Thus, in the region bounded by two infinite planes perpendicular to the line and at distances of the order of the separation or so away from Z_1 and Z_2, we have essentially an incident field, a scattered field, and a TEM mode. The total fields comprising these three parts satisfy Maxwell's equations and the boundary conditions on the surfaces of the conductors of the line. (Note that the TEM mode alone satisfies all these conditions.) At each end of the line the boundary condition is that the ratio of the "line voltage" to the "line current" is Z_1 or Z_2.

As soon as lumped elements, like Z_1 and Z_2, are introduced into the picture the circuit concepts, like voltage and current, will be involved. It has often been said that at low frequencies the voltage and current are respectively related to the integrals of the electric and magnetic fields. We now see that the boundary conditions for our problem posed in Figure 1 are of two different types, namely, one involving directly the fields on the surfaces of the line's conductors and the other involving instead the integrals of fields at the ends of the line. Thus, it is clear that our problem can not be treated purely on the basis of field theory. Since our primary interest is in the voltage and current induced in the terminating impedances Z_1 and Z_2, voltage and current should then be treated as important variables from the outset, and equations should be derived for them from Maxwell's field equations. But, what should the appropriate definitions be for the voltage and current for a transmission line in terms of the fields?

Let us first deal with the definition of voltage, V, which always seems to be less questionable. We define (see Fig. 1b)

$$V(z) \equiv -\int_a^b \underline{E}(x,y,z)\cdot \underline{d\ell} \tag{1}$$

where $\underline{E}$ is the total electric field, $\underline{d\ell}$ is a vector line element lying in the transverse xy-plane, a and b are points belonging to conductors 1 and 2 respectively, and the integral is from a to b along any path in the xy-plane. Throughout this note we will always make the assumption that all wavelengths of interest are larger than the cross-sectional dimensions of the line. Equation (1) seems to be the voltage definition that best fits one's physical picture. But the current definition in terms of the magnetic field is, as always, less clear-cut. Do we want equal and oppositely directed currents in the two conductors at every cross section of the line? Or, do we want in our transmission-line equations the current defined to be the line integral of the total tangential magnetic field on only one of the conductors? Here, we will use the former definition for at least two reasons. First, we want a "balanced" line because at each end of the line the currents on the conductors must be equal and oppositely directed. To state it in another way, the currents of the "common mode" (i.e., the currents on the two conductors have the same magnitude and direction) have no direct effect on a terminating impedance. Second, for a "balanced" line the definition for the line's inductance is unambiguous. We will elaborate the second point shortly. Combining the above considerations we are naturally led to the following definition for the transmission line current I:

$$I(z) \equiv \frac{1}{L}\int_a^b [\underline{e}_z \times \underline{B}^{TEM}(x,y,z)]\cdot \underline{d\ell} \tag{2}$$

where $\underline{B}^{TEM}$ is the magnetic field intensity of the TEM mode whose total current flowing on all the conductors at every cross section of the line is zero, and L is the inductance of the line. Since the longitudinal (the z-directed) displacement current of a TEM mode is identically zero, the integral (2), just as the integral (1), is independent of the path of integration. Thus, L is a true constant and, hence, is also the proportionality constant between the magnetic energy and the current squared of the line.

Section II contains the derivation of the equations for a balanced transmission line from Maxwell's equations. It is shown in Section III that for a uniform cylindrical transmission line immersed in a plane wave, only one two-dimensional static problem needs to be solved for determining all the coefficients and source terms in the transmission-line equations. Detailed calculations are given in Section IV for a two-conductor line. Section V considers a possible extension of the theory to cables that do not support a TEM mode (for example, coaxial cables with finitely conducting sheaths) and to cables that have small apertures in their sheaths.

II. Equations for a Balanced Line

With equations (1) and (2) for the definitions of voltage and current we can now proceed to derive a coupled set of equations for them from Maxwell's equations. The time-harmonic factor $e^{-i\omega t}$ will be suppressed throughout.

Let us begin with the Maxwell equation

$$\nabla\times\underline{E} = i\omega\underline{B}$$

$$= i\omega\underline{B}^{TEM} + i\omega\underline{B}' \tag{3}$$

where we have split the total $\underline{B}$ into two parts in accord with our considerations in the Introduction:

$$\underline{B} = \underline{B}^{TEM} + \underline{B}' \tag{4}$$

Scalarly multiplying (3) by $\underline{n}d\ell$, which is equal to $\underline{d\ell}\times\underline{e}_z$ (see Fig. 1b), and noting that

$$d\ell\underline{n}\cdot\nabla\times\underline{E} = -\frac{\partial}{\partial z}\underline{E}\cdot\underline{d\ell} + d\ell\underline{n}\cdot(\nabla_t\times\underline{E})$$

$$\underline{n}\cdot(\nabla_t\times\underline{E}) = (\underline{n}\times\nabla_t)\cdot\underline{E} = (\underline{e}_z\times\underline{n})\cdot\nabla_t E_z = \frac{\partial E_z}{\partial\ell}$$

where $\nabla_t = \nabla - \underline{e}_z\frac{\partial}{\partial z}$, we have

$$-\frac{\partial}{\partial z}\underline{E}\cdot\underline{d\ell} = i\omega\underline{n}\cdot\underline{B}^{TEM}d\ell + i\omega\underline{n}\cdot\underline{B}'d\ell - \frac{\partial E_z}{\partial\ell}d\ell$$

Integrating this equation we get, with $E_z = 0$ on the perfect conductors 1 and 2,

$$-\frac{d}{dz}\int_a^b\underline{E}\cdot\underline{d\ell} = i\omega\int_a^b\underline{n}\cdot\underline{B}^{TEM}d\ell + i\omega\int_a^b\underline{n}\cdot\underline{B}'d\ell$$

which, according to (1) and (2), becomes

$$\frac{dV}{dz} = i\omega LI + v \tag{5}$$

$$v \equiv i\omega\Phi_s \equiv i\omega\int_a^b(\underline{e}_z\times\underline{B}')\cdot\underline{d\ell} \tag{6}$$

Here, the distributed voltage source, v(z), is proportional to the magnetic flux, Φ_s, per unit length linking the conductors of the line. Φ_s is the total flux minus the flux of the TEM mode and can be calculated quite accurately from an appropriate magnetostatic boundary-value problem, which will be stated in the next section.

Let us now turn to the other Maxwell equation

$$\nabla\times\underline{B} = - i\omega\mu\varepsilon\underline{E}$$

or

$$\nabla\times\underline{B}^{TEM} = - i\omega\mu\varepsilon\underline{E} + i\omega\mu\varepsilon\underline{E}' \tag{7}$$

where

$$\underline{E}' \equiv \underline{E} - \underline{E}^{TEM} \tag{8}$$

Scalarly multiplying (7) by $\underline{d\ell}$ and noting that (see Fig. 1b)

$$\underline{d\ell}\cdot\nabla\times\underline{B}^{TEM} = - d\ell\underline{n}\cdot[\underline{e}_z\times\nabla\times\underline{B}^{TEM}] = \frac{\partial}{\partial z}\,\underline{n}\cdot\underline{B}^{TEM}\,d\ell$$

we have

$$\frac{\partial}{\partial z}\,\underline{n}\cdot\underline{B}^{TEM}\,d\ell = - i\omega\mu\varepsilon\underline{E}\cdot\underline{d\ell} + i\omega\mu\varepsilon\underline{E}'\cdot\underline{d\ell}$$

Integrating this equation we get

$$\frac{d}{dz}\int_a^b \underline{n}\cdot\underline{B}^{TEM}\,d\ell = - i\omega\mu\varepsilon\int_a^b \underline{E}\cdot\underline{d\ell} + i\omega\mu\varepsilon\int_a^b \underline{E}'\cdot\underline{d\ell}$$

which, by means of (1) and (2), becomes

$$\frac{dI}{dz} = i\omega CV + i_{sc} \tag{9}$$

$$i_{sc} = - i\omega CV_{oc} \equiv i\omega C\int_a^b \underline{E}'\cdot\underline{d\ell} \tag{10}$$

where the line capacitance, C, per unit length is related to the line inductance L by $LC = \mu\varepsilon$. In antenna theory the integral (10) of the transverse electric field between two conductors is called the open-circuited voltage, V_{oc}, induced between two isolated conductors by an incident electric field. By isolated we mean that the static total charge induced on each conductor is zero. To calculate V_{oc} we will solve only an electrostatic boundary-value problem, which will be stated in the next section. Conceptually, it may be easier for some to visualize the induced charge on one conductor (say, conductor 1 in Figure 1), rather than V_{oc}, as the distributed current source across the line. To do so we make use of the equivalence of Thévenin's and Norton's theorems:

$$i_{sc} = Y_{in} V_{oc}$$

which is exactly the first equation of (10) because the input admittance Y_{in} is equal to $-i\omega C$ at low-frequency limit. This short-circuited current can, in turn, be visualized as the induced charge flowing from, say, conductor 1 to conductor 2 (Fig. 1), or vice versa, along a thin wire connecting the two conductors. From the continuity equation between currents and charges we immediately have

$$i_{sc} = i\omega q_1 \tag{11}$$

Figure 2 shows the direction of this distributed short-circuited current source together with the v, L and C for a section of the line. Thus, instead of calculating V_{oc} one can calculate q_1 from an electrostatic problem with the two conductors at the same potential, which may as well be set equal to zero.

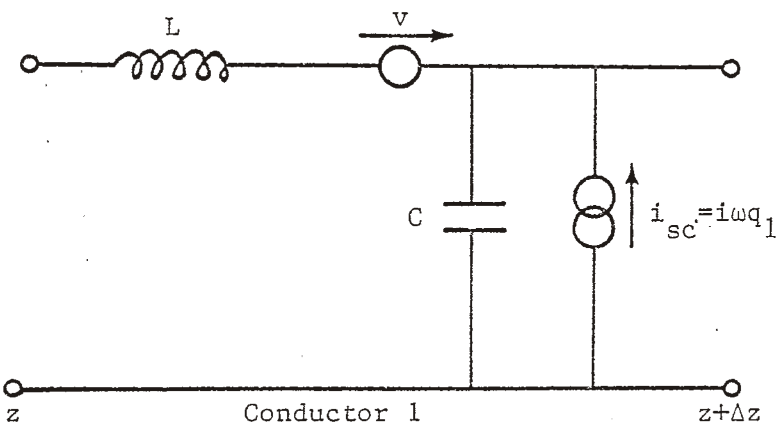

Figure 2. Circuit representation for one section of the line (in agreement with Reference 1).

III. Calculations of the Line Parameters and Two Source Terms

Referring to the transmission-line equations (5) and (9) we see that there are three quantities needed to be calculated: (a) the capacitance C (or the inductance L) per unit length, (b) the distributed current source i_{sc} per unit length across the line, and (c) the distributed voltage source v per unit length along the line. For a uniform cylindrical transmission line such as a two-parallel-wire line or a coaxial cable with infinitely long slits in its sheath, one needs to solve only one electrostatic problem for these three quantities. We will now prove this assertion for two different kinds of external fields.

A. Uniform external fields

By uniform external fields we mean that the static (electric or magnetic) field at infinity is uniform. Mathematically, it means that conditions (ii) and (iii) are satisfied in Problem (b) defined below.

(a) The capacitance problem

Figure 3 shows the cross section of an arbitrary cylindrical transmission line but uniform in the direction of energy propagation. To find the capacitance C per unit length of this line, we will solve the following problem for the electrostatic potential function ϕ_c (see Fig. 3):

(i) $$\nabla^2\phi_c = 0, \qquad \text{exterior to } S_1 \text{ and } S_2$$

(ii) $$\phi_c = \begin{cases} V_1 \text{ on } S_1 \\ V_2 \text{ on } S_2 \end{cases}$$

(V_1 and V_2 are constants to be determined)

(iii) $$\int_{S_2} \frac{\partial\phi_c}{\partial n}\, dS = -\int_{S_1} \frac{\partial\phi_c}{\partial n}\, dS = -Q_c/\varepsilon$$

(Q_c/ε can be set equal to unity)

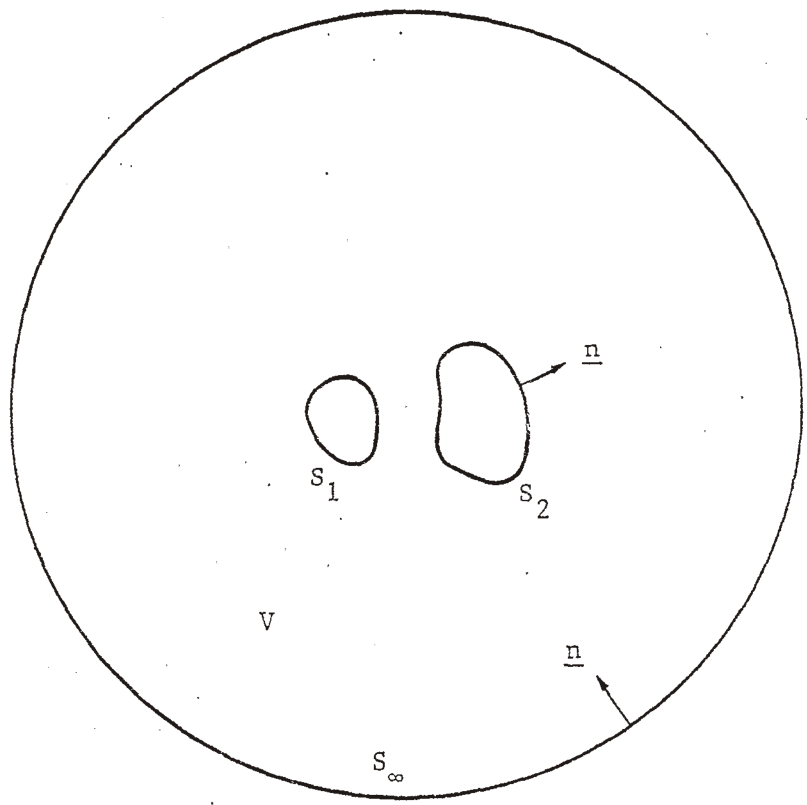

Figure 3. Cross section of a uniform cylindrical line.

(iv) $\phi_c = 0(1/\rho), \quad \rho \to \infty$

($\rho = |\underline{\rho}|$ and $\underline{\rho}$ is a two-dimensional vector)

The capacitance C is then determined from the relation

$$C = \frac{Q_c}{V_2 - V_1} \tag{12}$$

One promising numerical method to solve the problem posed in (i)-(iv) is to set up an integral equation for the charge density when (1) $V_2 = 1$ and $V_1 = -1$, and when (2) $V_2 = 1$ and $V_1 = 1$. Then the integral equation is solved by some appropriate algorithm for (1) and (2). By a proper linear superposition of these two solutions one can then obtain the solution to the capacitance problem.

(b) Calculation of i_{sc}

As is evident from equation (10) i_{sc} is directly related to V_{oc} which can be obtained by solving the following boundary-value problem (Fig. 3):

(i) $\nabla^2\phi = 0, \quad$ exterior to S_1 and S_2

(ii) $\int_{S_2} \frac{\partial\phi}{\partial n}\, dS = \int_{S_1} \frac{\partial\phi}{\partial n}\, dS = 0$

(iii) $\phi = \phi^{inc} + 0(1/\rho), \quad \rho \to \infty$

$\phi^{inc} = - \underline{E}_o \cdot \underline{\rho}$

(iv) $\phi = \begin{cases} \phi_1 & \text{on } S_1 \\ \phi_2 & \text{on } S_2 \end{cases}$

Then the open-circuited voltage V_{oc} is determined from the relation

$$V_{oc} = \phi_2 - \phi_1 \tag{13}$$

We will now show that V_{oc} can be obtained directly from the solution of Problem (a) alone. To show this let us start with the equation

$$\phi_c \nabla^2 \phi - \phi \nabla^2 \phi_c = 0 \tag{14}$$

which is valid within V bounded by S_1, S_2 and S_∞ (a large circle). An application of Guass' theorem gives

$$\int_{S_1+S_2} \left(\phi_c \frac{\partial \phi}{\partial n'} - \phi \frac{\partial \phi_c}{\partial n'} \right) dS' + \int_{S_\infty} \left(\phi_c \frac{\partial \phi}{\partial n'} - \phi \frac{\partial \phi_c}{\partial n'} \right) dS' = 0 \tag{15}$$

where $\underline{n}'$ is the inward unit normal into V. Now, it is easy to see that

$$\int_{S_1+S_2} \left(\phi_c \frac{\partial \phi}{\partial n'} - \phi \frac{\partial \phi_c}{\partial n'} \right) dS' = (\phi_2 - \phi_1) Q_c / \varepsilon \tag{16}$$

and

$$\begin{aligned} \int_{S_\infty} \left(\phi_c \frac{\partial \phi}{\partial n'} - \phi \frac{\partial \phi_c}{\partial n'} \right) dS' &= \int_{S_\infty} \left(\phi_c \frac{\partial \phi^{inc}}{\partial n'} - \phi^{inc} \frac{\partial \phi_c}{\partial n'} \right) dS' \\ &= - \int_{S_1+S_2} \left(\phi_c \frac{\partial \phi^{inc}}{\partial n'} - \phi^{inc} \frac{\partial \phi_c}{\partial n'} \right) dS' \\ &= - \frac{1}{\varepsilon} \int_{S_1+S_2} \phi^{inc} \sigma_c dS' \end{aligned} \tag{17}$$

where we have used the fact that

$$\int_{S_1+S_2} \frac{\partial \phi^{inc}}{\partial n'} dS' = 0$$

and the definition that

$$\sigma_c = - \varepsilon \frac{\partial \phi_c}{\partial n}, \qquad \text{on } S_1 \text{ and } S_2$$

Using (16) and (17) in (15) and $\phi^{inc} = - \underline{E}_o \cdot \underline{\rho}$ we finally arrive at

$$V_{oc} = \phi_2 - \phi_1 = - \underline{E}_o \cdot \underline{h} \qquad (18)$$

where

$$\underline{h} = \frac{\int_{S_1+S_2} \underline{\rho} \sigma_c dS}{\int_{S_2} \sigma_c dS} \qquad (19)$$

Obviously, h can be interpreted as the mean distance between the total charges Q_c and $-Q_c$ in the capacitance problem. That is to say, $\underline{h}$ is the low-frequency limit of the effective height in antenna theory.[6]

(c) Calculation of v

The magnetic flux Φ_s, which gives rise to v, can be calculated by considering two magnetostatic problems defined by the vector potentials $\underline{A}_L$ and $\underline{A}$, in complete analogy with the electrostatic problems defined by ϕ_c and ϕ considered above. Since the problems under consideration are two-dimensional, it is sufficient to choose $\underline{A}_L = \underline{e}_z A_L$ and $\underline{A} = \underline{e}_z A$. Now, A_L is the solution of the problem defined by (i)-(iv) below, while A is the solution of the problem defined by (v)-(viii) below. (See Figure 3).

(i) $\nabla^2 A_L = 0,$ exterior to S_1 and S_2

(ii) A_L = constant on S_1 and S_2

(iii) $\int_{S_2} \frac{\partial A_L}{\partial n} dS = - \int_{S_1} \frac{\partial A_L}{\partial n} dS = \mu I$

(I is the total current flowing in one conductor)

(iv) $A_L = O(1/\rho), \quad \rho \to \infty$

(v) $\nabla^2 A = 0,$ exterior to S_1 and S_2

(vi) $$\int_{S_2} \frac{\partial A}{\partial n}\, dS = \int_{S_1} \frac{\partial A}{\partial n}\, dS = 0$$

(vii) $$A = A^{inc} + 0(1/\rho), \qquad \rho \to \infty$$

$$A^{inc} = \underline{B}_o \cdot (\underline{e}_z \times \underline{\rho}) \qquad \text{so that } \nabla \times \underline{e}_z A^{inc} = \underline{B}_o .$$

(viii) $$A = \begin{cases} A_2 \text{ on } S_2 \\ A_1 \text{ on } S_1 \end{cases}$$

The flux Φ_s is then determined from the relation

$$\Phi_s = A_2 - A_1 \tag{20}$$

in consistence with the sign of v in Figure 2. With A_L substituted for ϕ_c and A for ϕ in Problem (b) one immediately gets

$$\Phi_s = A_2 - A_1 = - \frac{\underline{B}_o \cdot \int_{S_1+S_2} (\underline{e}_z \times \underline{\rho}) K_L dS}{\int_{S_2} K_L dS} \tag{21}$$

where K_L is the current density defined by

$$K_L = \frac{1}{\mu} \frac{\partial A_L}{\partial n} \qquad \text{on } S_1 \text{ and } S_2$$

A comparison between the problem defined for ϕ_c and that for A_L reveals that σ_c and K_L must have the same distribution on S_1 and S_2. Hence, equation (20) can be written as

$$\Phi_s = - \underline{B}_o \cdot (\underline{e}_z \times \underline{h}) \tag{22}$$

where $\underline{h}$ is given by (19). Clearly, $\underline{e}_z \times \underline{h}$ can be interpreted as the vector effective area per unit length.

Thus, we have proved our assertion that for uniform external fields, all three quantities C, i_{sc} and v can be obtained by solving only one electrostatic problem, i.e., the capacitance problem.

B. Nonuniform external fields

By nonuniform external fields we mean that the sources of the electrostatic and magnetostatic fields are located on the conductors of the line. These sources are of course charges and currents. In the case of a coaxial cable with narrow slits or small holes in its sheath exposed to an incident plane wave, the dominant fields that leak into the cable and eventually induce currents in the load impedances may be the fields due to the sheath's charges and currents induced by the incident wave as if the small apertures were absent. In the case of a two-conductor transmission line of unequal size (for example, an umbilical cable attached to a missile[2]), the longitudinal component of the electric field of the incident wave may induce unequal amount of charges and currents on the two conductors. The difference of these induced charges and currents would contribute, respectively, to the source terms i_{sc} and v in addition to the contributions from the transverse components of the electric and magnetic fields of the incident wave as discussed in A. To deal with this kind of nonuniform external fields we proceed as follows, again restricting our considerations to two-dimensional geometries.

In calculating i_{sc} we will solve the problem as posed in Problem (b) above except that conditions (ii) and (iii) are now replaced by (Fig. 3)

(ii)' $$\int_{S_2} \varepsilon \frac{\partial\phi}{\partial n}\, dS = -\, Q_2^{inc}, \qquad \int_{S_1} \varepsilon \frac{\partial\phi}{\partial n}\, dS = -\, Q_1^{inc}$$

(iii)' $$\phi \to \ln \rho, \qquad \rho \to \infty$$

Following the same procedure as in Problem (b) we get

$$V_{oc} = \phi_2 - \phi_1 = \frac{V_2 Q_2^{inc} + V_1 Q_1^{inc}}{Q_c} \tag{23}$$

Using $Q_c = (V_2 - V_1)C$, we get from (23)

$$i_{sc} = - i\omega C V_{oc} = - \left(\frac{V_2}{V_2-V_1}\right) i\omega Q_2^{inc} - \left(\frac{V_1}{V_2-V_1}\right) i\omega Q_1^{inc} \tag{24}$$

Here, V_2 and V_1 are obtained from the capacitance problem.

Let us turn to calculating v. Instead of conditions (vi) and (vii) as stated in Problem (c) above we now have (Fig. 3)

$$\text{(vi)}' \qquad \int_{S_2} \frac{\partial A}{\partial n}\, dS = \mu I_2^{inc}, \qquad \int_{S_1} \frac{\partial A}{\partial n}\, dS = \mu I_1^{inc}$$

$$\text{(vii)}' \qquad A \to \ln \rho, \qquad \rho \to \infty$$

Following exactly the procedure as in Problem (c) we get

$$\begin{aligned} v = i\omega\Phi_s &= i\omega(A_2 - A_1) \\ &= \left(\frac{V_2}{V_2-V_1}\right) i\omega L I_2^{inc} + \left(\frac{V_1}{V_2-V_1}\right) i\omega L I_1^{inc} \end{aligned} \tag{25}$$

Here, as before, the inductance L is related to C by $LC = \mu\varepsilon$.

Thus, we have completed our proof that for nonuniform external fields, all three quantities, C, i_{sc} and v, can be obtained by solving only one problem, i.e., the capacitance problem.

In general, the source terms i_{sc} and v in the transmission-line equations (9) and (5) consist of two parts, viz.,

$$\begin{aligned} i_{sc} &= i_{sc}^{(u)} + i_{sc}^{(n)} \\ v &= v^{(u)} + v^{(n)} \end{aligned} \tag{26}$$

where $i_{sc}^{(u)}$ and $i_{sc}^{(n)}$ are respectively given by (18) and (24), whereas $v^{(u)}$ and $v^{(n)}$ are respectively given by (22) and (25). The superscripts u and n serve to remind us of the uniform and nonuniform external fields. In the next section we will calculate these source terms for a two-parallel-conductor transmission line.

IV. Two Parallel Cylinders

In this section we will apply the theory set out in the previous two sections to a two-conductor transmission line immersed in a time-harmonic plane wave. This means that we will calculate the line's capacitance C, and the two source terms v and i_{sc}. Our calculations apply to an incident plane wave of arbitrary polarization and to two parallel cylinders of unequal size.

Let us first calculate the source terms $v^{(u)}$ and $i_{sc}^{(u)}$ due to uniform external fields. Referring to Figure 4 we can write the incident electric field $\underline{E}^{inc}$ as

$$\underline{E}^{inc} = \underline{E}_o e^{i\underline{k}\cdot\underline{r}} = \underline{E}_o e^{ik_x x} e^{ik_y y} e^{ikz\cos\theta_o} \equiv \underline{E}_{ot} e^{ikz\cos\theta_o} \tag{27}$$

and the incident magnetic field $\underline{H}^{inc}$ as

$$\underline{H}^{inc} = \underline{H}_o e^{i\underline{k}\cdot\underline{r}} = \underline{H}_{ot} e^{ikz\cos\theta_o} \tag{28}$$

where $\underline{E}_{ot}$ and $\underline{H}_{ot}$ are functions of x and y and can be considered as constant under the low-frequency assumption. Hence, the two source terms can be written as

$$v^{(u)} = v_o e^{ikz\cos\theta_o}$$
$$i_{sc}^{(u)} = i_o e^{ikz\cos\theta_o} \tag{29}$$

Here, v_o and i_o will be calculated by the procedure described in Section III.A.

The capacitance C per unit length of two infinite parallel cylinders whose cross sections are shown in Figure 5 is well known and given by (see, for example, Smythe[3])

$$\frac{2\pi\varepsilon}{C} = \cosh^{-1}\left(\frac{D^2-R_1^2-R_2^2}{2R_1R_2}\right) \tag{30}$$

To calculate v_o and i_o we need only to find $\underline{h}$, the vector distance between two mean line charges. To do this we will use expressions given in Chapter IV of Smythe and Figure 5. We begin with the complex potential W given by

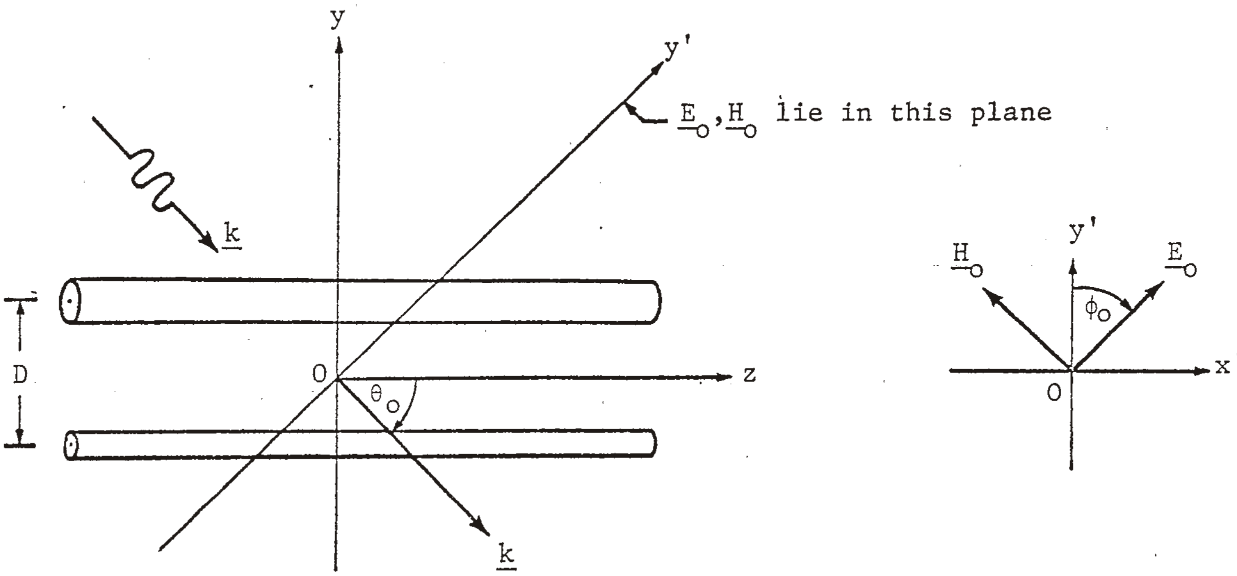

Figure 4. Two parallel cylinders in a plane wave (the x and x'-axes point into the page).

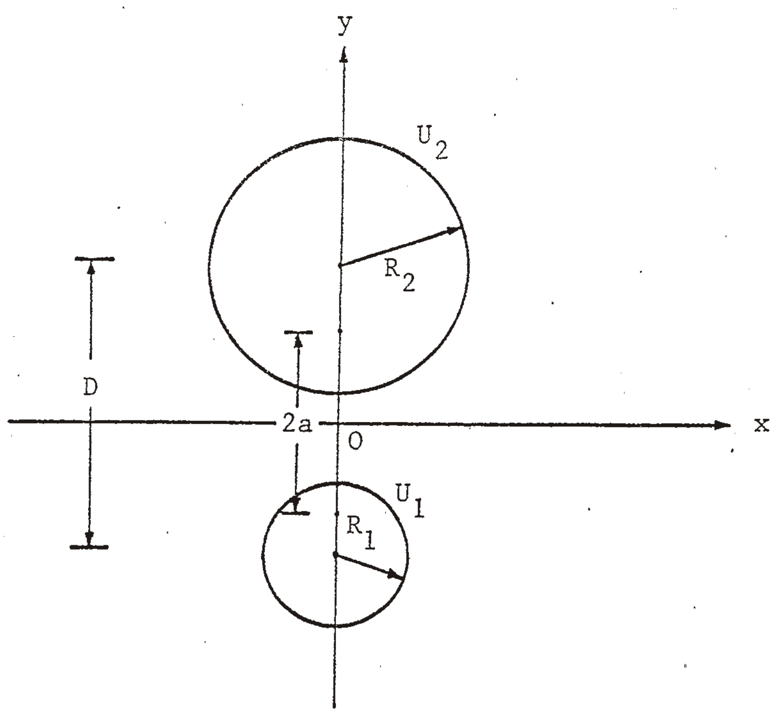

Figure 5. Cross section of Figure 4 (the z-axis points out of the page).

$$W = U + jV = \ln \frac{z+ja}{z-ja} \tag{31}$$

where $z = x + jy$. It is clear that equation (31) is the complex potential of two line charges of charges $2\pi\varepsilon$ at $y = a$, and $-2\pi\varepsilon$ at $y = -a$. On physical grounds one would expect that

$$\underline{h} = 2a\underline{e}_y \tag{32}$$

We will now prove that equation (32) is indeed true, and we will express a in terms of the two radii R_1 and R_2 and the separation D between the centers of the cylinders. First, let us write down the equations of the two circles in Figure 5:

$$\begin{aligned} x^2 + (y - a \coth U_1)^2 &= a^2 \operatorname{csch}^2 U_1 \\ x^2 + (y - a \coth U_2)^2 &= a^2 \operatorname{csch}^2 U_2 \end{aligned} \tag{33}$$

Obviously,

$$R_1 = a|\operatorname{csch} U_1| \tag{34a}$$

$$R_2 = a \operatorname{csch} U_2 \tag{34b}$$

$$D = a[|\coth U_1| + \coth U_2] \tag{34c}$$

From (34a) and (34b) we have

$$\begin{aligned} |\coth U_1| &= \sqrt{1 + (R_1/a)^2} \\ \coth U_2 &= \sqrt{1 + (R_2/a)^2} \end{aligned} \tag{35}$$

Substituting (35) into (34c) and solving for a we get

$$\frac{2a}{D} = \sqrt{1 - \left(\frac{R_1+R_2}{D}\right)^2} \cdot \sqrt{1 - \left(\frac{R_1-R_2}{D}\right)^2} \tag{36}$$

Next, we will find the charge density σ per unit length. From (31) we have

$$\sigma = \varepsilon\left|\frac{dW}{dz}\right| = \frac{2a\varepsilon}{|z^2+a^2|} = \frac{2a\varepsilon}{\sqrt{x^2+(y+a)^2}\ \sqrt{x^2+(y-a)^2}}$$

On the circle with radius R_1, i.e., $x^2 = R_1^2 - (y - a \coth U_1)^2$, we have

$$x^2 + (y^2 + a^2) = R_1^2 - (y - a \coth U_1)^2 + (y + a)^2$$

$$= R_1^2 - \left(y - \sqrt{R_1^2 + a^2}\right)^2 + (y + a)^2$$

$$= 2y\left(\sqrt{R_1^2 + a^2} + a\right)$$

$$x^2 + (y - a)^2 = 2y\left(\sqrt{R_1^2 + a^2} - a\right)$$

Thus, the charge density σ_1 on the cylinder with radius R_1 divided by the total charge Q_1, $Q_1 = -2\pi\varepsilon$, is

$$\frac{\sigma_1}{Q_1} = -\frac{1}{2\pi R_1}\cdot\frac{a}{|y|} \qquad (37)$$

Similarly, the charge density σ_2 on the cylinder with radius R_2 divided by the total charge Q_2, $Q_2 = 2\pi\varepsilon$, is

$$\frac{\sigma_2}{Q_2} = \frac{1}{2\pi R_2}\cdot\frac{a}{y} \qquad (38)$$

It is interesting to note the simplicity of the expressions (37) and (38) for the charge densities. If the origin 0 of the coordinate system (Fig. 5) is chosen to be the mid-point between the centers of the cylinders, expressions (37) and (38) will be modified by a simple linear translation.

We now calculate the equivalent (or, effective) height $\underline{h}$ by formula (19). Clearly, $\underline{h}$ is directed along the y-axis. Thus, substituting (37) and (38) in (19) we obtain

$$\underline{h} = \underline{e}_y \frac{1}{Q_2} \int_{S_1+S_2} y\sigma dS = \underline{e}_y \int_{S_1} \frac{-y\sigma_1}{Q_2} dS + \underline{e}_y \int_{S_2} \frac{y\sigma_2}{Q_2} dS = 2a\underline{e}_y \tag{39}$$

which is identical to (32), as expected. Substitution of (36) into (39) gives

$$\frac{h}{D} = \sqrt{1 - \left(\frac{R_1+R_2}{D}\right)^2} \cdot \sqrt{1 - \left(\frac{R_1-R_2}{D}\right)^2} \tag{40}$$

For two equal cylinders of radius R, or equivalently, for one cylinder of radius R above a perfectly conducting ground plane at a distance D/2, equation (40) reduces to

$$\frac{h}{D} = \sqrt{1 - (\frac{2R}{D})^2} \tag{41}$$

Let us now summarize our results according to (29), (18) and (22):

$$\begin{aligned} V^{(u)} &= i\omega B_{ox} h e^{ikz \cos \theta_o} \\ i_{sc}^{(u)} &= i\omega C E_{oy} h e^{ikz \cos \theta_o} \end{aligned} \tag{42}$$

where h and C are given by (40) and (30).

We now put (42) in the transmission-line equations (5) and (9) and obtain second-order differential equations for V and I as:

$$\begin{aligned} \frac{d^2V}{dz^2} + k^2V &= -(E_{oy} + Z_o H_{ox} \cos \theta_o) k^2 h e^{ikz \cos \theta_o} \\ \frac{d^2I}{dz^2} + k^2I &= -(E_{oy} \cos \theta_o + Z_o H_{ox}) \frac{k^2 h}{Z_c} e^{ikz \cos \theta_o} \end{aligned} \tag{43}$$

where Z_c is the characteristic impedance of the line and is equal to $\sqrt{L/C}$ and $Z_o = 120\pi$ ohms. Let ϕ_o be the angle between $\underline{E}_o$ and the y'-axis (Fig. 4). Then

$$\frac{E_{oy}}{H_{ox}} = \frac{\underline{e}_y \cdot \underline{E}_o}{\underline{e}_x \cdot \underline{H}_o} = \frac{E_o \cos \phi_o \cos \theta_o}{-H_o \cos \phi_o} = -Z_o \cos \theta_o$$

Hence, equations (43) become

$$\frac{1}{k^2}\frac{d^2V}{dz^2} + V = 0 \tag{44a}$$

$$\frac{1}{k^2}\frac{d^2I}{dz^2} + I = \frac{E_o h}{Z_c} \sin^2\theta_o \cos\phi_o e^{ikz\cos\theta_o} \tag{44b}$$

There are two limiting cases of interest: (1) when $\phi_o = \pi/2$ or $\theta_o = 0$ (Fig. 4) the right-hand side of (44b) is identically zero; (2) when $\phi_o = 0$ and $\theta_o = \pi/2$, the right-hand side attains its maximum value. The results of these cases become clear when one thinks of the incident magnetic field linking the two conductors as producing the distributed voltage source and the incident electric field across the two conductors as producing the distributed current source. Alternatively, one can write (44) as a set of first-order coupled equations:

$$\begin{aligned} \frac{dV}{dz} &= i\omega LI - ikZ_oH_oh\cos\phi_o e^{ikz\cos\theta_o} \\ \frac{dI}{dz} &= i\omega CV + ik\frac{E_oh}{Z_c}\cos\phi_o\cos\theta_o e^{ikz\cos\theta_o} \end{aligned} \tag{45}$$

Here, we wish to point out that equations (45) also apply to a two-parallel-plate transmission line with its capacitance numerically equal to (30) and the plate's separation equal to h.

Up to now we have been considering the source terms $v^{(u)}$ and $i_{sc}^{(u)}$ due to uniform external fields. We now go on to consider the source terms $v^{(n)}$ and $i_{sc}^{(n)}$ due to nonuniform external fields which arise, respectively, from the total currents and charges (per unit length) on the conductors induced by the longitudinal component (i.e., the component along the line) of the electric field of the incident wave. If the conductors are of equal size and wavelengths are much greater than D, it is clear that $i_{sc}^{(n)}$ and $v^{(n)}$ given by (24) and (25) are zero because $Q_1^{inc} = Q_2^{inc}$, $I_1^{inc} = I_2^{inc}$ and $V_1 = -V_2$. For cylinders of unequal size one needs to calculate the factors $V_2/(V_2 - V_1)$ and $V_1/(V_2 - V_1)$. It follows easily from equations (33) to (36) that

$$\frac{V_2}{V_2-V_1} = \frac{U_2}{U_2-U_1} = \frac{C}{2\pi\varepsilon} \operatorname{csch}^{-1}(\alpha R_2) \equiv p_2$$

$$\frac{V_1}{V_2-V_1} = \frac{U_1}{U_2-U_1} = -\frac{C}{2\pi\varepsilon} \operatorname{csch}^{-1}(\alpha R_1) \equiv -p_1 \tag{46}$$

where C is given by (30) and

$$\alpha = \frac{2D}{\sqrt{D^2-(R_1+R_2)^2}\,\sqrt{D^2-(R_1-R_2)^2}}$$

Thus, $i_{sc}^{(n)}$ and $v^{(n)}$ will be completely determined by (24) and (25) once the Q^{inc}'s and I^{inc}'s are known. It would be extremely valuable if one could estimate for general incidence the relative quantitative importance of $i_{sc}^{(u)}$ versus $i_{sc}^{(n)}$, and $v^{(u)}$ versus $v^{(n)}$. But this seems too difficult, if not impossible, to do.

Thus, for two parallel cylinders of unequal size and for an incident wave of arbitrary incidence and polarization, the total voltage and current sources are given by (26), (42), (24), (25) and (46):

$$v = v^{(u)} + v^{(n)}$$

$$= -ikZ_oH_oh\cos\phi_o e^{ikz\cos\theta_o} + i\omega L(p_2I_2^{inc} - p_1I_1^{inc})$$

$$i = i^{(u)} + i^{(n)}$$

$$= ik\frac{E_oh}{Z_c}\cos\phi_o\cos\theta_o e^{ikz\cos\theta_o} - i\omega(p_2Q_2^{inc} - p_1Q_1^{inc})$$

where I_2^{inc}, I_1^{inc}, Q_2^{inc} and Q_1^{inc} are the total axial currents and charges (per unit length) induced on the cylinders. Of course, these currents and charges are related by the continuity equation.

V. Cables With Imperfect Sheaths

In the previous sections we have considered the excitations by external fields of uniform cylindrical transmission lines which can support a TEM mode. A two-conductor line and a coaxial cable with long slits in its sheath are two typical examples. For this class of transmission lines it has been shown in Section III that only one two-dimensional static problem need be solved for determining all the coefficients and source terms in the transmission-line equations. We will now discuss briefly the extension of our previous considerations to other classes of transmission lines and cables and relegate the detailed calculations to a future study. Specifically, we want to consider two classes: (a) cables which can support a TEM mode but have isolated inhomogeneities in the direction of energy propagation, a coaxial cable with small isolated holes in its sheath being one example; (b) cables which do not support a TEM mode but have a "quasi-TEM mode" as the dominant mode, a coaxial line with a highly conducting sheath being one example. For class (a), two static (electrostatic and magnetostatic) problems need be solved for determining the two source terms, since the problem is no longer two-dimensional but, rather, three-dimensional. For class (b), the current used in the transmission-line equations will be defined in terms of the magnetic field of the dominant mode that can be excited within the cable. In fact, the magnetic fields of all other modes, except that of the axisymmetric TM mode, are irrelevant in the current definition, since the current of interest is the total longitudinal current flowing along the cable. Of course, if the cable's sheath is perfect, the dominant mode will be the TEM mode.

To begin our discussion let us write down from Section II two important equations generalized for cables with imperfect sheaths (Fig. 6a):

$$-\frac{d}{dz}\int_a^b \underline{E}\cdot d\underline{\ell} = i\omega \int_a^b \underline{n}\cdot\underline{B}^{(o)} d\ell - \int_a^b \frac{\partial E_z^{(o)}}{\partial \ell}\, d\ell + \int_a^b \left(i\omega\underline{n}\cdot\underline{B}' - \frac{\partial E_z'}{\partial \ell}\right) d\ell \qquad (47)$$

$$\frac{d}{dz}\int_a^b \underline{n}\cdot\underline{B}^{(o)} d\ell = -\, i\omega\mu\varepsilon \int_a^b \underline{E}\cdot d\underline{\ell} + \int_a^b i\omega\mu\varepsilon\underline{E}'\cdot d\underline{\ell} + \int_a^b \frac{\partial B_z^{(o)}}{\partial n}\, d\ell \qquad (48)$$

These equations are a direct consequence of Maxwell's equations after performing

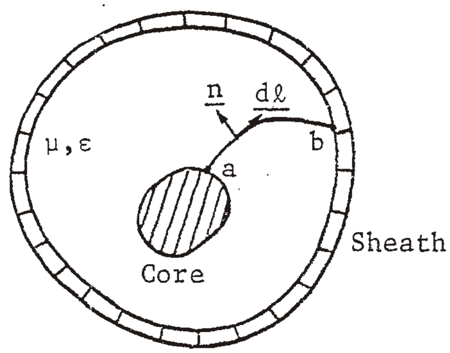

Figure 6a. Cross section of a cylindrical cable (the z-axis points into the page).

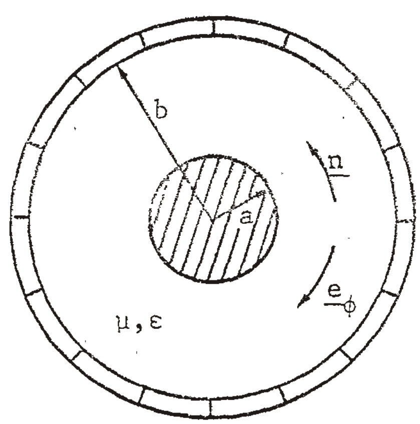

Figure 6b. Cross section of a coaxial line (the z-axis points into the page).

some vector algebra. Here, we have split the total fields $\underline{E}$ and $\underline{B}$ as

$$\underline{E} = \underline{E}^{(o)} + \underline{E}', \qquad \underline{B} = \underline{B}^{(o)} + \underline{B}'$$

with the superscript o indicating the quantity of the dominant mode. If the dominant mode is TEM and if the transmission line is perfectly conducting, equations (47) and (48) will reduce to the corresponding equations in Section II.

For coaxial cables, the appropriate reference frame is the cylindrical coordinate system (ρ,ϕ,z). Referring to Figure 6 we have $\underline{n} = -\underline{e}_\phi$, $\underline{d\ell} = \underline{e}_\rho d\rho$. Hence, it is clear that the last term in (48) can be discarded, since the dominant mode used in defining the transmission line current is ϕ-independent. If one writes

$$\int_a^b B_\phi^{(o)} d\rho = - LI$$

where $\underline{I}$ is the total axial current in the inner conductor and flows along the negative z-axis, and L is a positive constant and has a value equal to $(\mu/2\pi)\ln(b/a)$ henries per meter for a coaxial line with perfectly conducting walls (see Figs. 6), then equations (47) and (48) become, with equation (1) for the voltage definition,

$$\frac{dV}{dz} = i\omega LI - E_z^{(o)}(b) + E_z^{(o)}(a) + v \qquad (49)$$

$$\frac{dI}{dz} = i\omega CV + i_{sc} \qquad (50)$$

Here, $C = \mu\varepsilon/L$ and

$$v = - i\omega \int_a^b B_\phi' d\rho - E_z'(b) + E_z'(a) \qquad (51.a)$$

$$i_{sc} = i\omega C \int_a^b E_\rho' d\rho \qquad (51.b)$$

Let us emphasize again that $\underline{B}'$ and $\underline{E}'$ are the total fields minus the fields of the dominant mode and are calculated by solving a scattering problem for an unterminated line.

For the case of a coaxial line with a small aperture at $z = z_a$ in the otherwise perfectly conducting sheath one can write, to a high degree of accuracy,[7]

$$
\begin{aligned}
v(z) &= V_a \delta(z - z_a) \\
i_{sc}(z) &= I_a \delta(z - z_a)
\end{aligned}
\tag{52}
$$

where

$$V_a = - i\omega \int_{-\infty}^{\infty} dz \int_a^b B_\phi' d\rho$$

$$I_a = i\omega C \int_{-\infty}^{\infty} dz \int_a^b E_\rho' d\rho = i\omega Q_1$$

Q_1 is the total induced charge on the inner conductor, as has been discussed at the end of Section II. It is conceptually simpler to think of the localized sources (52) than the distributed sources (51). This consideration can be of course extended to many isolated small holes or small holes with periodic distribution (e.g., a coaxial cable with a braid shield) provided that mutual interactions among the holes are adequately taken into account. Detailed calculations for V_a and I_a are now being carried out.[4]

For a coaxial line with a highly conducting sheath the following standard assumptions are usually made to equations (49) and (50):[5]

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a}$$

$$E_z^{(o)}(a) = - Z_a I, \qquad E_z^{(o)}(b) = Z_{bb} I$$

$$v(z) = - E_z'(b)$$

$$i_{sc}(z) = 0$$

The quantity Z_{bb} is the impedance at the inside surface of the outer conductor and has the value equal to the longitudinal electric field there divided by the total axial current in the inner conductor. To see how the general expressions (51) for the source terms reduce to their simple form in the present case one recalls that for all frequencies of practical interest, the skin effects of the sheath make the fields, B'_ϕ, E'_ρ, $E_z^{(o)}(a)$, within the sheath insignificant compared to $E_z^{(o)}(b)$. In this connection, Chapter XI of reference [3] can be consulted.

Acknowledgment

Thanks go to Dr. C. E. Baum for initiating this program and providing generously his suggestions; to Drs. R. W. Latham and L. Marin for some enlightening discussions; and to Mrs. G. Peralta for typing the manuscript and drawing the illustrations.

References

1. Latham, R. W., "An approach to certain cable shielding calculations," Interaction Notes, Note 90, January 1972.
2. Harrison, C. W., "Missile with attached umbilical cable as a receiving antenna," Interaction Notes, Note 13, February 1963.
3. Smythe, W. R., Static and Dynamic Electricity, McGraw-Hill, New York, 1950.
4. Latham, R. W., "Small holes in cable shields," Interaction Notes, to be published.
5. Bates, C. P. and G. T. Hawley, "A model for currents and voltages induced within long transmission cables by an electromagnetic waves," IEEE Transactions on Electromagnetic Compatibility, Vol. EMC-13, No. 4, November 1971.
6. Baum, C. E., "Design of a pulse-radiating dipole antenna as related to high-frequency and low-frequency limits," Sensor and Simulation Notes, Note 69, January 1969.
7. Taylor, C. D. and C. W. Harrison, Jr., "On the excitation of a coaxial line through a small aperture in the outer sheath," Interaction Notes, Note 104, January 1972.