WL-EMP-IN-139

INTERACTION NOTE

Note 139

January 1972

INTEGRAL EQUATIONS FOR CURRENTS INDUCED ON A WIRE MODEL OF A PARKED AIRCRAFT

induced currents, effects of radiation, aircraft

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ABSTRACT

A set of coupled integral equations of the Hallén type are derived for a wire model of a parked aircraft illuminated by a time-harmonic electromagnetic field. The model is above a perfectly conducting ground plane of infinite extent and is grounded. The grounding strap, the wings, the nose and fuselage sections, and the horizontal and vertical stabilizers are included in the model.

CURRENTS INDUCED ON WIRE MODEL OF A PARKED AIRCRAFT

It is of interest to determine the currents induced on a parked aircraft illuminated by an electromagnetic wave, at least to a reasonably good approximation. To this end, the aircraft is represented by a thin-wire model, such as that depicted in Figure 1a, and a set of coupled, Hallén-type integral equations is formulated. From these equations, one may determine the total axial currents induced on the wires.

In Figure 1a, one identifies those wire segments which represent various members of an aircraft: wings, fuselage, nose section, vertical stabilizer, and horizontal stabilizer. The parking apron is represented by a perfectly conducting ground plane of infinite extent. In addition, the model provides for the possibility that the aircraft may be grounded by a strap (wire) connected between the ground plane and the common junction of the wings, nose section, and fuselage.

Hallén-type integral equations for induced current on a general crossed-wire structure in free space have been formulated by Taylor [1] whose work is based upon an earlier, important contribution to thin-wire theory by Mei [2]. Taylor utilizes in his analysis an auxiliary scalar function introduced by Mei, and more recently it has been demonstrated [3] that integral equation formulation for thin-wire structures can be based entirely upon the familiar magnetic vector potential and electric scalar potential whenever the wires are not curved. Taylor et.al [4] and Crow and Shumpert [6,7] have studied the problem of induced currents on aircraft with thin-wire structure models extensively. However, they have not considered the present case involving the presence of a ground plane or the added complication due to the grounding strap.

The aircraft model of Figure 1a is illustrated again in Figure 1b where more attention is given to the details of the coordinate system and where only the wire axes are illustrated. The radii and lengths of the members of the airfcraft model are tabulated below.

Element	Radius	Length (axial)
fuselage	a^f	f
nose	a^n $(= a^f = a^{nf})$	n
wings	a^w	w
vertical stabilizer	a^s	h
horizontal stabilizer	a^t	t
grounding strap	a^g	g

The circularly cylindrical wires of the model are perfectly conducting tubes having walls of vanishing thickness. To facilitate the present analysis, each element of the model is either parallel or perpendicular to each other element as well as to the infinite ground plane.

Formulation

From the outset, one assumes that currents reside on the surfaces of the perfectly conducting wire elements (cylindrical tubes) and that the current density may be treated as being essentially uniform around the periphery of an element, which, of course, is less accurate near to than remote from the junctions.

In this note, the integral equation formulation is based upon the familiar magnetic vector potential $\overline{A}$ and electric scalar potential Φ. As usual, to develop a Hallén-type integral equation, one independently establishes two expressions for a given component of the vector potential in the direction of a wire element and evaluated on the surface of this wire: (1) one expression is obtained from the solution of an inhomogenous differential equation which the component of vector potential satisfies and (2) the other is obtained directly from the familiar potential integral for $\overline{A}$. These two expressions must be equal and, hence, are equated. Next, one imposes the condition that the tangential component of the electric field on the surface of the element be zero and evaluates arbitrary constants of integration which arise in the solution of the inhomogenous differential equation for the component of vector potential along the element. The constants of integration must assume values which render the resulting integral equations in compliance with boundary conditions. As an alternative to explicit evaluation of constants, one finds it more convenient in the present analysis to add to the set of integral equations auxiliary equations which constrain the constants, and, thus, to arrive at a consistent system of equations which, in principle, can be solved.

The physical principles which, in general, form the basis for evaluating the constants or for deriving additional equations which constrain the constants are listed below.

I. The magnetic vector potential component in any direction must be continuous; that is, for example, A_x, A_y, and A_z each must be continuous everywhere along a wire surface, and, in particular, at wire junctions.

II. The scalar potential must be continuous along any wire surface. The constants are constrained to enforce continuity of scalar potential, in particular, at wire junctions.

III. At a junction of wires, the sum of the currents must be zero; this is Kirchhoff's current law, which only says that charges may not accumulate at a junction.

IV. The conduction current density must be zero at free wire ends.

V. Often one can reduce the complexity of a set of equations by recognizing symmetry properties of the wire structure under study; for example, knowing that the scalar potential is zero at a particular point on a wire structure, one usually is able to evaluate a constant of integration directly and, thereby, can lessen the labor of obtaining solutions.

Condition I above follows directly from the discussion in [8], and II and III are stated in [9]; IV simply requires that the current be zero at the free ends of the wire segments.

<u>Expressions for Components of $\bar{E}$ in Terms of $\bar{A}$</u>

To begin the analysis, one determines in three dimensions how the electric field $\overline{E}$ is related to the vector potential $\overline{A}$, both of which are said to be "produced" by the currents in the region. In particular, relationships are derived which delineate how a component of $\overline{E}$, evaluated along a line parallel to the direction of this component, depends upon $\overline{A}$ and, subsequently, how this component of $\overline{E}$ varies as a function of the currents. In infinite free-space, the electric field $\overline{E}(\overline{r})$ at a point $\overline{r}$ can be written in terms of the magnetic vector potential $\overline{A}(\overline{r})$ at $\overline{r}$ as

$$\overline{E}(\overline{r}) = - j\omega\overline{A}(\overline{r}) - j\frac{\omega}{k^2}\text{grad}\left(\text{div }\overline{A}(\overline{r})\right) \qquad (1)$$

where, of course, ω is the angular frequency and k is 2π/wavelength. One may obtain the z-component of the electric field directly:

$$E_z(\overline{r}) = -j\frac{\omega}{k^2}\left[k^2A_z(\overline{r}) + \frac{\partial}{\partial z}\left(\text{div }\overline{A}(\overline{r})\right)\right]. \qquad (2)$$

At this point, one finds it convenient to define a transverse (to the z direction) vector and a divergence operator transverse to the z direction in a Cartesian system:

$$\overline{A}_{t_z}(\overline{r}) = \overline{A}(\overline{r}) - A_z(\overline{r})\ \hat{u}_z = A_x(\overline{r})\ \hat{u}_x + Ay(\overline{r})\ \hat{u}_y \quad (3a)$$

and

$$\mathrm{div}_{t_z}\left[\overline{A}_{t_z}(\overline{r})\right] = \mathrm{div}\ \overline{A}(\overline{r}) - \frac{\partial}{\partial z}A_z(\overline{r})$$

$$= \frac{\partial}{\partial x}A_x(\overline{r}) + \frac{\partial}{\partial y}A_y(\overline{r}) \ . \tag{3b}$$

The notation t_z implies transverse* to the z-coordinate. Vectors transverse to the other two coordinates can be defined in an analagous way as can divergence operators transverse to these coordinates:

$$\mathrm{div}_{t_x}\overline{A}_{t_x}(\overline{r}) = \frac{\partial}{\partial y}A_y(\overline{r}) + \frac{\partial}{\partial z}A_z(\overline{r}) \tag{3c}$$

and

$$\mathrm{div}_{t_y}\overline{A}_{t_y}(\overline{r}) = \frac{\partial}{\partial x}A_x(\overline{r}) + \frac{\partial}{\partial z}A_z(\overline{r}) \ , \tag{3d}$$

where, of course, t_x and t_y denote transverse to x and to y, respectively.

Making use of the definitions of (3) as well as the expression of (2), one may write the differential equation,

$$\frac{\partial^2}{\partial z^2}A_z + k^2 A_z = j\frac{k^2}{\omega}E_z - \frac{\partial}{\partial z}\left[\mathrm{div}_{t_z}\overline{A}_{t_z}\right] \ . \tag{4}$$

*The letter t also is used to denote horizontal stabilizer (tail) but this dual usage should not cause confusion.

The above procedure can be extended to obtain equations involving the other components of the electric field at a general point $\overline{r}$ in space; the resulting equations are collected below:

$$\frac{\partial^2}{\partial x^2} A_x + k^2 A_x = j\frac{k^2}{\omega} E_x - \frac{\partial}{\partial x}\left[\operatorname{div}_{t_x} \overline{A}_{t_x}\right], \tag{5a}$$

$$\frac{\partial^2}{\partial y^2} A_y + k^2 A_y = j\frac{k^2}{\omega} E_y - \frac{\partial}{\partial y}\left[\operatorname{div}_{t_y} \overline{A}_{t_y}\right], \tag{5b}$$

$$\frac{\partial^2}{\partial z^2} A_z + k^2 A_z = j\frac{k^2}{\omega} E_z - \frac{\partial}{\partial z}\left[\operatorname{div}_{t_z} \overline{A}_{t_z}\right]. \tag{5c}$$

Each equation, (5a) through (5c) above, is inhomogeneous and is the familiar harmonic differential equation for a specified component of the magnetic vector potential. In each case, the inhomogeneous term involves a component of the electric field, which is a known quantity in subsequent analysis, as well as a term including components of the vector potential $\overline{A}$. From solutions of Equations (5a) through (5c), one may obtain formal relationships among the unknown components of the vector potential which lead to relationships among the unknown components of current.

Next one considers Equation (5c) which involves the z-component of electric field E_z and seeks to obtain its formal solution along any line in space parallel to the z axis or, of course, along the z axis itself.

$$A_z(z) = C_z \cos kz + B_z' \sin kz$$
$$+ \frac{1}{k}\int_{\xi=z_0}^{z}\left\{ j\frac{k^2}{\omega} E_z(\xi) - \frac{\partial}{\partial\xi}\left[\mathrm{div}_{t_z} \bar{A}_{t_z}\right]_{\xi}\right\}\sin k(z-\xi)\, d\xi, \quad (6)$$

where C_z and B_z' are arbitrary constants. The lower limit z_0 of the above integral is any point within the region over which z varies and $\left[\mathrm{div}_{t_z} \bar{A}_{t_z}\right]_{\xi}$ is evaluated along the above-mentioned line with the subscript ξ indicating that it is a function only of ξ. Equations (5a) and (5b) yield expressions for A_x and A_y corresponding in form to (6). Equation (6) and the two for A_x and A_y can be cast into a more suitable form for later purposes, if the second term in the integral of each is integrated by parts once. On the next page are collected the desired equations for the components of $\bar{A}$.

The terms $v_x(x)$, $v_y(y)$, and $v_z(z)$ in Equations (7) are defined

$$v_x(x) = j\frac{k}{\omega}\int_{\xi=x_0}^{x} E_x(\xi)\sin k(x-\xi)\, d\xi \quad , \qquad (8a)$$

$$A_x(x) = C_x \cos kx + B_x' \sin kx + v_x(x) + \frac{1}{k}\left[\operatorname{div}_{t_x}\overline{A}_{t_x}\right]_{x_0} \sin k(x - x_0)$$

$$- \int_{\xi = x_0}^{x} \left[\operatorname{div}_{t_x}\overline{A}_{t_x}\right]_{\xi} \cos k(x - \xi)\, d\xi \text{ , along x-directed line;} \tag{7a}$$

$$A_y(y) = C_y \cos ky + B_y' \sin ky + v_y(y) + \frac{1}{k}\left[\operatorname{div}_{t_y}\overline{A}_{t_y}\right]_{y_0} \sin k(y - y_0)$$

$$- \int_{\xi = y_0}^{y} \left[\operatorname{div}_{t_y}\overline{A}_{t_y}\right]_{\xi} \cos k(y - \xi)\, d\xi , \quad \text{along y-directed line;} \tag{7b}$$

$$A_z(z) = C_z \cos kz + B_z' \sin kz + v_z(z) + \frac{1}{k}\left[\operatorname{div}_{t_z}\overline{A}_{t_z}\right]_{z_0} \sin k(z - z_0)$$

$$- \int_{\xi = z_0}^{z} \left[\operatorname{div}_{t_z}\overline{A}_{t_z}\right]_{\xi} \cos k(z - \xi)\, d\xi , \quad \text{along z-directed line.} \tag{7c}$$

$$v_y(y) = j\frac{k}{\omega} \int_{\xi=y_0}^{y} E_y(\xi) \sin k(y - \xi)\, d\xi \quad , \tag{8b}$$

$$v_z(z) = j\frac{k}{\omega} \int_{\xi=z_0}^{z} E_z(\xi) \sin k(z - \xi)\, d\xi \quad , \tag{8c}$$

and are readily determined since, for a given structure, E_x, E_y, and E_z are known on the x-directed, the y-directed, and the z-directed wire elements, respectively.

In the expression for each solution, Equation (7), one notices that care is taken to explicitly specify a line along which a given equation is valid. The reason for this stipulation will be evident later but it should be clear at this point that, since each component of the vector potential is, in general, a function of all three coordinates (x, y, z), one may obtain solutions of the form of (7a) through (7c) only along coordinate axes, or lines parallel to coordinate axes in space. Otherwise, some mechanism to include the variation in three dimensions must appear in each equation.

Scalar Potential from Vector Potential

An important step in the development of a set of integral equations for a wire structure under study is the evaluation of various arbitrary constants of integration, e.g., the C's and the B's in Equations (7), which occur in the solutions to the differential equations (5). To evaluate such constants, one must, of course, impose additional physical conditions or constraints upon the system of equations. One physical condition which can be used to evaluate the constants of integration is the fact that the scalar potential Φ must be continuous everywhere and, in particular, that it must be continuous at wire junctions (Condition II).

To determine a relationship associated with the scalar potential along a line parallel to the z-axis, one first calculates $\frac{\partial}{\partial z} A_z$ by differentiating (7c) and, after a few simple steps, arrives at

$$\frac{\partial}{\partial z} A_z(z) = - kC_z \sin kz + kB_z' \cos kz + \frac{\partial}{\partial z} v_z(z)$$

$$+ \left[\mathrm{div}_{t_z} \bar{A}_{t_z}\right]_{z_0} \cdot \cos k(z - z_0) - \left[\mathrm{div}_{t_z} \bar{A}_{t_z}\right]_z$$

$$+ k \int_{\xi=z_0}^{z} \left[\mathrm{div}_{t_z} \bar{A}_{t_z}\right]_{\xi} \sin k(z - \xi)\, d\xi \,. \qquad (9)$$

Notice that a minor rearrangement of terms in (9) and an appeal to (3b), together with the Lorentz condition, enable one to recognize within (9) an expression for $-j\frac{k^2}{\omega}\Phi(z)$, where $\Phi(z)$ is the total scalar potential evaluated along the z-directed line. Similar manipulation of (7a) and (7b) results in other corresponding equations for scalar potential, all of which are given on the following page (Equations (10)). In Equations (10), $\Phi(x)$, $\Phi(y)$, and $\Phi(z)$ each is the total scalar potential due to all induced charges on the entire wire model. However, one observes that each is written in terms of the arbitrary constants peculiar to the line along which it is evaluated. Subsequently, the x-, y-, and z-directed lines alluded to here are specified to reside on the surfaces of wire elements in the structure.

Particularization of Equations to Aircraft Model

Application of (7b) to the right-hand wing $y\varepsilon(-w,0)$ and to the left-hand wing $y\varepsilon(0,w)$ results in the two equations below for the vector potential along each of these members of the aircraft model:

$$A_y^{w-}(y) = C_y^{w-}\cos ky + B_y^{w-}\sin ky + v_y^{w-}(y) \qquad (11a)$$

$$-\int_{\xi=0}^{y}\left\{\operatorname{div}_{t_y}\overline{A}_{t_y}\right\}_{\xi}^{w-}\cos k(y-\xi)\,d\xi\ ,$$

$$y\varepsilon(-w,0);$$

$$-j\frac{k^2}{\omega}\Phi(x) = -kC_x \sin kx + kB'_x \cos kx + \frac{\partial}{\partial x} v_x(x) + \left(\mathrm{div}_{t_x}\bar{A}_{t_x}\right)_{x_0} \cos k(x-x_0)$$

$$+\,k\int_{\xi=x_0}^{x}\left(\mathrm{div}_{t_x}\bar{A}_{t_x}\right)_{\xi} \sin k(x-\xi)\,d\xi, \qquad \text{along x-directed line;} \qquad (10a)$$

$$-j\frac{k^2}{\omega}\Phi(y) = -kC_y \sin ky + kB'_y \cos ky + \frac{\partial}{\partial y} v_y(y) + \left(\mathrm{div}_{t_y}\bar{A}_{t_y}\right)_{y_0} \cos k(y-y_0)$$

$$+\,k\int_{\xi=y_0}^{y}\left(\mathrm{div}_{t_y}\bar{A}_{t_y}\right)_{\xi} \sin k(y-\xi)\,d\xi, \qquad \text{along y-directed line;} \qquad (10b)$$

$$-j\frac{k^2}{\omega}\Phi(z) = -kC_z \sin kz + kB'_z \cos kz + \frac{\partial}{\partial z} v_z(z) + \left(\mathrm{div}_{t_z}\bar{A}_{t_z}\right)_{z_0} \cos k(z-z_0)$$

$$+\,k\int_{\xi=z_0}^{z}\left(\mathrm{div}_{t_z}\bar{A}_{t_z}\right)_{\xi} \sin k(z-\xi)\,d\xi, \qquad \text{along z-directed line.} \qquad (10c)$$

and

$$A_y^{w+}(y) = C_y^{w+} \cos ky + B_y^{w+} \sin ky + v_y^{w+}(y)$$

$$- \int_{\xi=0}^{y} \left[\operatorname{div}_{t_y} \bar{A}_{t_y} \right]_{\xi}^{w+} \cos k(y-\xi)\, d\xi, \quad y\varepsilon(0,w). \qquad (11b)$$

The superscripts, w^- and w^+, in Equations (11) denote* that the quantity so labeled is evaluated on the surface of the right-hand and left-hand wing, respectively. The lower limits of integration y_0^{w-} and y_0^{w+} are selected to be zero to lessen subsequent equation complexity and the arbitrary constants B_y^{w-} and B_y^{w+} are defined (with $y_0^{w-} = y_0^{w+} = 0$ in Equation (7b)):

$$B_y^{w-} = B_y^{'w-} + \frac{1}{k} \left[\operatorname{div}_{t_y} \bar{A}_{t_y} \right]_0^{w-} \qquad (12a)$$

and

$$B_y^{w+} = B_y^{'w+} + \frac{1}{k} \left[\operatorname{div}_{t_y} \bar{A}_{t_y} \right]_0^{w+} . \qquad (12b)$$

From (8b) one sees that $v_y^{w\pm}(y) = 0$ and $\frac{\partial}{\partial y} v_y^{w\pm}(y) = 0$ at $y = 0$. Thus, in view of Condition I, one concludes that $C_y^{w-} = C_y^{w+}$ from evaluation of (11a) and (11b) at the wing junction; also, from Condition II and evaluation of $\Phi_y^{w-}(y)$

*Similar notation is used to represent quantities associated with other elements of the model.

and $\Phi_y^{w+}(y)$ (see Equation (10b)) at this junction, it is seen that $B_y^{w-} = B_y^{w+}$. Hence, Equations (11a) and (11b) reduce to a single equation which suffices for the y-component of vector potential along the wings in the region $y\varepsilon(-w,w)$. A similar procedure reveals that a single equation suffices for the y-component of vector potential along the two elements of the horizontal stabilizer $y\varepsilon(-t,t)$ and another single equation along the nose-fuselage element in the region $x\varepsilon(-n,f)$. In summary, then, there are five such equations for components of $\overline{A}$.

Associated with A_x^{fn} along the nose-fuselage section are constants C_x^{fn} and B_x^{fn} (Equation (7a)), where

$$B_x^{fn} = B_x^{'fn} + \frac{1}{k}\left[\operatorname{div}_{t_x} \overline{A}_{t_x}\right]_0^{fn} . \tag{13}$$

Also, associated with A_z^g along the grounding strap are the constants C_z^g and B_z^g with

$$B_z^g = B_z^{'g} + \frac{1}{k}\left[\operatorname{div}_{t_z} \overline{A}_{t_z}\right]_0^{g} . \tag{14}$$

With these defined constants in equations for $\Phi_z^g(z)$, $\Phi_y^w(y)$, and $\Phi_x^{fn}(x)$, written directly from Equations (10), the requirement (Condition II) that scalar potential be continuous at the wing-fuselage-nose-grounding-strap intersection implies $\Phi_z^g(0) = \Phi_y^w(0) = \Phi_x^{fn}(0)$, which, in turn, means that $B_z^g = B_y^w = B_x^{fn}$. With these three equal constants replaced by a single symbol B, the vector potentials along those wires intersecting at the front junction reduce to those expressions given in (15). In Equations (15), the terms v_x^{fn}, v_y^w, and v_z^g are given below for clarity:

$$v_x^{fn}(x) = j\frac{k}{\omega}\int_{\xi=0}^{x} E_x^{fn}(\xi) \sin k(x-\xi)\, d\xi, \quad x\varepsilon(-n,f) \qquad (16a)$$

$$v_y^w(y) = j\frac{k}{\omega}\int_{\xi=0}^{y} E_y^w(\xi) \sin k(y-\xi)\, d\xi, \quad y\varepsilon(-w,w) \qquad (16b)$$

$$v_z^g(z) = j\frac{k}{\omega}\int_{\xi=0}^{z} E_z^g(\xi) \sin k(z-\xi)\, d\xi, \quad z\varepsilon(0,g) \qquad (16c)$$

where E_x^{fn}, E_y^w, and E_z^g are evaluated along the surface of nose-fuselage, wings, and grounding strap, respectively.

Now attention is focused upon the tail junction. As before, one enforces continuity of scalar potential at the junction and requires that

$$\Phi_x^{fn}(f) = \Phi_y^t(0) = \Phi_z^s(0) \qquad (17)$$

$$A_x^{fn}(x) = C_x^{fn} \cos kx + B \sin kx + v_x^{fn}(x) - \int_{\xi=0}^{x} \left[\operatorname{div}_{t_x} \bar{A}_{t_x} \right]_{\xi}^{fn} \cos k(x - \xi)\, d\xi, \qquad x \varepsilon (-n, f) \tag{15a}$$

$$A_y^{w}(y) = C_y^{w} \cos ky + B \sin ky + v_y^{w}(y) - \int_{\xi=0}^{y} \left[\operatorname{div}_{t_y} \bar{A}_{t_y} \right]_{\xi}^{w} \cos k(y - \xi)\, d\xi, \qquad x \varepsilon (-w, w) \tag{15b}$$

$$A_z^{g}(z) = C_z^{g} \cos kz + B \sin kz + v_v^{g}(z) - \int_{\xi=0}^{z} \left[\operatorname{div}_{t_z} \bar{A}_{t_z} \right]_{\xi}^{g} \cos k(z - \xi)\, d\xi, \qquad z \varepsilon (0, g) \tag{15c}$$

where, of course, the superscripts t and s denote horizontal stabilizer (tail) and vertical stabilizer, respectively. The requirement (17) necessitates that

$$-kC_x^{fn} \sin kf + kB \cos kf + \frac{\partial}{\partial x} v_x(f)$$

$$+ k \int_{\xi=0}^{f} \left[\mathrm{div}_{t_x} \bar{A}_{t_x}\right]_{\xi}^{fn} \sin k(f-\xi)\, d\xi = k\, B_y^t = k\, B_z^s$$

where B_y^t and B_z^s each is defined in a manner corresponding to that of B_y^w and of B_z^g. With $B_y^t = B_z^s = D$, the auxiliary equation above becomes

$$D = -\, C_x^{fn} \sin kf + B \cos kf + \frac{1}{k} \frac{\partial}{\partial x} v_x(f)$$

$$+ \int_{\xi=0}^{f} \left[\mathrm{div}_{t_x} \bar{A}_{t_x}\right]_{\xi}^{fn} \sin k(f-\xi)\, d\xi \;. \tag{18}$$

In addition, one has two more equations like (15) but for the tail region; these are given in (19).

At this point in the development, one sees that (15), (18) and (19) comprise six equations which relate all the unknown currents (through vector potential). To these are added two Kirchhoff-law equations (Condition III) and six

$$A_y^t(y) = C_y^t \cos ky + D \sin ky + v_y^t(y) - \int_{\xi=0}^{y} \left[\mathrm{div}_{t_y} \bar{A}_{t_y} \right]_{\xi}^{t} \cos k(y-\xi)\,d\xi,$$

$$y \varepsilon (-t,t) \qquad (19a)$$

$$A_z^s(z) = C_z^s \cos kz + D \sin kz + v_z^s(z) - \int_{\xi=0}^{z} \left[\mathrm{div}_{t_z} \bar{A}_{t_z} \right]_{\xi}^{s} \cos k(z-\xi)\,d\xi,$$

$$z \varepsilon (-h,0) \qquad (19b)$$

equations requiring current to be zero at free wire ends (Condition IV):

$$I_x^n(0) + I_y^{w-}(0) - I_y^{w+}(0) - I_x^f(0) - I_z^g(0) = 0, \quad (20a)$$

$$I_x^f(f) + I_z^s(0) + I_y^{t-}(0) - I_y^{t+}(0) = 0, \quad (20b)$$

and

$$I_x^n(-n) = 0, \quad (21a)$$

$$I_y^{w-}(-w) = 0, \quad (21b)$$

$$I_y^{w+}(w) = 0, \quad (21c)$$

$$I_y^{t-}(-t) = 0, \quad (21d)$$

$$I_y^{t+}(t) = 0, \quad (21e)$$

$$I_z^s(-h) = 0. \quad (21f)$$

The total number of equations is readily found to be fourteen, but, on the other hand, the unknowns total fifteen: eight unknown currents plus C_x^{fn}, C_y^w, C_z^g, C_y^t, C_z^s, B, and D.

The last needed equation is due to the observation that the charges on the structure plus those on its image produce zero scalar potential everywhere on the ground plane. In particular, $\Phi_z^g = 0$ where the grounding strap joins the ground plane; hence,

$$0 = - C_z^g \sin kg + B \cos kg + \frac{1}{k} \frac{\partial}{\partial z} v_z^g(g)$$
$$+ \int_{\xi=0}^{g} \left[\mathrm{div}_{t_z} \bar{A}_{t_z} \right]_{\xi}^{g} \sin k(g-\xi) d\xi \ . \tag{22}$$

Potential Integrals

Various components of the vector potential $\bar{A}$ are needed in the previously derived equations. Before presenting the potential integrals for $\bar{A}$, one finds it convenient to define the following total axial currents (See Figure 1):

$$I_x^{fn}(x) = \begin{cases} I^n(x) & , x \varepsilon (-n, 0) \\ I^f(x) & , x \varepsilon (0, f) , \end{cases} \tag{23a}$$

$$I_y^w(y) = \begin{cases} I^{w-}(y) & , y \varepsilon (-w, 0) \\ I^{w+}(y) & , y \varepsilon (0, w) , \end{cases} \tag{23b}$$

$$I_y^t(y) = \begin{cases} I^{t-}(y) & , y \varepsilon (-t, 0) \\ I^{t+}(y) & , y \varepsilon (0, t) . \end{cases} \tag{23c}$$

In terms of the above-defined currents, the needed components of $\overline{A}$ at any point in space are

$$A_x(x,y,z;a) = \frac{\mu}{4\pi}\int_{x'=-n}^{f} I_x^{fn}(x')\Big[K(x,y,z;a;x',0,0) - K(x,y,z;a;x',0,2g)\Big]\,dx', \qquad (24a)$$

$$A_y(x,y,z;a) = \frac{\mu}{4\pi}\int_{y'=-w}^{w} I_y^{w}(y')\Big[K(x,y,z;a;0,y',0) - K(x,y,z;a;0,y',2g)\Big]\,dy'$$

$$+ \frac{\mu}{4\pi}\int_{y'=-t}^{t} I_y^{t}(y')\Big[K(x,y,z;a;f,y',0) - K(x,y,z;a;f,y',2g)\Big]\,dy', \qquad (24b)$$

and

$$A_z(x,y,z;a) = \frac{\mu}{4\pi}\int_{z'=0}^{g} I_z^{g}(z')\Big[K(x,y,z;a;0,0,z') + K(x,y,z;a;0,0,2g-z')\Big]\,dz'$$

$$+ \frac{\mu}{4\pi}\int_{z'=-h}^{0} I_z^{s}(z')\Big[K(x,y,z;a;f,0,z') + K(x,y,z;a;f,0,2g-z')\Big]\,dz', \qquad (24c)$$

where the kernel K is defined

$$K(x,y,z;a;x',y',z') = \frac{e^{-jkR}}{R} \tag{25}$$

with

$$R = \left[a^2 + (x-x')^2 + (y-y')^2 + (z-z')^2\right]^{\frac{1}{2}}. \tag{26}$$

Expressions (24) are substituted into Equations (15), (18), (19) and (22); these, together with (20) and (21), completely describe the current on the aircraft model of Figure 1. One notes that the components of vector potential evaluated on the surfaces of the wire elements, Equations (15), (18), (19), and (22), can be replaced by the explicit potential integrals of (24):

$$A_x^{fn}(x) = A_x(x,0,0;a^{fn}), \tag{27a}$$

$$A_y^{w}(y) = A_y(0,y,0;a^{w}), \tag{27b}$$

$$A_z^{g}(z) = A_z(0,0,z;a^{g}), \tag{27c}$$

$$A_y^{t}(y) = A_y(f,y,0,a^{t}), \tag{27d}$$

$$A_z^{s}(z) = A_z(f,0,z;a^{s}); \tag{27e}$$

and

$$\left\{ \operatorname{div}_{t_x} \bar{A}_{t_x} \right\}_{\xi}^{fn} = \frac{\partial}{\partial y} A_y(\xi,0,0;a^{fn}) + \frac{\partial}{\partial z} A_z(\xi,0,0;a^{fn}), \qquad (28a)$$

$$\left\{ \operatorname{div}_{t_y} \bar{A}_{t_y} \right\}_{\xi}^{w} = \frac{\partial}{\partial x} A_x(0,\xi,0;a^{w}) + \frac{\partial}{\partial z} A_z(0,\xi,0;a^{w}), \qquad (28b)$$

$$\left\{ \operatorname{div}_{t_y} \bar{A}_{t_y} \right\}_{\xi}^{t} = \frac{\partial}{\partial x} A_x(f,\xi,0;a^{t}) + \frac{\partial}{\partial z} A_z(f,\xi,0;a^{t}), \qquad (28c)$$

$$\left\{ \operatorname{div}_{t_z} \bar{A}_{t_z} \right\}_{\xi}^{g} = \frac{\partial}{\partial x} A_x(0,0,\xi;a^{g}) + \frac{\partial}{\partial y} A_y(0,0,\xi;a^{g}), \qquad (28d)$$

$$\left\{ \operatorname{div}_{t_z} \bar{A}_{t_z} \right\}_{\xi}^{s} = \frac{\partial}{\partial x} A_x(f,0,\xi;a^{s}) + \frac{\partial}{\partial y} A_y(f,0,\xi;a^{s}). \qquad (28e)$$

In the above, notation such as $\frac{\partial}{\partial x} U(x_1,y_1,z_1)$ implies $\frac{\partial}{\partial x} U(x,y_1,z_1)$ evaluated at some point $x = x_1$.

With (27) and (28) substituted for appropriate terms in Equations (15), (18), (19), and (22), one has a set of integral equations, which, when constrained by the conditions represented in (20) and (21), are the equations governing the currents on the wire model of Figure 1. Notice that there are eight unknown currents and seven constants of integration, the five C's plus B and D, totaling fifteen unknowns. Correspondingly, there are five integral equations, (15) and (19), two auxiliary equations, (18) and (22), and eight equations constraining the currents, (20) and (21); these total fifteen as one expects and represent a system of equations which, in principle, should yield solutions.

The desired, final equations are presented in (29), where the kernels are given by

$$G_{fn}(x,y,z;a) = \frac{\mu}{4\pi}[K(x,y,z;a;x',0,0) - K(x,y,z;a;x',0,2g)], \quad (30a)$$

$$G_{w}(x,y,z;a) = \frac{\mu}{4\pi}[K(x,y,z;a;0,y',0) - K(x,y,z;a;0,y',2g)], \quad (30b)$$

$$G_{t}(x,y,z;a) = \frac{\mu}{4\pi}[K(x,y,z;a;f,y',0) - K(x,y,z;a;f,y',2g)], \quad (30c)$$

$$G_{g}(x,y,z;a) = \frac{\mu}{4\pi}[K(x,y,z;a;0,0,z') + K(x,y,z;a;0,0,2g-z')], \quad (30d)$$

and

$$G_{s}(x,y,z;a) = \frac{\mu}{4\pi}[K(x,y,z;a;f,0,z') + K(x,y,z;a;f,0,2g-z')]. \quad (30e)$$

Equations (29), subject to the boundary conditions, Equations (20) and (21), completely characterize the axial currents on structure of Figure 1. $\bar{E}^{i}$, the components of which appear in

Equations (29), is the impressed electric field which exists in the region above the ground plane ($z < g$) in the presence of the ground plane but with the wire structure removed. That is, $\bar{E}^i$ is the sum of the illuminating electric field and that reflected from the ground plane in the absence of the wire structure.

Conclusions

Equations from which may be calculated axial currents on the aircraft model of Figure 1 are given in this note. They are based upon the usual thin-wire assumptions and must be viewed in this light. The equations can be modified to include the affects of end caps in a rather direct manner [10 - 13] and, thus, their applicability can be extended to thicker wires. Also, if desired, the analysis can be readily generalized [3,7] to handle the case of swept-wing aircraft models.

The double integrals which appear in Equations (29) on the following pages can all be integrated once and, thereby, the resulting set of integral equations can be greatly simplified. However, due to desirable compactness in notation, the single integral forms are not given here.

$$\int_{x'=-n}^{f} I_x^{fn}(x') G_{fn}(x,0,0;a^{fn})dx' + \int_{\xi=0}^{x} \left\{ \int_{y'=-w}^{w} I_y^{w}(y') \frac{\partial}{\partial y} G_w(\xi,0,0;a^{fn})dy' + \int_{y'=-t}^{t} I_y^{t}(y') \frac{\partial}{\partial y} G_t(\xi,0,0;a^{fn})dy' \right.$$

$$\left. + \int_{z'=0}^{g} I_z^{g}(z') \frac{\partial}{\partial z} G_g(\xi,0,0;a^{fn})dz' + \int_{z'=-h}^{0} I_z^{s}(z') \frac{\partial}{\partial z} G_s(\xi,0,0;a^{fn})dz' \right\} \cos k(x-\xi)d\xi$$

$$= C_x^{fn} \cos kx + B \sin kx - j\frac{k}{\omega} \int_{\xi=0}^{x} E_x^{i}(\xi,0,0;a^{fn}) \sin k(x-\xi)d\xi \quad , \; x\varepsilon(-n,f) \qquad (29a)$$

$$\int_{y'=-w}^{w} I_y^{w}(y') G_w(0,y,0;a^{w})dy' + \int_{y'=-t}^{t} I_y^{t}(y') G_t(0,y,0;a^{w})dy' + \int_{\xi=0}^{y} \left\{ \int_{x'=-n}^{f} I_x^{fn}(x') \frac{\partial}{\partial x} G_{fn}(0,\xi,0;a^{w})dx' \right.$$

$$\left. + \int_{z'=0}^{g} I_z^{g}(z') \frac{\partial}{\partial z} G_g(0,\xi,0;a^{w})dz' + \int_{z'=-h}^{0} I_z^{s}(z') \frac{\partial}{\partial z} G_s(0,\xi,0;a^{w})dz' \right\} \cos k(y-\xi)d\xi$$

$$= C_y^{w} \cos ky + B \sin ky - j\frac{k}{\omega} \int_{\xi=0}^{y} E_y^{i}(0,\xi,0;a^{w}) \sin k(y-\xi)d\xi \quad , \; y\varepsilon(-w,w) \qquad (29b)$$

$$\int_{y'=-w}^{w} I_y^w(y')G_w(f,y,0;a^t)dy' + \int_{y'=-t}^{t} I_y^t(y')G_t(f,y,0;a^t)dy' + \int_{\xi=0}^{y}\left\{\int_{x'=-n}^{f} I_x^{fn}(x')\,\frac{\partial}{\partial x}\,G_{fn}(f,\xi,0;a^t)dx'\right.$$

$$\left. + \int_{z'=0}^{g} I_z^g(z')\,\frac{\partial}{\partial z}\,G_g(f,\xi,0;a^t)dz' + \int_{z'=-h}^{0} I_z^s(z')\,\frac{\partial}{\partial z}\,G_s(f,\xi,0;a^t)dz'\right\}\cos k(y-\xi)d\xi$$

$$= C_y^t \cos ky + D \sin ky - j\frac{k}{\omega}\int_{\xi=0}^{y} E_y^i(f,\xi,0;a^t)\sin k(y-\xi)d\xi\ ,\ y\varepsilon(-t,t) \qquad (29c)$$

$$\int_{z'=0}^{g} I_z^g(z')G_g(0,0,z;a^g)dz' + \int_{z'=-h}^{0} I_z^s(z')G_s(0,0,z;a^g)dz' + \int_{\xi=0}^{z}\left\{\int_{x'=-n}^{f} I_x^{fn}(x')\,\frac{\partial}{\partial x}\,G_{fn}(0,0,\xi;a^g)dx'\right.$$

$$\left. + \int_{y'=-w}^{w} I_y^w(y')\,\frac{\partial}{\partial y}\,G_w(0,0,\xi;a^g)dy' + \int_{y'=-t}^{t} I_y^t(y')\,\frac{\partial}{\partial y}\,G_t(0,0,\xi;a^g)dy'\right\}\cos k(z-\xi)d\xi$$

$$= C_z^g \cos kz + B \sin kz - j\frac{k}{\omega}\int_{\xi=0}^{z} E_z^i(0,0,\xi;a^g)\sin k(z-\xi)d\xi\ ,\ z\varepsilon(0,g) \qquad (29d)$$

$$\int_{z'=0}^{g} I_z^g(z')G_g(f,0,z;a^s)dz' + \int_{z'=-h}^{0} I_z^s(z')G_s(f,0,z;a^s)dz' + \int_{\xi=0}^{z}\left\{\int_{z'=-n}^{f} I_x^{fn}(x')\ \frac{\partial}{\partial x}\ G_{fn}(f,0,\xi;a^s)dx'\right.$$

$$\left. + \int_{y'=-w}^{w} I_y^w(y')\ \frac{\partial}{\partial y}\ G_w(f,0,\xi;a^s)dy' + \int_{y'=-t}^{t} I_y^t(y')\ \frac{\partial}{\partial y}\ G_t(f,0,\xi;a^s)dy'\right\} \cos k(z-\xi)d\xi$$

$$= C_z^s \cos kz + D \sin kz - j\frac{k}{\omega}\int_{\xi=0}^{z} E_z^i(f,0,\xi;a^s)\ \sin k(z-\xi)d\xi \quad , \quad z\epsilon(-h,0) \tag{29e}$$

$$\int_{\xi=0}^{f}\left\{\int_{y'=-w}^{w} I_y^w(y')\ \frac{\partial}{\partial y}\ G_w(\xi,0,0;a^{fn})dy' + \int_{y'=-t}^{t} I_y^t(y')\ \frac{\partial}{\partial y}\ G_t(\xi,0,0;a^{fn})dy' + \int_{z'=0}^{g} I_z^g(z')\ \frac{\partial}{\partial z}\ G_g(\xi,0,0;a^{fn})dz'\right.$$

$$\left. + \int_{z'=-h}^{0} I_z^s(z')\ \frac{\partial}{\partial z}\ G_s(\xi,0,0;a^{fn})dz'\right\} \sin k(f-\xi)d\xi$$

$$= D + C_x^{fn} \sin kf - B \cos kf + j\frac{k}{\omega}\int_{\xi=0}^{f} E_x^i(\xi,0,0;a^{fn})\ \cos k(f-\xi)d\xi \tag{29f}$$

$$\int_{\xi=0}^{g}\left\{\int_{x'=-n}^{f} I_x^{fn}(x')\,\frac{\partial}{\partial x}\,G_{fn}(0,0,\xi;a^g)\,dx' + \int_{y'=-w}^{w} I_y^{w}(y')\,\frac{\partial}{\partial y}\,G_w(0,0,\xi;a^g)\,dy'\right.$$

$$\left. + \int_{y'=-t}^{t} I_y^{t}(y')\,\frac{\partial}{\partial y}\,G_t(0,0,\xi;a^g)\,dy'\right\} \sin k(g-\xi)\,d\xi$$

$$= C_z^g \sin kg - B\cos kg + j\frac{k}{\omega}\int_{\xi=0}^{g} E_z^i(0,0,\xi;a^g)\cos k(g-\xi)\,d\xi \qquad (29g)$$

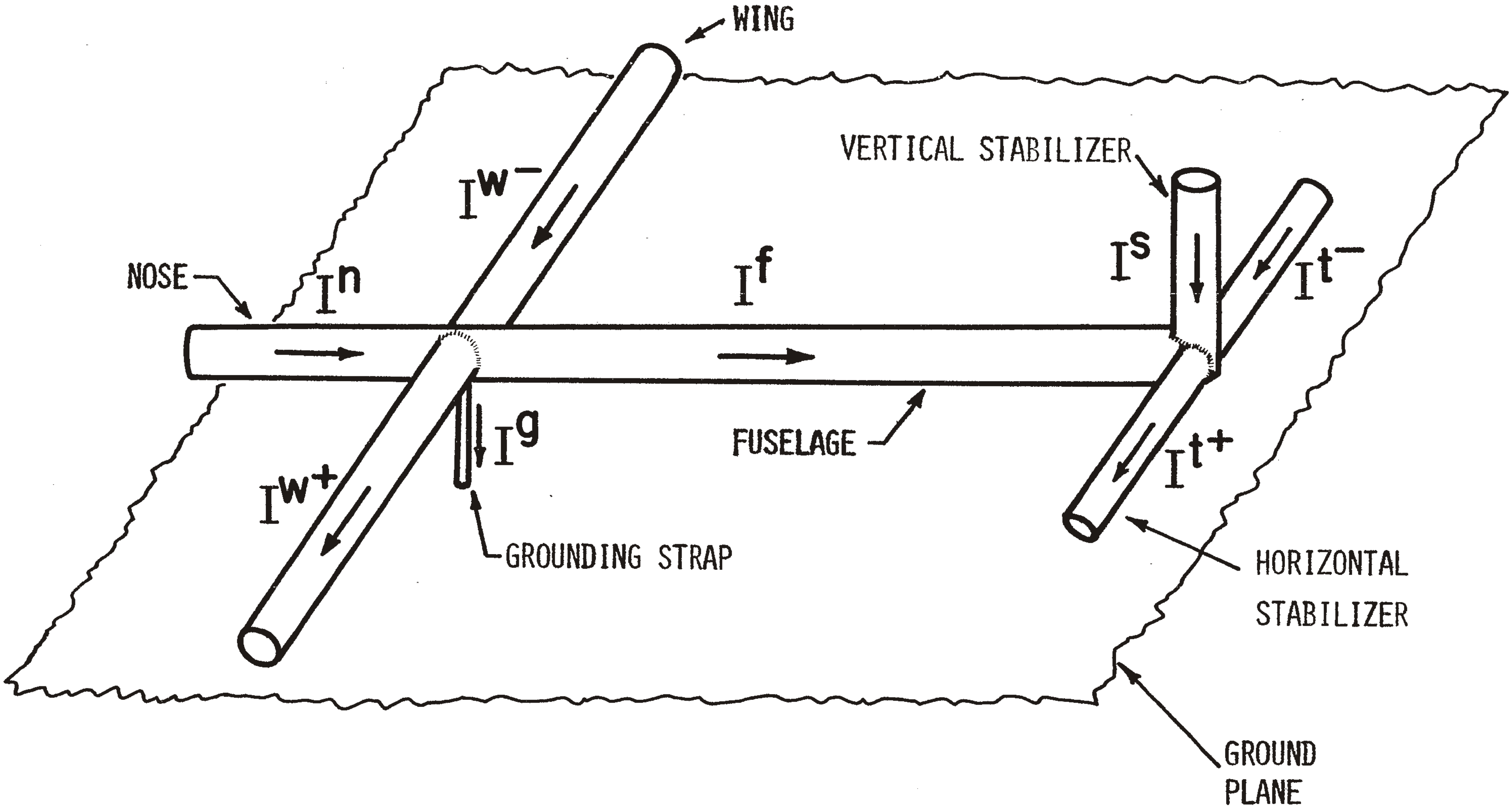

Figure 1a Wire Model of Parked Aircraft

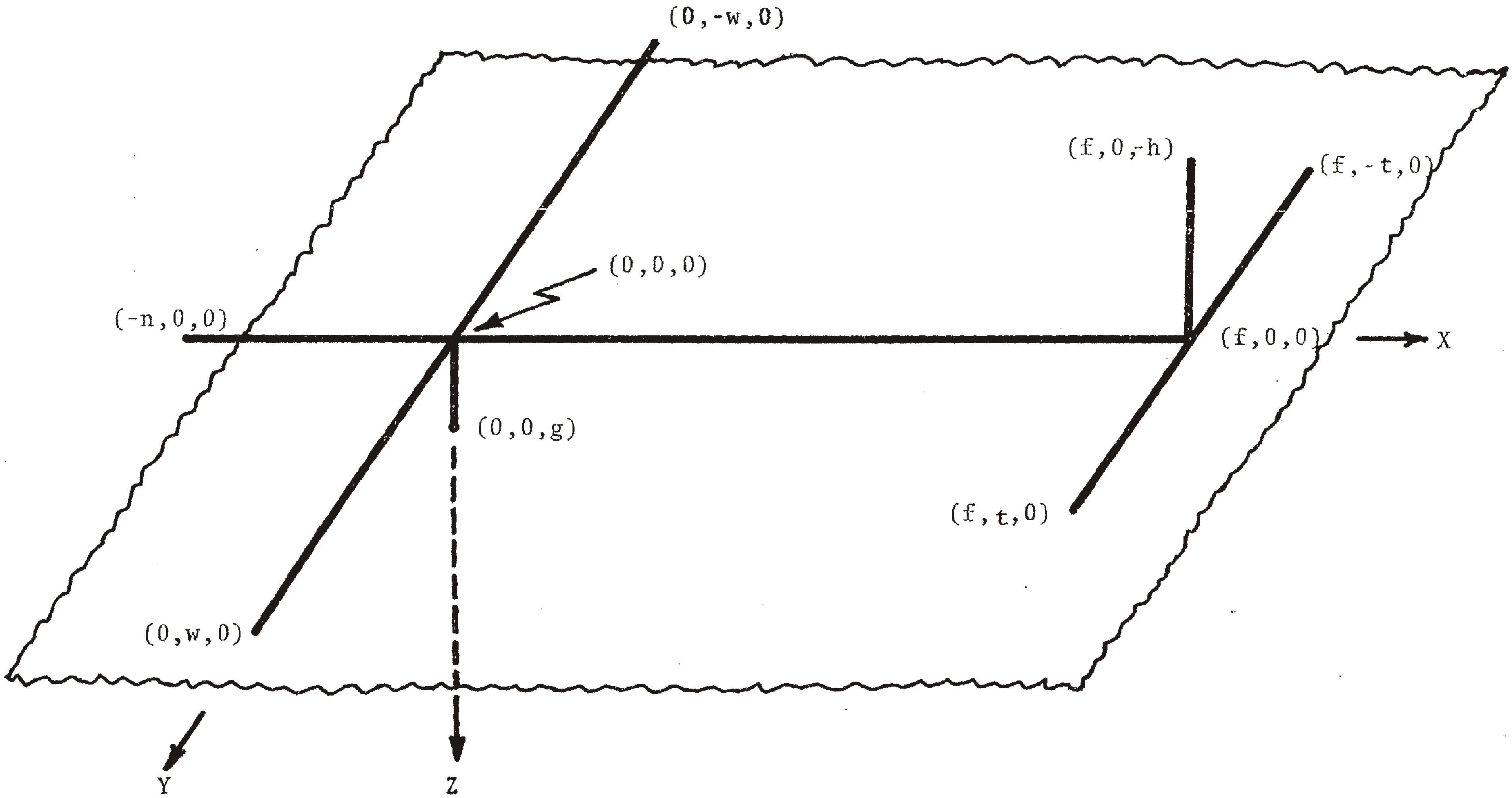

Figure 1b Aircraft Model(Wire Axes)

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