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The Interaction of Electromagnetic Pulses with an Infinitely Long Conducting Cylinder Above a Perfectly Conducting Ground

by

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ABSTRACT

The scattering of plane electromagnetic waves by an infinitely long conducting cylinder over perfectly conducting ground is treated first, for two independent cases of polarization. With the use of addition theorems for cylindrical waves, two coupled systems of equations are obtained for the expansion coefficients of the scattered waves, for each case of polarization. Rigorous expressions are derived for the total axial current and for the potential of the cylinder with respect to the ground. Reduction is then made to the case of a thin wire. Approximate expressions are obtained for the leading expansion coefficients. Integral expressions are derived for the time dependent axial current which is induced when delta function and step function pulses are incident on the wire. They are put into a form from which asymptotic values for large times are obtained. The calculation of numerical results is then considered. Procedures for calculating the scattering coefficients and for calculating the current in the case of a step function pulse are discussed. A figure depicting the results of a sample calculation is presented. In a number of appendices we write down the Fourier transforms of the pulses of interest, derive the addition theorems for scalar and vector cylindrical waves, sketch the derivation of rough estimates for the natural resonances of the system, and treat the case when the conductivity of the cylinder is finite.

cylinders, scattering, polarization

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I INTRODUCTION

The objective of this work is to investigate the interaction of a plane electromagnetic pulse with an infinitely long, solid, conducting cylinder when it is located above and parallel to a perfectly conducting ground. In a succeeding note we will consider a finitely conducting ground.

The pulses that are of interest include the delta, step, and exponential functions of $t - \frac{\hat{n} \cdot \vec{r}}{c}$. By means of the Fourier transform each pulse in the time domain can be expanded in a continuous spectrum of plane waves. This is done for the pulses mentioned above in Appendix A. Our procedure will be to solve the problem of scattering of a plane electromagnetic wave by the cylinder in the presence of the perfectly conducting ground. Then we shall multiply the solution by the appropriate $f(\omega)$ and take the inverse transform to get the required time-dependent solution.

Expressions will be derived for the coefficients in the expansion of the scattered electromagnetic field and from these we shall obtain expressions for the total axial current. The potential between the wire and the ground is also of interest and we shall present expressions for the potential for two independent cases of polarization.

We shall initially present a rigorous treatment of the scattering of plane electromagnetic waves by a perfectly conducting cylinder over a perfectly conducting ground. The solution so obtained is valid for all sizes of the cylinder at arbitrary distances from the ground. This solution is therefore easily generalized to the cases of two, three, or more wires over the ground by using the same techniques. The solution will then be reduced to the case of thin wires at quite high heights above the ground.

In the case of a perfectly conducting ground, the problem may be looked at in two equivalent ways, namely, as an infinite cylinder in the presence of a perfectly conducting ground on which a plane wave from a source at an infinite distance impinges, or as a problem of two cylinders in the presence of two sources, the original source and its mirror image in the plane of the air-ground interface. The latter method is useful in formulating the solution of the problem, especially when we consider the effects of the waves scattered by one cylinder acting on the other cylinder. In order to apply the boundary conditions at the surface of one cylinder, it is necessary to express the waves scattered by the other cylinder in terms of wave functions appropriate to the first cylinder. This is achieved by means of the translational addition theorems for cylindrical waves. Since the addition theorems that are available in the literature [1,2] are not in a form that is suitable for our purpose, we derive the required theorems in Appendix B.

When the cylinder is perfectly conducting, two independent cases of polarization can be distinguished, namely, waves with the electric field perpendicular to the axis of the cylinder, or transverse electric waves, and waves with the magnetic field perpendicular to the axis, or transverse magnetic waves. However, when the conductivity of the cylinder is finite, a superposition of the transverse electric and magnetic waves is necessary to satisfy the boundary conditions at the cylindrical surface, except in the case of axially symmetric waves. Now it is a fact that for moderate and higher frequencies, a good conductor acts just like a perfect conductor with the idealized surface current replaced by an equivalent surface current that is actually distributed through a small thickness (the skin depth) at the surface. When the frequency becomes low enough so that the skin depth of a good conductor becomes an appreciable fraction of the radius of the cylinder, it is obviously not physically correct to use the idealization of a perfect conductor. In fact, the use of the perfectly conducting assumption in our reduction to thin wires is found to lead to the unphysical result of a finite axial current in the wire in the limit of zero frequency. Fortunately, however, in this case the axially symmetric waves are the most important ones and we can accordingly easily consider a finitely conducting wire and thereby obtain the physically correct result for the low frequency part of the spectrum.

The scattering problem under consideration here is one of the simplest examples of multiple scattering. The problems of multiple scattering, on the other hand, is to some extent a linear version of the many body problem. Even in the simple problem of interaction involved here, there are features that are common to all many body problems. We shall, for example, in the case of a thin wire, in effect sum over an infinite subseries of the complete perturbation series. The fact that we must integrate over an infinite range of frequency, however, makes the present problem a far from simple one.

Although we shall not make use of their work, we should mention that the scattering of a plane electromagnetic wave by two spheres has been treated in a fine piece of work by Bruning and Lo [3] with the use of the translational addition theorem for spherical waves. The spherical addition theorem is much more complicated than the cylindrical one and leads to an extensive program of numerical computation for spheres of even moderate size.

II EXPANSIONS OF THE INCIDENT AND REFLECTED WAVES IN VECTOR WAVE FUNCTIONS

The wave forms of the incident pulses are expressed as Fourier expansions of plane waves in Appendix A. We shall investigate, therefore, the scattering of plane electromagnetic waves that are incident on an infinitely long, conducting cylinder in the presence of a perfectly conducting ground. The incident plane waves will be specified by their propagation vector $\vec{k}$ with spherical components k, γ, α and their polarization. The position vector $\vec{r}$ has components r, θ, φ. A time dependence $e^{-i\omega t}$ will be assumed and suppressed until we consider incident pulses.

We shall expand the waves in terms of cylindrical vector wave functions. To obtain the expansions of the incident waves we proceed as follows. Let Θ_i be the angle between $\vec{r}$ and $\vec{k}$. Then

$$\cos \Theta_i = \cos \theta \cos \gamma + \sin \theta \sin \gamma \cos (\varphi - \alpha)$$

and

$$\begin{aligned} \vec{k} \cdot \vec{r} = kr \cos \Theta_i &= kz \cos \gamma + kx \sin \gamma \cos \alpha + ky \sin \gamma \sin \alpha \\ &= kz \cos \gamma + k\rho \sin \gamma \cos (\varphi - \alpha) \\ &= k_z z + k_x x + k_y y \quad . \qquad (1) \end{aligned}$$

Furthermore

$$\nabla e^{i\vec{k}\cdot\vec{r}} \times \hat{e}_z = ik \sin \gamma \, (\sin \alpha \, \hat{e}_x - \cos \alpha \, \hat{e}_y) \, e^{ikr \cos \Theta_i} \qquad (2)$$

Consider an incident plane wave with the electric vector perpendicular to the axis of the cylinder (which is in the z-direction):

$$\vec{E}^{\,inc} = E_0 \,(\sin\alpha\,\hat{e}_x - \cos\alpha\,\hat{e}_y)\, e^{ikr\cos\Theta_i} . \tag{3}$$

According to Eq. (2) this may be written as

$$\vec{E}^{\,inc} = \frac{E_0}{i\,k\,\sin\gamma}\, \nabla e^{ikr\cos\Theta_i} \times \hat{e}_z . \tag{4}$$

Now from Eq. (1)

$$e^{ikr\cos\Theta_i} = e^{ik\rho\,\sin\gamma\,\cos(\varphi-\alpha)}\; e^{ikz\cos\gamma}$$

and [1,2,8]

$$e^{ik\rho\,\cos\varphi} = \sum_{m=0}^{\infty} (2 - \delta_{om})\, i^m J_m(k\rho)\,\cos m\varphi .$$

Therefore

$$e^{ikr\cos\Theta_i} = \sum_{m=0}^{\infty} (2 - \delta_{om})\, i^m J_m\,(k\rho\,\sin\gamma)\,\cos m(\varphi-\alpha)\; e^{ikz\cos\gamma} . \tag{5}$$

Let us define the vector wave functions

$$\vec{M}^{(i)}_{\substack{e\\o}m}(\vec{r},\gamma) = \nabla \times \psi^{(i)}_{\substack{e\\o}m}(\vec{r},\gamma)\,\hat{e}_z$$

$$= \nabla\, \psi^{(i)}_{\substack{e\\o}m}(\vec{r},\gamma) \times \hat{e}_z , \tag{6}$$

$$\vec{N}^{(i)}_{\substack{e\\o}m}(\vec{r},\gamma) = \frac{1}{k}\,\nabla \times \vec{M}^{(i)}_{\substack{e\\o}m}(\vec{r},\gamma) , \tag{7}$$

where

$$\psi^{(i)}_{\substack{e\\o}m}(\vec{r},\gamma) = Z^{(i)}_m\,(k\,\rho\,\sin\gamma)\,\begin{matrix}\cos\\ \sin\end{matrix}\, m\,\varphi\; e^{ikz\cos\gamma} , \tag{8}$$

in which the $Z^{(i)}_m$ are the cylindrical functions:

$$Z_m^{(1)} = J_m \,, \qquad Z_m^{(2)} = N_m \,,$$
$$Z_m^{(3)} = H^{(1)} \,, \qquad Z_m^{(4)} = H_m^{(2)} \,. \tag{9}$$

In component form the cylindrical vector wave functions are

$$\vec{M}^{(i)}_{\substack{e\\o}m}(\vec{r},\gamma) = \Big\{ \mp \frac{m}{\rho} Z_m^{(i)}(k\rho \sin\gamma) \substack{\sin\\ \cos} m\varphi\, \hat{e}_\rho$$
$$- \frac{d}{d\rho} Z_m^{(i)}(k\rho \sin\gamma) \substack{\cos\\ \sin} m\varphi\, \hat{e}_\varphi \Big\} e^{ikz\cos\gamma} \tag{10}$$

$$\vec{N}^{(i)}_{\substack{e\\o}m}(\vec{r},\gamma) = \Big\{ i \cos\gamma \frac{d}{d\rho} Z_m^{(i)}(k\rho \sin\gamma) \substack{\cos\\ \sin} m\varphi\, \hat{e}_\rho$$
$$\mp \frac{i\, m \cos\gamma}{\rho} Z_m^{(i)}(k\rho \sin\gamma) \substack{\sin\\ \cos} m\varphi\, \hat{e}_\varphi$$
$$+ k \sin^2\gamma\, Z_m^{(i)}(k\rho \sin\gamma) \substack{\cos\\ \sin} m\varphi\, \hat{e}_z \Big\} e^{ikz\cos\gamma} \,. \tag{11}$$

These vector wave functions satisfy the vector Helmholtz equation

$$(\nabla \times \nabla \times - k^2)\vec{A} = 0$$

that must be satisfied by the electric and magnetic field vectors. The sets of functions $\vec{M}^{(i)}_{\delta m}$, $\vec{N}^{(i)}_{\delta m}$ satisfy orthogonality relations*, but we will not write them down as we shall not need them.

From Eqs. (4), (5), (6), and (8) we get the vector wave function expansion of the incident field:

$$\vec{E}^{inc} = \frac{E_0}{i\,k \sin\gamma} \sum_{m=0}^{\infty} (2 - \delta_{om})\, i^m \Big[\cos m\alpha\, \vec{M}^{(i)}_{em}(\vec{r},\gamma)$$
$$+ \sin m\alpha\, \vec{M}^{(i)}_{om}(\vec{r},\gamma) \Big] \,, \quad \big(\vec{E}^{inc} \perp \hat{e}_z\big) \,. \tag{12}$$

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*c.f. Tai [9]

The expansion of the incident magnetic field is then

$$\vec{H}^{inc} = \frac{1}{ik}\sqrt{\frac{\epsilon_0}{\mu_0}}\,\nabla\times\vec{E}^{inc}$$

$$= -\sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{E_0}{k\sin\gamma}\sum_{m=0}^{\infty}(2-\delta_{om})\,i^m\cdot\Big[\cos m\alpha\,\vec{N}^{(1)}_{em}\,(\vec{r},\gamma)$$

$$+\sin m\alpha\,\vec{N}^{(1)}_{om}\,(\vec{r},\gamma)\Big]\ . \qquad (13)$$

For the other independent case of polarization with the magnetic vector perpendicular to the cylindrical axis, we have

$$\vec{H}^{inc} = \sqrt{\frac{\epsilon_0}{\mu_0}}\,E_0\,(\sin\alpha\,\hat{e}_x - \cos\alpha\,\hat{e}_y)\,e^{ikr\cos\Theta_i}$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{E_0}{i\,k\,\sin\gamma}\,\nabla\times e^{ikr\cos\Theta_i}\,\hat{e}_z$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{E_0}{i\,k\,\sin\gamma}\sum_{m=0}^{\infty}(2-\delta_{om})\,i^m\Big[\cos m\alpha\,\vec{M}^{(1)}_{em}\,(\vec{r},\gamma)$$

$$+\sin m\alpha\,\vec{M}^{(1)}_{om}\,(\vec{r},\gamma)\Big]\ , \qquad (14)$$

and

$$\vec{E}^{inc} = \frac{i}{k}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\nabla\times\vec{H}^{inc}$$

$$= \frac{E_0}{k}\Big[-k_z\cos\alpha\,\hat{e}_x - k_z\sin\alpha\,\hat{e}_y + (k_x\cos\alpha + k_y\sin\alpha)\,\hat{e}_z\Big]$$

$$\cdot\, e^{ikr\cos\Theta_i}$$

$$= \frac{E_0}{k \sin \gamma} \sum_{m=0}^{\infty} (2 - \delta_{om}) i^m \left[\cos m \alpha \, \vec{N}_{em}^{(1)} (\vec{r},\gamma) \right.$$

$$\left. + \sin m \cdot \alpha \, \vec{N}_{om}^{(1)} (\vec{r},\gamma) \right] , \quad (\vec{H}^{inc} \perp \hat{e}_z) \quad . \qquad (15)$$

Let us now turn to the waves reflected from the perfectly conducting ground. At the same time we note that in setting up this problem, one can invoke the method of images and consider the incident wave and its mirror image impinging on both the cylinder and its image, the images being taken, of course, with respect to the surface of the ground. In this picture we have, in place of the reflected wave, the incident wave from the image source. The latter will be identical with the reflected wave in the region above the ground.

The coordinate systems in the original problem and in the one resulting from the use of images are shown in Figures 1 and 2. The y-axis has been chosen to be perpendicular to the air-ground interface. In the coordinate system centered on the image cylinder the "y"-coordinate will be primed. Angles and the position vector in this coordinate system will also be primed. The x- and z-coordinates are the same in both systems.

For the case of transverse electric waves, when the electric field is perpendicular to the axis of the cylinder, the electric field of the reflected wave is

$$\vec{E}^{ref} = E_0' (- \sin \alpha \, \hat{e}_x - \cos \alpha \, \hat{e}_y) \, e^{ikr \cos \Theta_r} \qquad (16)$$

where

$$E_0' = E_0 \, e^{-i2kh \sin \gamma \sin \alpha} \qquad (17)$$

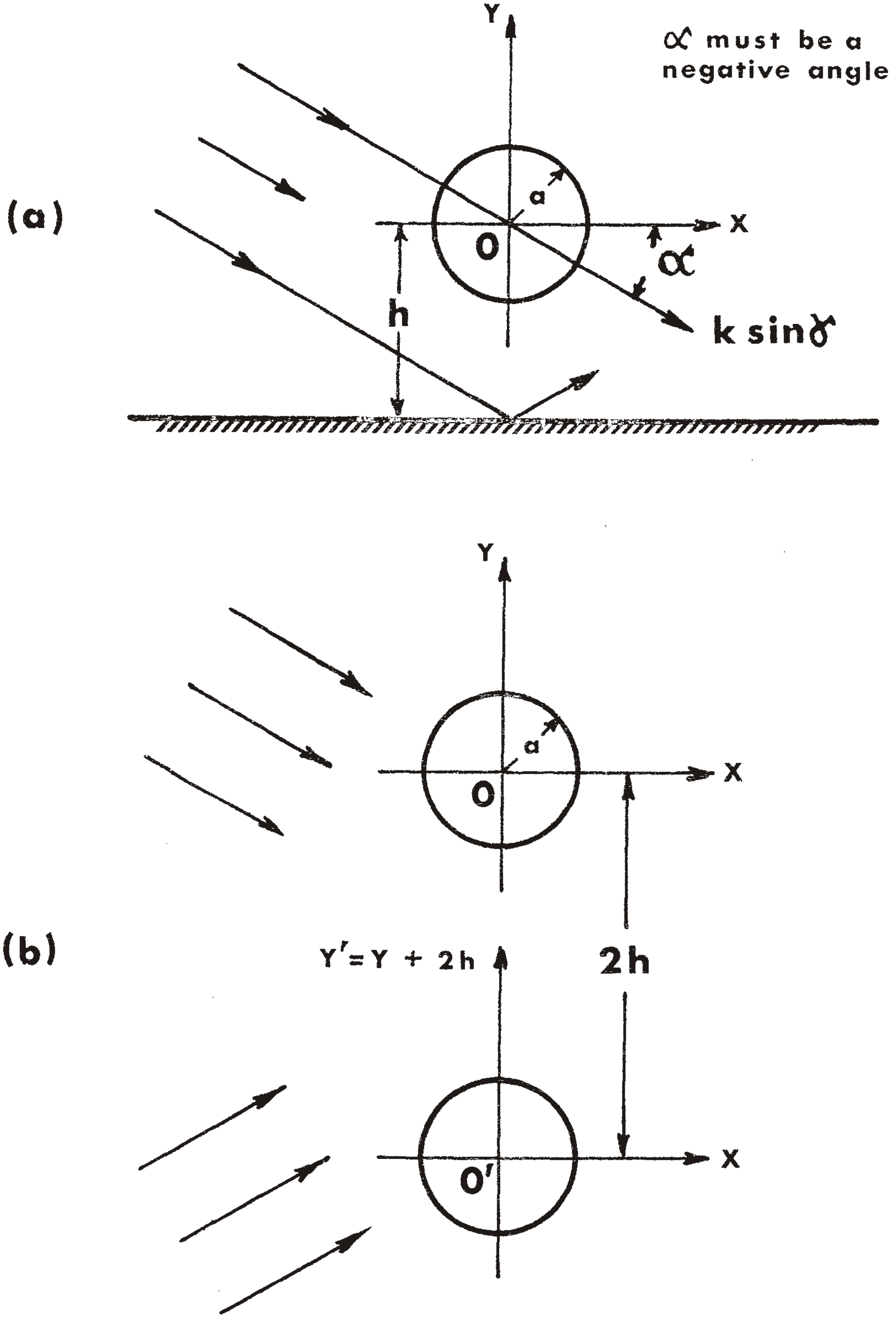

FIGURE 1 (a) PLANE WAVE INCIDENT ON A CONDUCTING CYLINDER OVER A PERFECTLY CONDUCTING GROUND.

(b) PLANE WAVES FROM ORIGINAL SOURCE AND IMAGE SOURCE INCIDENT ON ORIGINAL CYLINDER AND IMAGE CYLINDER.

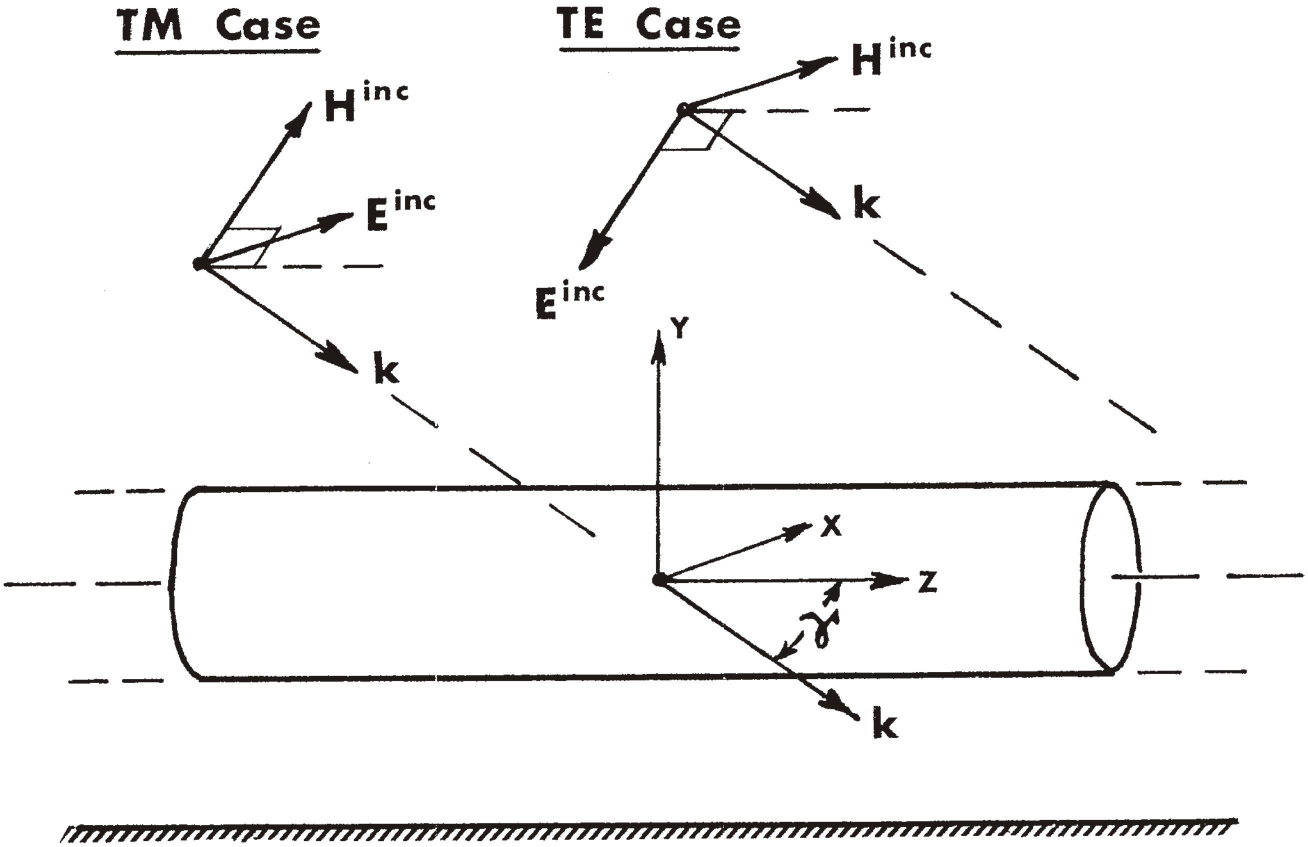

FIGURE 2 GEOMETRY FOR SCATTERING BY AN INFINITELY LONG CYLINDER OVER A CONDUCTING GROUND.

and

$$k\, r \cos \Theta_r = k_x x - k_y y + k_z z$$

$$= k \rho \sin \gamma \cos (\varphi + \alpha) + kz \cos \gamma \quad . \tag{18}$$

The vector wave function expansion of the reflected field is obtained in the same way as was that of the incident field above. It is

$$\vec{E}^{ref} = \frac{E_0'}{i\, k \sin \gamma} \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \Big[\cos m \alpha\, \vec{M}^{(1)}_{em} (\vec{r},\gamma)$$

$$- \sin m \alpha\, \vec{M}^{(1)}_{om} (\vec{r},\gamma)\Big] \ , \tag{19}$$

$$\vec{H}^{ref} = - \sqrt{\frac{\epsilon_0}{\mu_0}}\ \frac{E_0'}{k \sin \gamma} \sum_{m=0}^{\infty} (2 - \delta_{om})\, i^m \Big[\cos m \alpha\, \vec{N}^{(1)}_{em} (\vec{r},\gamma)$$

$$- \sin m \alpha\, \vec{N}^{(1)}_{om} (\vec{r},\gamma)\Big] \ , (\vec{E} \perp \hat{e}_z) \quad . \tag{20}$$

In the picture of Figure 1b, we have

$$\vec{E}'^{,inc} = E_0 (- \sin \alpha\, \hat{e}_x - \cos \alpha\, \hat{e}_y)\, e^{ikr' \cos \Theta_i'}$$

$$= \frac{E_0}{i\, k \sin \gamma} \sum_{m=0}^{\infty} (2-\delta_{0m})\, i^m \Big[\cos m \alpha\, \vec{M}^{(1)}_{em} (\vec{r}', \gamma)$$

$$- \sin m \alpha\, \vec{M}^{(1)}_{om} (\vec{r}', \gamma)\Big] \quad . \tag{21}$$

Here we have

$$k\, r' \cos \Theta_i' = k_x x - k_y y' + k_z z$$

$$= k_x x - k_y y + k_z z - 2k_y h$$

$$= k\rho \sin\gamma \cos(\varphi+\alpha) + kz\cos\gamma - 2kh\sin\gamma \sin\alpha$$

$$= k\, r \cos\Theta_r - 2\, k\, h \sin\gamma \sin\alpha \quad . \tag{22}$$

Therefore

$$\vec{E}'^{\,inc} = \vec{E}^{ref} \, , \qquad y \geq 0 \quad . \tag{23}$$

For transverse magnetic waves, with the magnetic field perpendicular to the axis of the cylinder, we have

$$\vec{H}^{ref} = \sqrt{\frac{\epsilon_0}{\mu_0}}\, E_0' \,(\sin\alpha\, \hat{e}_x + \cos\alpha\, \hat{e}_y)\, e^{ikr\cos\Theta_r}$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}}\, \frac{E_0'}{i\,k\sin\gamma} \sum_{m=0}^{\infty} (2-\delta_{Om})\, i^m \Big[-\cos m\alpha\, \vec{M}^{(1)}_{em}(\vec{r},\gamma)$$

$$+ \sin m\alpha\, \vec{M}^{(1)}_{om}(\vec{r},\gamma)\Big] \, , \tag{24}$$

$$\vec{E}^{ref} = \frac{E_0'}{k}\,[k_z \cos\alpha\, \hat{e}_x - k_z \sin\alpha\, \hat{e}_y - (k_x \cos\alpha + k_y \sin\alpha)\, \hat{e}_z]$$

$$\cdot\, e^{ikr\cos\Theta_r}$$

$$= \frac{E_0'}{k\sin\gamma} \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \Big[-\cos m\alpha\, \vec{N}^{(1)}_{em}(\vec{r},\gamma)$$

$$+ \sin m\alpha\, \vec{N}^{(1)}_{om}(\vec{r},\gamma)\Big] \, , \quad (\vec{H} \perp \hat{e}_z) \quad . \tag{25}$$

III THE SCATTERED WAVES

The plane waves of the previous section impinge on the cylinder giving rise to scattered waves. In the picture of Figure 1b, both of the incident waves are scattered by each cylinder and the scattered waves from one cylinder interact also with the other cylinder. In the actual case of a perfectly reflecting ground, there is besides the scattered wave moving outward from the cylinder, another cylindrical wave due to the effect of the ground that has the form of a wave scattered by the image cylinder.

We shall expand both types of scattered waves in cylindrical vector wave functions of the third kind which, for our choice of time, $e^{-i\omega t}$, yield outgoing waves. The coefficients in the expansions of both types of scattered waves are determined simultaneously by imposing the boundary condition of the vanishing of the tangential components of the total electric field on the surface of the cylinder and on the surface of the ground, or equivalently, on the surfaces of both cylinders. This procedure involves the use of the translational addition theorem for the cylindrical vector wave functions that is derived in Appendix B. It leads to a system of equations for the expansion coefficients.

1. Transverse Electric Waves

When the waves are polarized with the electric field perpendicular to the axis of the cylinder, we write for the scattered waves

$$\vec{E}^{scat} = \frac{E_0}{i\,k\,\sin\gamma} \sum_{m=0}^{\infty} (2-\delta_{0m})\, i^m \Big\{ b_{em} \cos m\alpha\, \vec{M}^{(3)}_{em}(\vec{r},\gamma) + b_{om} \sin m\alpha\, \vec{M}^{(3)}_{om}(\vec{r},\gamma) \Big\} , \qquad (26)$$

and

$$\vec{E}'^{\,scat} = \frac{E_0}{i\,k\,\sin\gamma} \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \Big\{ b'_{em} \cos m\alpha\, \vec{M}^{(3)}_{em} (\vec{r}',\gamma)$$

$$- b'_{om} \sin m\alpha\, \vec{M}^{(3)}_{om} (\vec{r}',\gamma) \Big\} . \tag{27}$$

The first expression is for the wave moving outward from the cylinder above the ground and the second one is for the wave centered on the image cylinder as is made evident by its dependence on the primed coordinates.

On examining the x-components of the scattered electric fields of Eqs. (26) and (27), we find that if we set the primed coefficients equal to the unprimed ones, that is,

$$b'_{\substack{e\\o}m} = b_{\substack{e\\o}m} , \tag{28}$$

then the boundary condition of vanishing tangential electric field on the surface of the perfectly conducting ground is automatically satisfied. The expansion coefficients $b_{\substack{e\\o}m}$ are now completely determined by the boundary condition on the cylinder:

$$E^{inc}_{\varphi}(a,\varphi,z) + E^{ref}_{\varphi}(a,\varphi,z) + E^{scat}_{\varphi}(a,\varphi,z) + [\hat{e}_{\varphi} \cdot \vec{E}'^{\,scat}(\vec{r}')]_{\rho=a} = 0 . \tag{29}$$

Substituting Eqs. (12), (19), (26), and (27) in Eq. (29), we get

$$E_0 \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \Big[\cos m\alpha\, M^{(1)}_{em\varphi}(a,\varphi,z;\gamma)$$

$$+ \sin m\alpha\, M^{(1)}_{om\varphi}(a,\varphi,z;\gamma)\Big] +$$

$$+ E_0' \sum_{m=0}^{\infty} (2 - \delta_{om})\, i^m \Big[\cos m\alpha\, M^{(1)}_{em\varphi}(a,\varphi,z;\gamma)$$

$$- \sin m\alpha\, M^{(1)}_{om\varphi}(a,\varphi,z;\gamma) \Big]$$

$$+ E_0 \sum_{m=0}^{\infty} (2 - \delta_{om})\, i^m \Big[b_{em} \cos m\alpha\, M^{(3)}_{em\varphi}(a,\varphi,z;\gamma)$$

$$+ b_{om} \sin m\alpha\, M^{(3)}_{om\varphi}(a,\varphi,z;\gamma) \Big]$$

$$+ E_0 \sum_{m=0}^{\infty} (2 - \delta_{om})\, i^m \Bigg[b_{em}' \cos m\alpha \sum_{n,\mu,j} A_n(e,m|j,\mu)(-1)^n \cdot$$

$$M^{(1)}_{j,n+\mu,\varphi}(a,\varphi,z;\gamma)$$

$$- b_{om}' \sin m\alpha \sum_{n,\mu,j} (-1)^n A_n(o,m|j,\mu) \cdot M^{(1)}_{j,n+\mu,\varphi}(a,\varphi,z;\gamma) \Bigg]$$

$$= 0 \quad , \tag{30}$$

where we have used the addition theorem of Eq. (B-16) in writing the last term. Let us consider this term further. Since μ takes on the values m and -m, there will be terms of the form

$$\sum_{m'=0}^{\infty} d_{m'} \sum_{n=0}^{\infty} (-1)^n A_n M_{n+m'}$$

and

$$\sum_{m'=0}^{\infty} d_{m'} \sum_{n=0}^{\infty} (-1)^n A_n M_{n-m'}$$

where we have changed m to m', and $d_{m'}$ represents the product of factors depending only on m'. In the summations over $M_{n+m'}$ we let $m = n + m'$. Then

$$\sum_{m'=0}^{\infty} d_{m'} \sum_{n=0}^{\infty} (-1)^n A_n M_{n+m'} = \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} (-1)^{m-m'} A_{m-m'} d_{m'} M_m . \quad (31)$$

In the summations over $M_{n-m'}$, we let $m = n - m'$ and after some manipulation we obtain

$$\sum_{m'=0}^{\infty} d_{m'} \sum_{n=0}^{\infty} (-1)^n A_n M_{n-m'}$$

$$= \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} (-1)^{m+m'} A_{m+m'} d_{m'} M_m$$

$$+ \sum_{m=0}^{\infty} (1-\delta_{om}) \sum_{m'=m}^{\infty} (-1)^{m'-m} A_{m'-m} d_{m'} M_{-m} \quad (32)$$

where M_{-m} is either $(-1)^m M_m$ or $(-1)^{m+1} M_m$ depending upon whether M_m stands for $M^{(1)}_{em\varphi}(a,\varphi,z)$ or $M^{(1)}_{om\varphi}(a,\varphi,z)$, respectively. When these results are utilized in Eq. (30), and explicit expressions are inserted for $M^{(1),(3)}_{\substack{e\\o}m}$, there results a very long expression with fifteen different summations that we shall not write down here. Setting the coefficients of cos m φ and of sin m φ in that expression equal to zero, we get two coupled systems of equations that determine the coefficients b_{em} and b_{om}. With

$$\lambda = k \sin \gamma \quad (33)$$

and a prime over a Bessel or Hankel function denoting differentiation with respect to the argument of the function, these equations may be written in the form:

$$
\begin{aligned}
b_{em} = & - (1 + e^{-i\,2\,\lambda\;h\;\sin\alpha})\;\frac{J_m^{\prime}(\lambda a)}{H_m^{(1)\prime}(\lambda a)} \\
& - \frac{1}{2}\,\frac{(1+\delta_{om})}{\cos m\alpha}\,\frac{J_m^{\prime}(\lambda a)}{H_m^{(1)\prime}(\lambda a)} \left\{ \sum_{m'=0}^{m} (2-\delta_{om'})\; i^{m-m'}\; A_{m-m'}\;(e,m'|e,m')\;\cos m'\alpha\; b_{em'} \right. \\
& \qquad + \sum_{m'=0}^{\infty} (2-\delta_{om'})\; i^{m-m'}\; A_{m+m'}\;(e,m'|e,-m')\;\cos m'\alpha\; b_{em'} \\
& \qquad + (-1)^m (1-\delta_{om}) \sum_{m'=m}^{\infty} (2-\delta_{om'})\; i^{m-m'}\; A_{m'-m}\;(e,m'|e,-m')\;\cos m'\alpha\; b_{em'} \\
& \qquad - \sum_{m'=0}^{m} (2-\delta_{om'})\; i^{m-m'}\; A_{m-m'}\;(o,m'|e,m')\;\sin m'\alpha\; b_{om'} \\
& \qquad - \sum_{m'=0}^{\infty} (2-\delta_{om'})\; i^{m-m'}\; A_{m+m'}\;(0,m'|e,-m')\;\sin m'\alpha\; b_{om'} \\
& \qquad \left. -(-1)^m (1-\delta_{om}) \sum_{m'=m}^{\infty} (2-\delta_{om'})\; i^{m-m'}\; A_{m'-m}\;(o,m'|e,-m')\;\sin m'\alpha\; b_{om'} \right\} ,
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
b_{om} = &- \left(1 - e^{-i\,2\,\lambda\,h\,\sin\alpha}\right) \frac{J'_m(\lambda a)}{H_m^{(1)\prime}(\lambda a)} \\
&+ \frac{1}{2}\,\frac{1}{\sin m\alpha}\,\frac{J'_m(\lambda a)}{H_m^{(1)\prime}(\lambda a)} \left\{ \sum_{m'=0}^{m} (2-\delta_{om'})\, i^{m-m'}\, A_{m-m'}\,(o,m'|o,m')\,\sin m'\alpha\; b_{om'} \right. \\
&\qquad + \sum_{m'=0}^{\infty} (2-\delta_{om'})\, i^{m-m'}\, A_{m+m'}\,(o,m'|o,-m')\,\sin m'\alpha\; b_{om'} \\
&\qquad + (-1)^{m+1}\,(1-\delta_{om}) \sum_{m'=m}^{\infty} (2-\delta_{om'})\, i^{m-m'}\, A_{m'-m}\,(o,m'|o,-m')\,\sin m'\alpha\; b_{om'} \\
&\qquad - \sum_{m'=0}^{m} (2-\delta_{om'})\, i^{m-m'}\, A_{m-m'}\,(e,m'|o,m')\,\cos m'\alpha\; b_{em'} \\
&\qquad - \sum_{m'=0}^{\infty} (2-\delta_{om'})\, i^{m-m'}\, A_{m+m'}\,(e,m'|o,-m')\,\cos m'\alpha\; b_{em'} \\
&\qquad \left. - (-1)^{m+1}\,(1-\delta_{om}) \sum_{m'=m}^{\infty} (2-\delta_{om'})\, i^{m-m'}\, A_{m'-m}\,(e,m'|o,-m')\,\cos m'\alpha\; b_{em'} \right\} . \qquad (35)
\end{aligned}
$$

2. Transverse Magnetic Waves

When the waves are polarized with the magnetic field perpendicular to the axis of the cylinder, corresponding to the expansions of the incident and reflected fields in Eqs. (14), (15), (24) and (25), the expansions of the scattered fields must have the forms:

$$\vec{E}^{scat} = \frac{E_0}{k \sin\gamma} \sum_{m=0}^{\infty} (2-\delta_{om}) i^m \left[c_{em} \cos m\alpha\, \vec{N}^{(3)}_{em}(\vec{r},\gamma) + c_{om} \sin m\alpha\, \vec{N}^{(3)}_{om}(\vec{r},\gamma) \right] , \tag{36}$$

$$\vec{E}'^{scat} = \frac{E_0}{k \sin\gamma} \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \left[-c'_{em} \cos m\alpha\, \vec{N}^{(3)}_{em}(\vec{r}',\gamma) + c'_{om} \sin m\alpha\, \vec{N}^{(3)}_{om}(\vec{r}',\gamma) \right] , \tag{37}$$

$$\vec{H}^{scat} = \frac{1}{i k \sin\gamma} \sqrt{\frac{\epsilon_0}{\mu_0}}\, E_0 \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \left[c_{em} \cos m\alpha\, \vec{M}^{(3)}_{em}(\vec{r},\gamma) + c_{om} \sin m\alpha\, \vec{M}^{(3)}_{om}(\vec{r},\gamma) \right] , \tag{38}$$

$$\vec{H}'^{scat} = \frac{1}{i k \sin\gamma} \sqrt{\frac{\epsilon_0}{\mu_0}}\, E_0 \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \left[-c'_{em} \cos m\alpha\, \vec{M}^{(3)}_{em}(\vec{r}',\gamma) + c'_{om} \sin m\alpha\, \vec{M}^{(3)}_{om}(\vec{r}',\gamma) \right] . \tag{39}$$

Again, on examining the tangential components of the electric field, it is found that if we set the primed expansion coefficients equal to the unprimed ones,

$$c'_{\substack{e\\o}m} = c_{\substack{e\\o}m} , \tag{40}$$

then the boundary condition of vanishing tangential electric field on the surface of the perfectly conducting ground is completely satisfied. Setting now the z-component of the total electric field equal to zero on the surface of the cylinder, we thereby determine the coefficients c_{8m} and satisfy the boundary condition on both tangential components of the electric field. Thus we have

$$E_z^{inc}(a,\varphi,z) + E_z^{ref}(a,\varphi,z) + E_z^{scat}(a,\varphi,z) + \left[\hat{e}_z \cdot \vec{E}'^{\,scat}(\vec{r}\,',\gamma)\right]_{\rho=a} = 0 \tag{41}$$

from which we get

$$\begin{aligned}
&E_0 \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \left[\cos m\alpha\, N_{emz}^{(1)}(a,\varphi,z;\gamma) + \sin m\alpha\, N_{omz}^{(1)}(a,\varphi,z;\gamma)\right] \\
&+ E_0' \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \left[-\cos m\alpha\, N_{emz}^{(1)}(a,\varphi,z;\gamma) + \sin m\alpha\, N_{omz}^{(1)}(a,\varphi,z;\gamma)\right] \\
&+ E_0 \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \left[c_{em} \cos m\alpha\, N_{emz}^{(3)}(a,\varphi,z;\gamma) + c_{om} \sin m\alpha\, N_{omz}^{(3)}(a,\varphi,z;\gamma)\right] \\
&+ E_0 \sum_{m=0}^{\infty} (2-\delta_{om})\, i^m \Bigg[-c_{em} \cos m\alpha \sum_{n,\mu,j} (-1)^n A_n(e,m|j,\mu)\, N_{j,n+\mu,z}^{(1)}(a,\varphi,z;\gamma) \\
&\qquad\qquad + c_{om} \sin m\alpha \sum_{n,\mu,j} (-1)^n A_n(o,m|j,\mu)\, N_{j,n+\mu,z}^{(1)}(a,\varphi,z;\gamma)\Bigg] \\
&= 0
\end{aligned} \tag{42}$$

where we have utilized the addition theorem that results from taking the curl of Eq. (B-16) in the last term. Changing the summation variables and interchanging the order of summation just as we did in the case of transverse electric waves and then inserting the explicit expressions for the functions $N^{(1),(3)}_{\substack{e\\o}mz}(a,\varphi,z;\gamma)$, we again get a coupled system of equations for the expansion coefficients:

$$
\begin{aligned}
c_{em} = {} & -\left(1 - e^{-i\,2\,\lambda\,h\,\sin\alpha}\right) \frac{J_m(\lambda a)}{H_m^{(1)}(\lambda a)} \\
& + \frac{1}{2}\,\frac{(1+\delta_{om})}{\cos m\alpha}\,\frac{J_m(\lambda a)}{H_m^{(1)}(\lambda a)} \Bigg\{ \sum_{m'=0}^{m} (2-\delta_{om'})\, i^{m-m'} A_{m-m'}(e,m'|e,m') \cos m'\alpha\, c_{em'} \\
& \qquad + \sum_{m'=0}^{\infty} (2-\delta_{om'})\, i^{m-m'} A_{m+m'}(e,m'|e,-m') \cos m'\alpha\, c_{em'} \\
& \qquad + (-1)^m (1-\delta_{om})\, 2 \sum_{m'=m\geq 1}^{\infty} i^{m-m'} A_{m'-m}(e,m'|e,-m') \cos m'\alpha\, c_{em'} \\
& \qquad - 2 \sum_{m'=1}^{m} i^{m-m'} A_{m-m'}(o,m'|e,m') \sin m'\alpha\, c_{om'} \\
& \qquad - 2 \sum_{m'=1}^{\infty} i^{m-m'} A_{m+m'}(o,m'|e,-m') \sin m'\alpha\, c_{om'} \\
& \qquad - (-1)^m (1-\delta_{om})\, 2 \sum_{m'=m\geq 1}^{\infty} i^{m-m'} A_{m'-m}(o,m'|e,-m') \sin m'\alpha\, c_{om'} \Bigg\}, \\
& \hspace{20em} (m \geq 0), \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
c_{om} = {} & -(1 + e^{-i\,2\lambda\, h \sin\alpha})\,\frac{J_m(\lambda a)}{H_m^{(1)}(\lambda a)} \\
& - \frac{1}{2}\,\frac{1}{\sin m\alpha}\,\frac{J_m(\lambda a)}{H_m^{(1)}(\lambda a)} \Bigg\{ 2 \sum_{m'=1}^{m} i^{m-m'} A_{m-m'}(o,m' \,|\, o,m') \sin m'\alpha\, c_{om'} \\
& \qquad + 2 \sum_{m'=1}^{\infty} i^{m-m'} A_{m+m'}(o,m' \,|\, o,-m') \sin m'\alpha\, c_{om'} \\
& \qquad + (-1)^{m+1} (1 - \delta_{om})\, 2 \sum_{m'=m}^{\infty} i^{m-m'} A_{m'-m}(o,m' \,|\, o,-m') \sin m'\alpha\, c_{om'} \\
& \qquad - \sum_{m'=0}^{m} (2-\delta_{om'})\, i^{m-m'} A_{m-m'}(e,m' \,|\, o,m') \cos m'\alpha\, c_{em'} \\
& \qquad - \sum_{m'=0}^{\infty} (2-\delta_{om'})\, i^{m-m'} A_{m+m'}(e,m' \,|\, o,-m') \cos m'\alpha\, c_{em'} \\
& \qquad - (-1)^{m+1} (1 - \delta_{om})\, 2 \sum_{m'=m}^{\infty} i^{m-m'} A_{m'-m}(e,m' \,|\, o,-m') \cos m'\alpha\, c_{em'} \Bigg\} .
\end{aligned}
$$

$$(m \geq 1) \qquad\qquad (44)$$

IV AXIAL CURRENT

A quantity that is of interest in some applications is the total axial current induced on the cylinder. It is given by the expression

$$I = a \int_0^{2\pi} H_\varphi (a,\varphi,z)\, d\varphi \tag{45}$$

where H_φ is the φ-component of the total magnetic field.

In the case of <u>transverse electric waves</u>, there is no net axial current. This can be seen by noting from Eqs. (13), (20), and the curl of Eqs. (26) and (27), that H_φ is expressed by a number of summations over the functions $N^{(1),(3)}_{\substack{e\\o}m\varphi}(a,\varphi,z;\gamma)$. For the total axial current we thus have summations of the form [see Eq. (11)]

$$\sum_{m=0}^{\infty} (\ldots)\, m \int_0^{2\pi} \begin{matrix}\sin\\ \cos\end{matrix}\, m\,\varphi\, d\,\varphi = 0,$$

since each term vanishes. Physically this result is due to the fact that there is no z-component of the electric field.

For <u>transverse magnetic waves</u>, on the other hand, we have a z-component of the electric field and consequently a total axial current. It is given by Eq. (45) with

$$H_\varphi(a,\varphi,z) = H_\varphi^{inc}(a,\varphi,z) + H_\varphi^{ref}(a,\varphi,z) + H_\varphi^{scat}(a,\varphi,z) + [\hat{e}_\varphi \cdot \vec{H}'^{\,scat}(\vec{r}\,')]_{\rho=a}$$

$$= \frac{\sqrt{\epsilon_0/\mu_0}\, E_0}{i\, k \sin\gamma} \sum_{m=0}^{\infty} (2 - \delta_{om})\, i^m \cdot$$

$$\cdot \left[\cos m\alpha\, M^{(1)}_{em\varphi}(a,\varphi,z;\gamma) + \sin m\alpha\, M^{(1)}_{om\varphi}(a,\varphi,z;\gamma)\right] \quad +$$

$$+\frac{\sqrt{\epsilon_0/\mu_0}\;E_0'}{i\;k\;\sin\gamma}\sum_{m=0}^{\infty}(2-\delta_{om})\;i^m\;[-\cos m\alpha\,M^{(1)}_{em\varphi}(a,\varphi,z;\gamma)$$

$$+\sin m\alpha\,M^{(1)}_{om\varphi}(a,\varphi,z;\gamma)]$$

$$+\frac{\sqrt{\epsilon_0/\mu_0}\;E_0}{i\;k\;\sin\gamma}\sum_{m=0}^{\infty}(2-\delta_{om})\;i^m\;[\cos m\alpha\;c_{em}\;M^{(3)}_{em\varphi}(a,\varphi,z;\gamma)$$

$$+\sin m\alpha\;c_{om}\;M^{(3)}_{om\varphi}(a,\varphi,z;\gamma)]$$

$$+\frac{\sqrt{\epsilon_0/\mu_0}\;E_0}{i\;k\;\sin\gamma}\sum_{m'=0}^{\infty}(2-\delta_{om'})\;i^{m'}\Bigg[-\cos m'\alpha\;c_{em'}\sum_{n,\mu,j}(-1)^n A_n(e,m'|j,\mu)\;\cdot$$

$$\cdot\;M^{(1)}_{j,n+\mu,\varphi}(a,\varphi,z;\gamma)+\sin m'\alpha\;c_{om'}\sum_{n,\mu,j}(-1)^n A_n(o,m'|j,\mu)\;\cdot$$

$$\cdot\;M^{(1)}_{j,n+\mu,\varphi}(a,\varphi,z;\gamma)\Bigg] \qquad (46)$$

where we have used again the addition theorem of Eq. (B-16). When Eq. (46) is inserted in Eq. (45) the only terms that survive the integration over φ are the non-vanishing φ-independent terms that contain $M^{(1),(3)}_{e0\varphi}$. Substituting in the resulting relation the explicit expressions for $M^{(1),(3)}_{e0\varphi}$ from Eq. (10) and for translation coefficients from Eq. (B-19) with A_n given by Eq. (B-11), we obtain for the total axial current the result

$$\sqrt{\frac{\mu_0}{\epsilon_0}}\;\frac{I(\omega)}{2\pi a\,E_0}=\Bigg\{-i(1-e^{-i\,2\,k\,h\,\sin\gamma\,\sin\alpha})\;J_1(k\,a\,\sin\gamma)$$

$$-i\,H_1^{(1)}(k\,a\,\sin\gamma)\;c_{e0}+i\,\frac{1}{2}\,H_0^{(1)}(2\,k\,h\,\sin\gamma)\;J_1(k\,a\,\sin\gamma)\;c_{e0}$$

$$+i\,\frac{1}{2}\sum_{m'=0,2,4,\ldots}^{\infty}{}'\;(2-\delta_{om})^2\,(-1)^{m'}\,H^{(1)}_{m'}(2\,k\,h\,\sin\gamma)$$

$$\cdot\;J_1(k\,a\,\sin\gamma)\;\cos m'\alpha\;c_{em'}\qquad -$$

$$- 2 \sum_{m'=1,3,5\ldots}^{\infty}{}' (-1)^{m'} H_{m'}^{(1)} (2 k h \sin\gamma) J_1(k a \sin\gamma) \sin m' \alpha \, c_{om'} \Bigg\} e^{i k z \cos\gamma} , \tag{47}$$

where a prime over a summation signifies that the summations are over only even or odd values.

V POTENTIAL OF CYLINDER

The potential of the cylinder with respect to the ground is given by

$$V = \int_{-a}^{-h} E_y(-\frac{\pi}{2})\, dy = \int_{a}^{h} E_\rho(-\frac{\pi}{2})\, d\rho \quad , \tag{48}$$

in which we have ignored the other coordinates in the argument and indicated only that the electric field is to be integrated along the line $\varphi = -\frac{1}{2}\pi$. In terms of the primed coordinate system that is centered on the image cylinder, the potential of the cylinder is

$$V = \int_{2h-a}^{h} E_{y'}(\frac{\pi}{2})\, dy' = \int_{2h-a}^{h} E_{\rho'}(\frac{\pi}{2})\, d\rho' \quad , \tag{49}$$

where it is noted that in this sytem the field is integrated along the line $\varphi' = \frac{1}{2}\pi$.

1. Transverse Electric Waves

We shall write the ρ-component of the electric field in two parts. From Eqs. (1), (3), (10), (19), and (26) we have for the first part

$$\begin{aligned} & E_\rho^{inc}(-\frac{\pi}{2}) + E_\rho^{ref}(-\frac{\pi}{2}) + E_\rho^{scat}(-\frac{\pi}{2}) \\ & \quad = E_0 \cos\alpha\, e^{-ik\rho\sin\gamma\sin\alpha + ikz\cos\gamma} \\ & \quad + E_0' \cos\alpha\, e^{ik\rho\sin\gamma\sin\alpha + ikz\cos\gamma} \\ & \quad + \frac{2E_0}{ik\sin\gamma} \sum_{m=1}^{\infty} i^m \left[b_{em} \cos m\alpha\, M_{em\rho}^{(3)}(\rho, -\frac{\pi}{2}, z; \gamma) \right. + \end{aligned}$$

$$+ b_{om} \sin m\alpha \, M^{(3)}_{om\rho}(\rho, -\frac{\pi}{2}, z; \gamma)]$$

$$= E_0 e^{ikz\cos\gamma} \left\{ \cos\alpha \, e^{-i\lambda\rho\sin\alpha} + \cos\alpha \, e^{-i2\lambda h \sin\alpha} e^{i\lambda\rho\sin\alpha} + 2 \sum_{m=1}^{\infty}{}' \cos m\alpha \, b_{em} \, m \frac{H^{(1)}_m(\lambda\rho)}{\lambda\rho} - 2i \sum_{m=2}^{\infty}{}' \sin m\alpha \, b_{om} \, m \frac{H^{(1)}_m(\lambda\rho)}{\lambda\rho} \right\}, \quad (50)$$

where it is to be remembered that $\lambda = k \sin\gamma$ and that a prime over a summation sign signifies that the summation extends only over even or odd values of m. We shall write the second part in terms of the primed coordinates:

$$E^{\prime\,scat}_{\rho^\prime}(\frac{\pi}{2}) = \frac{2E_0}{ik\sin\gamma} \sum_{m=1}^{\infty} i^m [b_{em} \cos m\alpha \, M^{(3)}_{em\rho^\prime}(\rho^\prime, \frac{\pi}{2}, z; \gamma) - b_{om} \sin m\alpha \, M^{(3)}_{om\rho^\prime}(\rho^\prime, \frac{\pi}{2}, z; \gamma)]$$

$$= 2E_0 e^{ikz\cos\gamma} \left\{ - \sum_{m=1}^{\infty}{}' b_{em} \cos m\alpha \, m \frac{H^{(1)}_m(\lambda\rho^\prime)}{\lambda\rho^\prime} + i \sum_{m=2}^{\infty}{}' b_{om} \sin m\alpha \, m \frac{H^{(1)}_m(\lambda\rho^\prime)}{\lambda\rho^\prime} \right\}. \quad (51)$$

Inserting Eq. (50) in Eq. (48) and Eq. (51) in Eq. (49) and adding the resulting expressions, we find that the potential of the cylinder with respect to the ground in the case of transverse electric waves is

$$
\begin{aligned}
V = E_0\, e^{ikz\cos\gamma} \Bigg\{ & \frac{\cot\alpha}{i\lambda} \left[e^{-i\lambda a \sin\alpha} - e^{-i2\lambda h \sin\alpha} \cdot e^{i\lambda a \sin\alpha} \right] \\
& + 2 \sum_{m=1}^{\infty}{}' \cos m\alpha \; b_{em} \int_a^h m \frac{H_m^{(1)}(\lambda\rho)}{\lambda\rho}\, d\rho \\
& - 2i \sum_{m=2}^{\infty}{}' \sin m\alpha \; b_{om} \int_a^h m \frac{H_m^{(1)}(\lambda\rho)}{\lambda\rho}\, d\rho \\
& - 2 \sum_{m=1}^{\infty}{}' \cos m\alpha \; b_{em} \int_{2h-a}^h m \frac{H_m^{(1)}(\lambda\rho')}{\lambda\rho'}\, d\rho' \\
& + 2i \sum_{m=2}^{\infty}{}' \sin m\alpha \; b_{om} \int_{2h-a}^h m \frac{H_m^{(1)}(\lambda\rho')}{\lambda\rho'}\, d\rho' \Bigg\} .
\end{aligned}
\tag{52}
$$

This expression may be written as

$$
\begin{aligned}
V = E_0\, e^{ikz\cos\gamma} \Bigg\{ & \cos\alpha \int_a^{2h-a} e^{-i\lambda\rho\sin\alpha}\, d\rho \\
& + 2 \sum_{m=1}^{\infty}{}' \cos m\alpha \; b_{em} \int_a^{2h-a} m \frac{H_m^{(1)}(\lambda\rho)}{\lambda\rho}\, d\rho \\
& - 2i \sum_{m=2}^{\infty}{}' \sin m\alpha \; b_{om} \int_a^{2h-a} m \frac{H_m^{(1)}(\lambda\rho)}{\lambda\rho}\, d\rho \Bigg\} .
\end{aligned}
\tag{53}
$$

That is, the potential of the cylinder is given by the integral of the electric field of the incident and scattered fields from the cylinder to the image cylinder.

The integrals of the Hankel functions in the above expressions can be carried out with the aid of the recursion formulas

$$\int m \frac{H_m^{(1)}(z)}{z}\, dz = - H_m^{(1)}(z) + \int H_{m-1}^{(1)}(z)\, dz , \qquad (54)$$

$$H_{m-1}^{(1)}(z) = - 2 \frac{d}{dz} H_{m-2}^{(1)}(z) + H_{m-3}^{(1)}(z) . \qquad (55)$$

After repeated use of Eq. (55) one ends up with either $H_0^{(1)}$ or the integral of $H_0^{(1)}$, in the case of even or odd m, respectively.

2. Transverse Magnetic Waves

From Eqs. (1), (11), (15), (25), and (36), we now have

$$\begin{aligned} E_\rho^{inc}(-\tfrac{\pi}{2}) &+ E_\rho^{ref}(-\tfrac{\pi}{2}) + E_\rho^{scat}(-\tfrac{\pi}{2}) \\ &= E_0\, e^{i k z \cos\gamma} \Bigg\{ \cos\gamma \sin\alpha\, e^{-i\lambda\rho\sin\alpha} \\ &\quad + \cos\gamma \sin\alpha\, e^{-i 2\lambda h \sin\alpha}\, e^{i\lambda\rho\sin\alpha} \\ &\quad + i \cos\gamma {\sum_{m=0}^{\infty}}' (2-\delta_{0m}) \cos m\alpha\, c_{em} \frac{d H_m^{(1)}(\lambda\rho)}{d(\lambda\rho)} \\ &\quad + 2\cos\gamma {\sum_{m=1}^{\infty}}' \sin m\alpha\, c_{om} \frac{d H_m^{(1)}(\lambda\rho)}{d(\lambda\rho)} \Bigg\} , \qquad (56a) \end{aligned}$$

and

$$\begin{aligned} E_{\rho'}^{\prime scat}(\tfrac{\pi}{2}) = E_0\, e^{i k z \cos\gamma} \cos\gamma \Bigg\{ &-i {\sum_{m=0}^{\infty}}' (2-\delta_{om}) \cos m\alpha\, c_{em} \frac{d H_m^{(1)}(\lambda\rho')}{d(\lambda\rho')} \\ &-2 {\sum_{m=1}^{\infty}}' \sin m\alpha\, c_{om} \frac{d H_m^{(1)}(\lambda\rho')}{d(\lambda\rho')} \Bigg\} . \qquad (56b) \end{aligned}$$

The potential of the cylinder with respect to the ground in the case of transverse magnetic waves is then

$$V = -\frac{E_0\, e^{i k z \cos\gamma}}{k \sin\gamma} \cos\gamma \Big\{ i \left[e^{-i\lambda a \sin\alpha} - e^{-i 2\lambda h \sin\alpha} e^{i\lambda a \sin\alpha} \right]$$

$$+ i \sum_{m=0}^{\infty}{}' (2-\delta_{om}) \cos m\alpha \, c_{em} \left[H_m^{(1)}(\lambda a) - H_m^{(1)}(\lambda(2h-a)) \right]$$

$$+ 2 \sum_{m=1}^{\infty}{}' \sin m\alpha \, c_{om} \left[H_m^{(1)}(\lambda a) - H_m^{(1)}(\lambda(2h-a)) \right] \Big\} . \qquad (57)$$

However, by a simple argument, it can be shown that the potential V must vanish for transverse magnetic waves provided that both the cylinder and the ground plane are perfect conductors.

By taking components of Maxwell's equations for the total field (incident plus scattered), we find

$$\frac{\partial H_\varphi}{\partial z} = i\omega\epsilon_o E_\rho \quad ; \quad \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = i\omega\mu_o H_\varphi \quad .$$

Since all field components are proportional to exp(ikz cos γ), we have

$$H_\varphi = \frac{\omega\,\epsilon_o}{k \cos\gamma} E_\rho \quad ; \quad ik \cos\gamma\, E_\rho - i\omega\,\mu_o H_\varphi = \frac{\partial E_z}{\partial \rho} \quad ,$$

while elimination of H_φ between the latter results yields

$$E_\rho = \left(\frac{i \cos\gamma}{k \sin^2\gamma} \right) \frac{\partial E_z}{\partial \rho} \quad . \qquad (58a)$$

Finally, use of Eq. (58a) in Eq. (48) for the potential V yields

$$V = \left(\frac{i \cos\gamma}{k \sin^2\gamma} \right) \int_a^h \frac{\partial E_z}{\partial \rho} d\rho = \left(\frac{i \cos\gamma}{k \sin^2\gamma} \right) \left[E_z(h) - E_z(a) \right] = 0 \quad , \qquad (58b)$$

since E_z is identically zero on both the cylinder ($\rho = a$) and on the conducting ground plane ($\rho = h$).

VI PULSE SOLUTIONS FOR THIN WIRES

In the previous sections we have derived exact expressions for the various quantities of interest here when a given plane wave of frequency ω is incident on a cylinder in the presence of a perfectly conducting ground. To obtain the results for incident pulses we must multiply the above expressions by the appropriate spectral function $f(\omega)$ and by $e^{-i\omega t}$ and then integrate over all frequencies. Before we can proceed, however, we must have the values of the expansion coefficients. This entails solving the two sets of simultaneous equations in Eqs. (34) and (35) or in Eqs. (43) and (44) depending upon the polarization. The form of these equations immediately suggests an interative procedure which has, in fact, a direct physical interpretation, namely, that each further iteration takes into account an additional order of multiple scattering. This procedure is easily carried out by means of a computer and will be discussed further in a later section on numerical considerations.

The pulse solutions that we shall undertake to calculate here will be for a thin wire located at a large distance above the ground, so that $h/a \gg 1$. In this case we can obtain analytical approximations for the leading expansion coefficients and then obtain approximate expansions for the quantities of interest such as the total axial current induced by an incident pulse. The first approximation that we shall examine will be the first-order iterations for the leading coefficients required in the expression for the current. It will be found that the coefficient c_{e0} is the dominant one and we shall then turn to the approximation in which all the other coefficients are neglected and c_{e0} is determined as the solution of the remaining equation.

1. First-order iteration

Let is iterate once in the expression of Eq. (43) for c_{e0} to take into account the first-order effect of the ground. We find that

$$c_{e0} = -(1-e^{-i2\lambda h \sin \alpha}) \frac{J_o(\lambda a)}{H_o^{(1)}(\lambda a)} \cdot \left\{ 1 + \frac{J_o(\lambda a)}{H_o^{(1)}(\lambda a)} H_o^{(1)}(2\lambda h) \right\}$$

$$+ 2i(1+e^{-i2\lambda h \sin \alpha}) \frac{J_1(\lambda a)}{H_1^{(1)}(\lambda a)} \cdot \frac{\sin \alpha \, J_o(\lambda a) \, H_1^{(1)}(2\lambda h)}{H_o^{(1)}(\lambda a)} , \qquad (59)$$

where in Eq. (43) we have substituted the leading term in Eq. (44) for c_{o1}, namely,

$$c_{o1} = -(1+e^{-i2\lambda h \sin \alpha}) \frac{J_1(\lambda a)}{H_1^{(1)}(\lambda a)} . \qquad (60)$$

When the expressions of Eqs. (59) and (60) are substituted in Eq. (47) for the current, and use is made of the Wronskian relation

$$J_o(z) \, H_1^{(1)}(z) - H_o^{(1)}(z) \, J_1(z) = - \frac{2i}{\pi z} , \qquad (61)$$

the expression for the current becomes

$$\sqrt{\frac{\mu_0}{\epsilon_0}} \, \frac{I(\omega)}{2\pi a E_0} = \left\{ \frac{2}{\pi \lambda a} \cdot \frac{(1-e^{-i2\lambda h \sin \alpha})}{H_o^{(1)}(\lambda a)} \right.$$

$$+ \frac{2}{\pi \lambda a} \cdot \frac{(1-e^{-i2\lambda h \sin \alpha}) \cdot J_o(\lambda a) \, H_o^{(1)}(2\lambda h)}{H_o^{(1)^2}(\lambda a)} \quad -$$

$$- \frac{i\, 4(1 + e^{-i2\lambda h \sin \alpha}) \sin \alpha}{\pi \lambda a} \cdot \left. \frac{J_1(\lambda a)\, H_1^{(1)}(2\lambda h)}{H_o^{(1)}(\lambda a)\, H_1^{(1)}(\lambda a)} \right\} e^{ikz \cos \gamma} \; . \tag{62}$$

To obtain the total time dependent axial current, $I(\omega)e^{-i\omega t}$ must be multiplied by the appropriate spectral function and integrated over ω. We shall take the delta function pulse as an example.

<u>Delta function pulse</u>. We have

$$\sqrt{\frac{\mu_0}{\epsilon_0}} \; \frac{I(t)}{2\pi a E_0} = \frac{\sqrt{\mu_0 | \epsilon_0}}{2\pi a E_0} \int_{-\infty}^{\infty} \frac{I(\omega)}{2\pi} \, e^{-i\omega t} d\omega \; . \tag{63}$$

We introduce the following dimensionless variables that we shall employ from now on

$$u = \frac{ct - z \cos \gamma}{a \sin \gamma} \; , \tag{64}$$

$$v = - \frac{2h}{a} \sin \alpha = \frac{2h}{a} \left| \sin \alpha \right| , \tag{65}$$

$$\varkappa = \lambda a = k\, a \sin \gamma \; . \tag{66}$$

Putting the expression for $I(\omega)$ from Eq. (62) in Eq. (63), we obtain

$$\begin{aligned} \frac{\pi a \sin \gamma}{c} \sqrt{\frac{\mu_0}{\epsilon_0}} \; \frac{I(t)}{2\pi a E_0} &= \int_{-\infty}^{\infty} \frac{e^{-i\varkappa u}(1 - e^{i\varkappa v})}{\varkappa\, H_o^{(1)}(\varkappa)} \, d\varkappa \\ &+ \int_{-\infty}^{\infty} \frac{e^{-i\varkappa u}(1 - e^{ikv})}{\varkappa H_o^{(1)2}(\varkappa)} \; J_o(\varkappa) H_o^{(1)}(2\varkappa \tfrac{h}{a}) \, d\varkappa \\ &- 2i \sin \alpha \int_{-\infty}^{\infty} \frac{e^{-i\varkappa u}(1 + e^{i\varkappa v})\; J_1(\varkappa)\; H_1^{(1)}(2\varkappa \tfrac{h}{a})}{\varkappa\, H_o^{(1)}(\varkappa)\; H_1^{(1)}(\varkappa)} \, d\varkappa \; . \end{aligned} \tag{67}$$

An analysis of the last integral in Eq. (67), which comes from c_{o1}, indicates that it will be small compared with the other two, which come from c_{e0}, and it will therefore not be considered further. Let us examine the first integral in Eq. (67), which we shall call $\mathcal{J}_1$. We write

$$\mathcal{J}_1 = \mathcal{J}_{11}(u) - \mathcal{J}_{11}(u-v) \ , \tag{68}$$

with

$$\mathcal{J}_{11}(u) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{-i\varkappa u}}{(\varkappa + i\epsilon) H_0^{(1)}(\varkappa)} \, d\varkappa \ . \tag{69}$$

We remark that $H_0^{(1)}(\varkappa)$ is free of zeros on the principal branch $-\pi < \arg \varkappa < \pi$, and there is a branch cut from 0 to $-\infty$ along the negative real axis. For $u < -1$ we can close the contour of $\mathcal{J}_{11}(u)$ by an infinite semi-circle in the upper half $\varkappa$-plane and thereby obtain

$$\mathcal{J}_{11}(u) = 0, \qquad u < -1 \tag{70}$$

This simply states that the total axial current induced by the incident pulse is zero until the wave front of the pulse meets the surface of the cylinder at the given value of z. Similarly, of course,

$$\mathcal{J}_{11}(u-v) = 0, \quad u < 2\frac{h}{a} \left|\sin \alpha\right| - 1, \tag{71}$$

which is to say that $\mathcal{J}_{11}(u-v)$ vanishes until the reflected wave front reaches the surface of the cylinder at the given value of z.

For $u > -1$, the contour of $\mathcal{J}_{11}(u)$ must be closed in the lower half plane. A convenient way to proceed here is to note that the part of the integral in Eq. (69) which extends over the negative values of $\varkappa$ is the complex conjugate of the part over the positive values of $\varkappa$.

Utilizing this and completing the contour of the integral over positive $\varkappa$ by the quarter of the circle at infinity in the fourth quadrant, and then along the negative imaginary axis of $\varkappa$, we obtain

$$\mathcal{J}_{11}(u) = \pi^2 \int_0^\infty \frac{e^{-u\eta} I_o(\eta)}{\eta\,[K_o^2(\eta) + \pi^2 I_o^2(\eta)]}\, d\eta \qquad (72)$$

where I_o and K_o are the modified Bessel functions of order zero. The integral in Eq. (72) has been calculated previously by several authors [10, 11]. It is the result obtained for the current induced on a cylinder in free space [11]. We shall not consider it further here.

The second integral Eq. (67) can also be examined in the same way. It vanishes when $u < [2(h-a)/a] - 1$, and can be evaluated in various subsequent intervals of u. We shall not bother to write the results, down, however. Instead we shall go on to a more comprehensive expression for the current which contains the present integrals as the first terms in an infinite expansion.

2. φ-independent approximation

It was observed in the preceding section that the first approximation to c_{o1} leads to results that could be neglected when compared with those due to c_{e0}. All higher order coefficients can be expected to lead to results that are even more negligible in the present case. Carrying this argument to the extreme, we neglect all coefficients other than that of the φ-independent term, c_{e0}. We now solve Eq. (43) with m=0, when all the coefficients except c_{e0} are set equal to zero, and get

$$c_{e0} = - \frac{(1-e^{-i2\lambda h \sin \alpha})\, J_o(\lambda a)}{H_o^{(1)}(\lambda a) - H_o^{(1)}(2\lambda h)\, J_o(\lambda a)} \qquad (73)$$

Substituting this expression in Eq. (47) and setting all other coefficients equal to zero there, we obtain

$$\sqrt{\frac{\mu_0}{\epsilon_0}}\ \frac{I(\omega)}{2\pi a E_0} = \frac{2}{\pi\lambda a}\ \frac{(1-e^{-i2\lambda h \sin\alpha})e^{ikz\cos\gamma}}{H_o^{(1)}(\lambda a) - H_o^{(1)}(2\lambda h)\ J_o(\lambda a)}\ . \tag{74}$$

Consider the denominator of Eqs. (73) and (74). If we expand

$$\frac{1}{H_o^{(1)}(\kappa) - H_o^{(1)}(2\kappa\frac{h}{a})\ J_o(\kappa)} = \frac{1}{H_o^{(1)}(\kappa)}\left[1 - \frac{H_o^{(1)}(2\kappa\frac{h}{a})\ J_o(\kappa)}{H_o^{(1)}(\kappa)}\right]^{-1}$$

by the binomial expansion, we get an infinite series each term of which is what one would obtain if we had iterated Eq. (43) for c_{e0} with all other coefficients set equal to zero. Eq. (73) is therefore the sum of the iterated series for c_{e0} when all other coefficients are neglected. Furthermore, since Eq. (73) was obtained by solving Eq. (43) directly, and not summing the iterated series, it follows that the sum is valid for κ=0, when the series does not converge.

Let us consider Eq. (74) further. Since λ=k sin γ, we see that the limit of the right-hand side as $\omega \rightarrow 0$ is finite, namely,

$$\sqrt{\frac{\mu_0}{\epsilon_0}}\ \frac{I(\omega)}{2\pi a E_0} \xrightarrow[\omega\rightarrow 0]{} -\frac{2\frac{h}{a}\sin\alpha}{\log 2\frac{h}{a}}\ .$$

But as stated in the Introduction, this is an unphysical result. It is a consequence of our assuming that the cylinder is perfectly conducting, even at low frequencies where actually the field penetrates any real conductor. At low frequencies it is not physically sound to use the idealization of a perfect conductor. In Appendix D we derive an expression for I(ω) when the conductivity of the wire is finite. It goes correctly to zero when the frequency does. In the present section we shall continue to use Eq. (74) for convenience, but we emphasize that Eq. (D21) would lead to more physically correct results.

We shall now multiply Eq. (74) by the spectral functions for the delta function pulse and for the step function pulse insert the time dependence factor $e^{-i\omega t}$, and then integrate over the frequency to obtain the respective expressions for the total time-dependent axial current, defined by

$$I(t) = \int_{-\infty}^{\infty} f(\omega)\, I(\omega) e^{-i\omega t}\, d\omega \, . \tag{75}$$

Delta function pulse. In terms of the variables of Eqs. (64)-(66) we have

$$\frac{\pi a \sin\gamma}{c}\sqrt{\frac{\mu_0}{\varepsilon_0}}\;\frac{I(t)}{2aE_0} = \int_{-\infty}^{\infty} \frac{e^{-i\varkappa u}(1-e^{i\varkappa v})}{\varkappa\left[H_o^{(1)}(\varkappa)-H_o^{(1)}(2\varkappa\frac{h}{a})\, J_o(\varkappa)\right]}\, d\varkappa . \tag{76}$$

The integrand vanishes on the semicircle of infinite radius in the upper half plane when $u < -1$, so again we obtain the expected result that there is no current at given z until the wave front of the pulse meets the cylinder there. When $u > -1$ we must close the contour in the lower half plane. We write

$$\begin{aligned}\mathcal{I}_d &= \int_{-\infty}^{\infty} \frac{e^{-i\varkappa u}(1-e^{i\varkappa v})}{\varkappa[H_o^{(1)}(\varkappa) - H_o^{(1)}(2\varkappa\frac{h}{a})\; J_o(\varkappa)]}\, d\varkappa \\ &= \int_{0}^{\infty} \frac{e^{-i\varkappa u}(1-e^{i\varkappa v})}{\varkappa[H_o^{(1)}(\varkappa) - H_o^{(1)}(2\varkappa\frac{h}{a})\; J_o(\varkappa)]}\, d\varkappa \\ &\quad + \text{complex conjugate.}\end{aligned} \tag{77}$$

For $u > -1$, closing the contour around the infinite circle in the fourth quadrant and then along the negative imaginary axis, we obtain*

$$\mathcal{I}_d = 2\pi i \Sigma R_4 + \int_0^\infty \frac{e^{-u\eta}(1-e^{v\eta}) \qquad d\eta}{\eta[H_o^{(1)}(-i\eta) - H_o^{(1)}(-i2\frac{h}{a}\eta)\, J_o(-i\eta)]} \tag{78}$$

+ complex conjugate,

where ΣR_4 denotes the sum of the residues of the integrand at its poles in the fourth quadrant. These poles are the zeros of $H_o^{(1)}(\varkappa)-H_o^{(1)}(2\varkappa\frac{h}{a})J_o(\varkappa)$. Rough estimates of these zeros are sketched in Appendix C.

Now

$$H_o^{(1)}(-iz) = -\frac{2i}{\pi}K_o(z) + 2I_o(z) \tag{79}$$

Using this in the integral in Eq. (78) and combining it with its complex conjugate we obtain the following expression for the total axial current in the case of a delta function pulse:

$$\frac{\pi a \sin\gamma}{c}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{I(t)}{2\pi a E_0} = 2\pi i \Sigma R_4 + \text{complex conjugate}$$

$$+ \pi^2 \int_0^\infty \frac{e^{-u\eta}(e^{v\eta}-1)\,I_o(\eta)[I_o(2\frac{h}{a}\eta)-1]}{\eta\left\{[K_o(\eta)-K_o(2\frac{h}{a}\eta)\,I_o(\eta)]^2+\pi^2 I_o^2(\eta)[I_o(2\frac{h}{a}\eta)-1]^2\right\}}\, d\eta \tag{80}$$

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*The same integral as in Eq. (78) is obtained if, instead of following the present procedure, we start from Eq. (76) and continue the contour around the entire semicircle at infinity in the lower half plane. This leads to an integration around both sides of the branch cut from 0 to $-\infty$ along the negative real axis. If the integration variable is transformed to one running from 0 to ∞, a further integration around a contour in the first quadrant is required to bring the result to the form of Eq. (78).

When u → ∞, the residue terms vanish as they are all damped, and it is easily found that the integral in Eq. (80) goes to zero asymptotically as

$$\frac{\pi^2}{2}\,\frac{(2\frac{h}{a})^3\ \sin\alpha}{(\log\,2\frac{h}{a})^2}\cdot\frac{1}{u^3}\,,\qquad (u \to \infty). \tag{81}$$

Step function pulse. The spectral function by which we must multiply Eq. (74) is, from Appendix A,

$$\frac{1}{2\pi}\left(\frac{P}{-i\omega}+\pi\delta(\omega)\right) = \frac{1}{2\pi}\,\frac{a\,\sin\gamma}{c}\left\{\frac{P}{-i\varkappa}+\pi\delta(\varkappa)\right\}\ . \tag{82}$$

The total time dependent axial current is now given by

$$\pi\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{I(t)}{2aE_0} = \int_{-\infty}^{\infty}\left\{\frac{P}{-i\varkappa}+\pi\delta(\varkappa)\right\}\frac{e^{-i\varkappa u}(1-e^{i\varkappa v})\ d\varkappa}{\varkappa[H_0^{(1)}(\varkappa)-H_0^{(1)}(2\varkappa\frac{h}{a})J_0(\varkappa)]}\ . \tag{83}$$

The δ function term gives immediately

$$\frac{\pi(-i)v}{-\frac{2i}{\pi}\log\,2\frac{h}{a}} = \frac{\pi^2}{2}\cdot\frac{2\frac{h}{a}\,|\sin\alpha|}{\log\,2\frac{h}{a}}\ . \tag{84}$$

The principal value term is

$$\mathcal{J}_s = P\int_{-\infty}^{\infty}\frac{e^{-i\varkappa u}(1-e^{i\varkappa v})\quad d\varkappa}{-i\varkappa^2[H_0^{(1)}(\varkappa)-H_0^{(1)}(2\varkappa\frac{h}{a})J_0(\varkappa)]} \tag{85}$$

When u < -1, we can close the contour of the integral in Eq. (85) by the semicircle of infinite radius in the upper half plane and we find that

$$\mathcal{J}_s = -\frac{\pi^2}{2}\cdot\frac{2\frac{h}{a}\,|\sin\alpha|}{\log\,2\frac{h}{a}}\ . \tag{86}$$

Combining this with Eq. (84) we get the expected result that I(t) vanishes for u < -1.

When u > -1, we write

$$\mathcal{J}_s = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{e^{-i\varkappa u}(1-e^{i\varkappa v}) \qquad d\varkappa}{-i\varkappa^2 [H_o^{(1)}(\varkappa) - H_o^{(1)}(2\varkappa\frac{h}{a}) J_o(\varkappa)]} \tag{87}$$

+ complex conjugate,

and close the contour of the integral in the fourth quadrant by the portion of the circle at infinity, the negative imaginary axis, and then around a quarter of the small circle of radius ϵ at the origin. If the complex conjugate of this integral is carried out explicitly, the contour is in the first quadrant, and the two integrals together traverse a semicircle at the origin. Using the relation of Eq. (79) again we obtain

$$\mathcal{J}_s = 2\pi i \Sigma R_4 + \text{complex conjugate}$$

$$+ \frac{\pi^2}{2} \left|\sin \alpha\right| \frac{2\frac{h}{a}}{\log 2\frac{h}{a}}$$

$$+ \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{e^{-u\eta}(e^{v\eta}-1)[K_o(\eta) - K_o(2\frac{h}{a}\eta) I_o(\eta)] \quad d\eta}{\eta^2 \{[K_o(\eta) - K_o(2\frac{h}{a}\eta) I_o(\eta)]^2 + \pi^2 I_o^{\,2}(\eta)[I_o(2\frac{h}{a}\eta) - 1]^2\}} \tag{88}$$

where ΣR_4 denotes again the sum of the residues at the poles of the integrand in the fourth quadrant.

When u → ∞, the integral in Eq. (88) vanishes, as does the sum of the residue terms. Combining the result of Eq. (84) and the identical one that remains in Eq. (88) we find that

$$\sqrt{\frac{\mu_o}{\epsilon_o}} \frac{I(t)}{2\pi a E_0} \xrightarrow[u \to \infty]{} \frac{2\frac{h}{a} \left|\sin \alpha\right|}{\log 2\frac{h}{a}} \quad . \tag{89}$$

VII NUMERICAL CONSIDERATIONS

In this section we describe a program of numerical computations that has been formulated in support of the foregoing theoretical analysis. In the initial considerations it is assumed that an infinitely long conducting wire is located above and parallel to a perfectly conducting ground. In anticipation of extending this work to the more realistic situation that would involve multi-wires of finite length as well as an imperfectly conducting ground, we have formulated the numerical program in a more general way than might be necessary for the initial cases to be considered.

The first task of the numerical program is to determine the scattering coefficients that satisfy the appropriate boundary conditions on the wire surface and on the ground in response to a plane wave impinging upon the system. In our general formulation we retain the option of including as many of the scattering coefficients as are required for any given geometrical configuration. Whereas only the first-order scattering coefficient would provide a good approximation for the initial case of a single wire located well above ground, it is not clear that a single term would suffice for cases involving more than one wire in near proximity to each other, or to ground. In any event the computations are not unduly extended by keeping more than one term in the analysis.

Determination of the scattering coefficients provides the necessary input for determining the induced current and the potential difference between wire and ground in response to a plane wave. This leads to the next phase of calculations--determination of the system response to specific pulse shapes. To this end the plane wave response must be appropriately weighted by the spectral density function of the incoming pulse and integrated over the frequency domain.

The fast Fourier transform (FFT) algorithm is well known to be an efficient tool for numerical integration. This algorithm was employed in obtaining the numerical results reported in this paper. Initial results have been obtained for the specific case of a step-function pulse. It turns out that this pulse shape is difficult to treat because of a singularity in the relevant equation in the vicinity of zero frequency. Whereas the singularity is integrable, it is troublesome to handle with the FFT routine. To circumvent this difficulty, an approximate analytic result was obtained for the region of the singularity.

In the following we formulate the pertinent equations for determining the scattering coefficients in a form that is appropriate for numerical computation. Two cases are considered: I, the incident $\vec{H}$ field is perpendicular to the wire axis; II, the incident $\vec{E}$ field is perpendicular to the wire axis. Only in the first case is a current induced in the wire.

1. Transverse Magnetic Waves

We introduce some new notation that is convenient for the numerical work. Let $\varkappa$ = ka sin γ, and let

$$x_m = c_{em} H_m^{(1)}(\varkappa) \cos m\alpha \tag{90}$$

$$y_m = c_{om} H_m^{(1)}(\varkappa) \sin m\alpha \tag{91}$$

be variants of the scattering coefficients which must satisfy appropriately modified forms of Eqs. (43) and (44). We shall obtain solutions to these equations by the following iteration scheme:

$$x_m^{(p+1)} = \sum_{n=0}^{N-1} a_{mn} x_n^{(p)} + \sum_{n=1}^{N} b_{mn} y_n^{(p)} + f_m \tag{92}$$

$$y_m^{(p+1)} = \sum_{n=0}^{N-1} c_{mn} x_n^{(p)} + \sum_{n=1}^{N} d_{mn} y_n^{(p)} + g_m \tag{93}$$

where we take the initial guesses to be

$$x_m^{(0)} = f_m \tag{94}$$

$$y_m^{(0)} = g_m \tag{95}$$

and where

$$f_m = - [1 - \exp(i\varkappa v)]\, J_m(\varkappa) \cos m\alpha \tag{96}$$

$$g_m = - [1 + \exp(i\varkappa v)]\, J_m(\varkappa) \sin m\alpha \tag{97}$$

$$v = - 2(h/a) \sin \alpha \tag{98}$$

$$\left.\begin{aligned} a_{mn} &= 0 \quad , \qquad\qquad \text{if } n+m \text{ is odd} \\ &= \tfrac{1}{4}(1 + \delta_{0,m})(2 - \delta_{0,n})\, J_m(\varkappa) \left\{ (2 - \delta_{0,n+m}) \frac{H_{n+m}^{(1)}(2\varkappa h/a)}{H_n^{(1)}(\varkappa)} \right. \\ &\quad \left. + (1 + \delta_{m,n} - \delta_{0,m})(2 - \delta_{m,n}) \frac{H_{|n-m|}^{(1)}(2\varkappa h/a)}{H_n^{(1)}(\varkappa)} \right\} \\ &\qquad\qquad \text{if } n+m \text{ is even} \end{aligned}\right\} \tag{99}$$

$$\left.\begin{aligned} b_{mn} &= 0 \quad , \qquad\qquad \text{if } n+m \text{ is even} \\ &= - i(1 + \delta_{0,m})\, J_m(\varkappa) \left\{ \varepsilon(1 - \delta_{0,m}) \frac{H_{|n-m|}^{(1)}(2\varkappa h/a)}{H_n^{(1)}(\varkappa)} \right. \\ &\quad \left. + \frac{H_{n+m}^{(1)}(2\varkappa h/a)}{H_n^{(1)}(\varkappa)} \right\} \qquad \text{if } n+m \text{ is odd} \end{aligned}\right\} \tag{100}$$

where

$\varepsilon = +1$, if $n > m$

$\varepsilon = -1$, if $n < m$

$$c_{mn} = \begin{cases} 0 \quad , & \text{if } n+m \text{ is even} \\ (i/2)\,(2 - \delta_{0,n})\, J_m(\varkappa) \left\{ - \epsilon \dfrac{H^{(1)}_{|n-m|}(2\varkappa h/a)}{H^{(1)}_n(\varkappa)} + \dfrac{H^{(1)}_{n+m}(2\varkappa h/a)}{H^{(1)}_n(\varkappa)} \right\} & \text{if } n+m \text{ is odd} \end{cases} \tag{101}$$

$$d_{mn} = \begin{cases} 0 \quad , & \text{if } n+m \text{ is odd} \\ - J_m(\varkappa) \left\{ \dfrac{H^{(1)}_{|n-m|}(2\varkappa h/a)}{H^{(1)}_n(\varkappa)} - \dfrac{H^{(1)}_{n+m}(2\varkappa h/a)}{H^{(1)}_n(\varkappa)} \right\} & \text{if } n+m \text{ is even} \end{cases} \quad . \tag{102}$$

We note that the upper limit N of the summations in Eqs. (92) and (93) is determined by the geometry of the problem and the degree of accuracy required. For a first-order approximation we retain only the x_0 term. In this limit the iteration procedure reduces to

$$\begin{aligned} x_0^{(\infty)} &= f_o[1 + a_{00} + a_{00}^2 + \ldots] \\ &= f_o[1 - a_{00}]^{-1} \quad , \end{aligned}$$

or

$$x_0^{(\infty)} = \frac{-[1 - \exp(i\varkappa v)]\, J_0(\varkappa)\, H_0^{(1)}(\varkappa)}{[H_0^{(1)}(\varkappa) - J_0(\varkappa)\, H_0^{(1)}(2\varkappa h/a)]} \quad . \tag{103}$$

For smaller values of the ratio of wire height to wire radius, h/a, more terms must be retained in the iterative equations for the scattering coefficients. To this end the numerical program incorporates a subroutine for accurately determining the Bessel and Hankel functions of any order or argument.

In terms of the scattering coefficients x_m and y_m, the general expression of Eq. (47) for the induced current in dimensionless form is

$$\bar{I}(\omega) = \bar{I}(\varkappa) e^{ikz \cos \gamma}$$

with

$$\bar{I}(\varkappa) = -i \Bigg\{ [1 - \exp(i\varkappa v)] J_1(\varkappa) + \frac{H_1^{(1)}(\varkappa)}{H_0^{(1)}(\varkappa)} x_0 - J_1(\varkappa) \sum_{m=0}^{\frac{1}{2}(N-1)} (2 - \delta_{0,m}) \frac{H_{2m}^{(1)}(2\varkappa h/a)}{H_{2m}^{(1)}(\varkappa)} x_{2m} + 2i J_1(\varkappa) \sum_{m=1}^{\frac{1}{2}(N+1)} \frac{H_{2m-1}^{(1)}(2\varkappa h/a)}{H_{2m-1}^{(1)}(\varkappa)} y_{2m-1} \Bigg\} \tag{104}$$

where $\bar{I}$ denotes a dimensionless current defined by

$$\bar{I}(\varkappa) = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{I(\varkappa)}{2\pi a E_0} \tag{105}$$

2. Transverse Electric Waves

For this case we use variants of the scattering coefficients defined by

$$\bar{x}_m = b_{em} H_m^{(1)\prime}(\varkappa) \cos m\alpha \tag{106}$$

$$\bar{y}_m = b_{om} H_m^{(1)\prime}(\varkappa) \ \mathrm{sim} \ m\alpha \tag{107}$$

where the prime denotes differentiation with respect to the argument of the Hankel functions. The iterative equations to be solved in this case are

$$\bar{x}_m^{(p+1)} = \sum_{n=0}^{N-1} \bar{a}_{mn} \bar{x}_n^{(p)} + \sum_{n=1}^{N} \bar{b}_{mn} \bar{y}_n + \bar{f}_m \tag{108}$$

$$\bar{y}_m^{(p+1)} = \sum_{n=0}^{N-1} \bar{c}_{mn} \bar{x}_n^{(p)} + \sum_{n=1}^{N} \bar{d}_{mn} \bar{y}_n + \bar{g}_m \tag{109}$$

with initial guesses

$$\bar{x}_m(0) = \bar{f}_m$$

$$\bar{y}_m(0) = \bar{g}_m$$

and where

$$\bar{f}_m = - \left[1 + \exp(i\varkappa v)\right] J_m'(\varkappa) \cos m\alpha \tag{110}$$

$$\bar{g}_m = - \left[1 - \exp(i\varkappa v)\right] J_m'(\varkappa) \sin m\alpha \quad . \tag{111}$$

We will avoid writing out in detail the coefficients $\bar{a}_{mn}$, $\bar{b}_{mn}$, $\bar{c}_{mn}$, and $\bar{d}_{mn}$ for this case by noting that they can be obtained from the corresponding coefficients a_{mn}, b_{mn}, c_{mn}, and d_{mn} of Case I by a simple replacement of terms. For example, to obtain $\bar{a}_{mn}$ from a_{mn}, replace the term

$$\frac{J_m(\varkappa)}{H_n^{(1)}(\varkappa)} \qquad \text{by} \qquad -\frac{J_m'(\varkappa)}{H_n^{(1)\prime}(\varkappa)}$$

In like manner, we obtain $\bar{b}_{mn}$ from b_{mn}, etc.

Step Function Pulse

The current time response to the step-function pulse is obtained from the integral

$$\bar{I}(t) = \int_{-\infty}^{+\infty} \bar{I}(\omega)\, f(\omega)\, e^{-i\omega t}\, d\omega \tag{112}$$

where the bar over the current terms indicate a normalized current as defined by Eq. (105). The spectral density is (See Appendix A)

$$f(\omega) = \frac{1}{2\pi}\left(\frac{P}{-i\omega} + \pi\delta(\omega)\right)$$

where P indicates the principal value. In the vicinity of zero-frequency we will make use of the closed form expression for the current of Eq. (74):

$$\bar{I}(\omega) = \left\{\frac{[1 - \exp(-i2kh \sin\gamma \sin\alpha)]\; 2\, e^{ikz\cos\gamma}}{\pi ka \sin\gamma\, [H_0^{(1)}(ka \sin\gamma) - H_0^{(1)}(2kh \sin\gamma)\, J_0(ka \sin\gamma)]}\right\}. \tag{113}$$

In terms of variables, introduced in Eqs. (64) to (66),

$$u = \frac{ct - z\cos\gamma}{a \sin\gamma},$$

$$v = -2\,\frac{h}{a} \sin\alpha = 2\,\frac{h}{a}\,|\sin\alpha|,$$

$$\varkappa = ka \sin \gamma,$$

we can write Eq. (112) in the form

$$\bar{I}(u) = \frac{1}{2}\int_{-\infty}^{\infty} \delta\left(\frac{c\varkappa}{a \sin \gamma} \right) \left\{ \frac{\frac{2}{\pi\varkappa}(1 - e^{i\varkappa v})\, e^{-i\varkappa u}}{H_0^{(1)}(\varkappa) - H_0^{(1)}(2\varkappa h/a)\, J_0(\varkappa)} \right\} \frac{d\varkappa}{(a \sin \gamma/c)}$$

$$+ \frac{i}{\pi^2}\, P \int_{-\Delta\varkappa}^{\Delta\varkappa} \frac{(1 - e^{i\varkappa v})\, e^{-i\varkappa u}\, d\varkappa}{\varkappa^2 [H_0^{(1)}(\varkappa) - H_0^{(1)}(2\varkappa h/a)\, J_0(\varkappa)]}$$

$$+ \int_{-\infty}^{-\Delta\varkappa} + \int_{\Delta\varkappa}^{\infty} \left(\frac{i}{2\pi}\right) \bar{I}(\varkappa)\, e^{-i\varkappa u} \frac{d\varkappa}{\varkappa} \quad . \qquad (114)$$

For the principal value integral we will assume $\Delta\varkappa$ is sufficiently small so that

$$(\Delta\varkappa)\ 2h/a \ll 1 \quad .$$

Taking small value arguments for the Bessel and Hankel functions, we obtain

$$H_0^{(1)}(\varkappa) - H_0^{(1)}(2\varkappa h/a)\, J_0(\varkappa) \longrightarrow -\frac{2i}{\pi} \log\left(\frac{2h}{a}\right) . \qquad (115)$$

The delta function integral gives

$$\bar{I}(0) = \frac{v/2}{\log(2h/a)} \qquad (116)$$

For sufficiently small values of $\Delta\varkappa$, the principal value integral gives

$$I_{P.V.}(u) = \frac{v/\pi}{\log(\frac{2h}{a})} \left\{ \int_0^{\Delta\varkappa} \frac{\sin \varkappa u}{\varkappa} d\varkappa - \frac{v^2}{2} \frac{\sin \Delta\varkappa u}{u} \right\} . \qquad (117)$$

Combining Eqs. (114), (116), and (117), we get

$$\bar{I}(u) = \frac{v/2}{\log(\frac{2h}{a})} \left\{ 1 + \frac{2}{\pi} \left[\int_0^{\Delta\varkappa} \frac{\sin \varkappa u}{\varkappa} d\varkappa - \frac{v}{2} \frac{\sin \Delta\varkappa u}{u} \right] \right\}$$

$$+ \int_{\Delta\varkappa}^{\infty} + \int_{-\infty}^{-\Delta\varkappa} (\frac{i}{2\pi}) \bar{I}(\varkappa) e^{-i\varkappa u} \frac{d\varkappa}{\varkappa} \quad . \qquad (118)$$

The last integral in Eq. (118) was evaluated numerically by means of the FFT algorithm to obtain the results presented here.

The previously obtained asymptotic value of the current when u → ∞ can also be obtained from Eq. (118) as follows:

$$\lim \bar{I}_{P.V.}(u) = \frac{v/\pi}{\log(\frac{2h}{a})} \lim_{u \to \infty} \int_0^{\Delta\varkappa} \frac{\sin \varkappa u}{\varkappa} d\varkappa$$

$$= \frac{v/2}{\log(\frac{2h}{a})} \quad . \qquad (119)$$

The rapid oscillations in the last integral of Eq. (118) will wipe out any contribution from that term in the limit as u → ∞. Hence, we obtain from the first terms of Eq. (118)

$$\lim \bar{I}(u) = \frac{v/2}{\log(\frac{2h}{a})} \{ 1 + 1 \}$$

$$= \frac{v}{\log(\frac{2h}{a})} = \frac{2\frac{h}{a}|\sin \alpha|}{\log(\frac{2h}{a})} \quad , \qquad (120)$$

in agreement with Eq. (89).

Results

In Figure 3 we show the time history of the induced current as obtained for an initial test case with the following parameters:

Incoming pulse : Step-function in time

Polarization : $\vec{H}_{\perp}$ to wire axis
Height-to-radius ratio: $h/a = 100$
Equatorial angle : $\alpha = -45°$

We recall that the spherical coordinate angles α, γ define the direction of the incoming pulse. A negative value for α indicates a wave coming from above. The azimuthal angle γ is eliminated from the parameter study by incorporating it into the normalized time variable $u = (ct - z\cos\gamma)/a\sin\gamma$. The reduced time $u = 0$ indicates the time of arrival of the pulse at the position of the center of the wire for any value of z along the direction of the wire, in the absence of the wire.

In Figure 3 we have demarcated certain events in the time history of the current that can readily be identified by geometrical considerations. These events are identified as follows for any point z along the wire.

(i) $\underline{-1 \leq u < u_1}$ Initial current rise due to the direct incident wave and its associated scattered wave.

(ii) $\underline{u = u_1 = 2(h/a)|\sin\alpha| - 1}$ Arrival of the ground reflected incident wave at the wire surface.

(iii) $\underline{u = u_2 = 2(h/a) - 3}$ The scattered wave associated with the direct incident wave returns to the wire surface after a ground reflection.

(iv) $\underline{u = u_3 = (2h/a)[\,|\sin\alpha| + 1] - 3}$ The scattered wave associated with the ground reflected incident wave returns to the wire after a ground reflection.

In addition to the above events we can identify subsequent minor perturbations in the current response that can be identified with multiple scattering events.

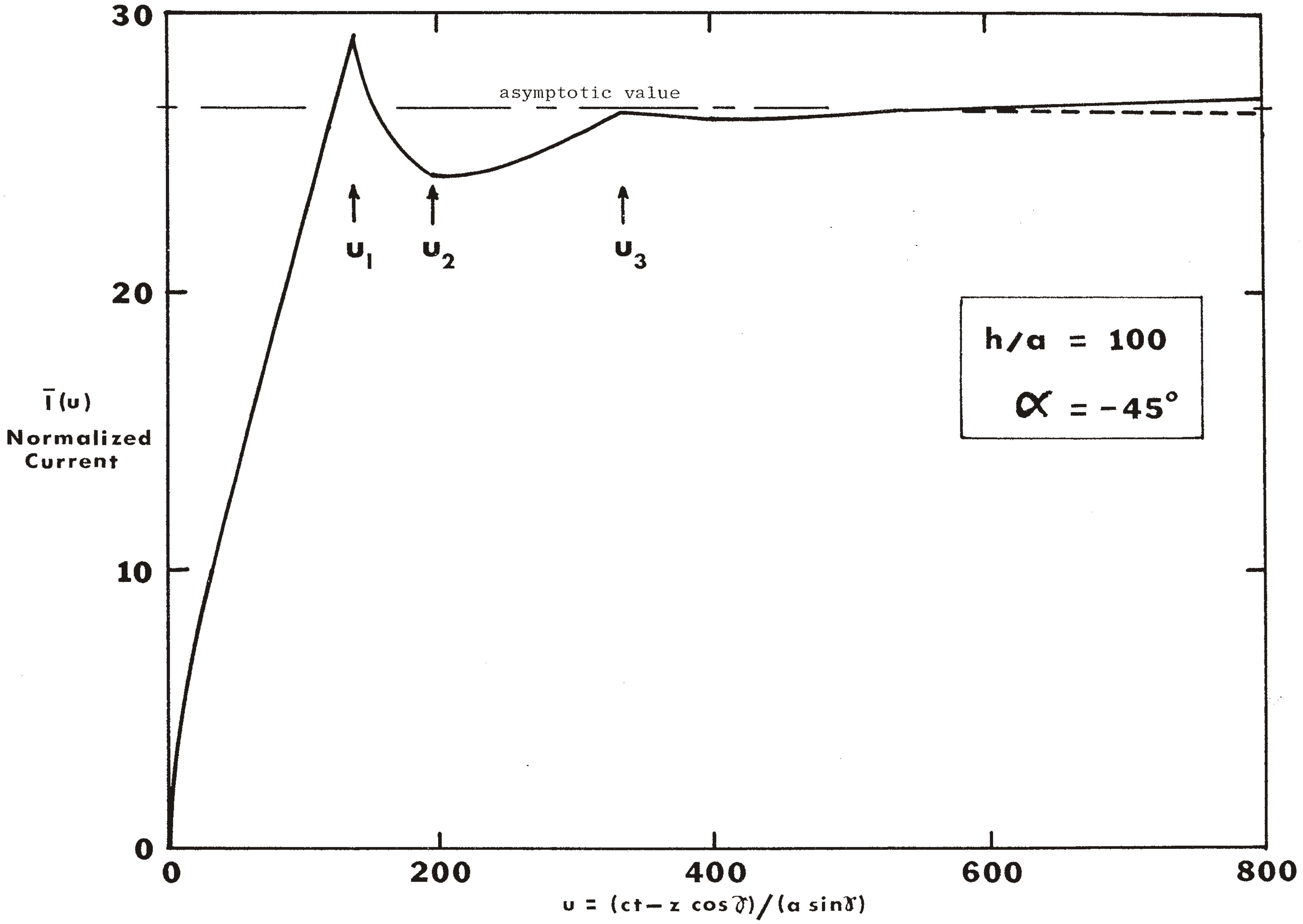

FIGURE 3 NORMALIZED AXIAL CURRENT RESPONSE TO AN INCIDENT STEP FUNCTION PULSE.

It must be noted, however, that the late time history as shown in the solid curve of Figure 3 has increasing error that can be identified with the aliasing phenomenon. The dashed curve shows improved late time results but with some loss of detail in the fine structure of the current response. This latter result was obtained be decreasing the frequency interval in the FFT integration without increasing the total number of points (2048) in the integration. The asymptotic value

$$I(\infty) \quad = \quad v[\log(2h/a)]^{-1}$$

is indicated at the right of Figure 3.

Additional calculations will be made of the current for various angles α and various height to radius ratios. These will be presented in a report supplemental to this one along with similar calculations of the potential between the wire and the ground. In calculating the potential, only the transverse electric case need be considered.

APPENDIX A

Fourier Transforms of Pulses

We let

$$F(t) = \int_{-\infty}^{\infty} \tilde{F}(\omega) e^{-i\omega t}\, d\omega \,.$$

Then

$$\tilde{F}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{i\omega t}\, dt \,.$$

For pulses that are functions of $t - \frac{\hat{n} \cdot \vec{r}}{c}$, we set

$$\tilde{F}(w) = f(\omega) e^{ik\hat{n} \cdot \vec{r}}$$

1. Delta Function Pulse

$$F(t) = \delta(t - \frac{\hat{n} \cdot \vec{r}}{c})$$

$$\tilde{F}(\omega) = \frac{1}{2\pi} e^{ik\hat{n} \cdot \vec{r}}$$

2. Exponential Pulse

$$F(t) = \exp[-\alpha(t - \frac{\hat{n} \cdot \vec{r}}{c})]\; H(t - \frac{\hat{n} \cdot \vec{r}}{c})$$

where H(t) is the Heaviside unit step function

$$H(t) = \begin{cases} 1, t > 0 \\ 0, t < 0 \end{cases}$$

$$\tilde{F}(\omega) = \frac{1}{2\pi(-i\omega + \alpha)} \; e^{ik\hat{n} \cdot \vec{r}}$$

3. Step Function Pulse

$$F(t) = H\left(t - \frac{\hat{n} \cdot \vec{r}}{c}\right)$$

$$\tilde{F}(\omega) = \lim_{\epsilon \to 0} \frac{1}{2\pi(-i\omega+\epsilon)}$$

Now[*]

$$\lim_{\epsilon \to 0} \frac{1}{\omega+i\epsilon} = P\frac{1}{\omega} - i\pi\delta(\omega)$$

where P is the principal value operator which indicates that a Cauchy principal value is to be taken in subsequent integrations. Thus,

$$\tilde{F}(\omega) = \frac{1}{2\pi}\left\{\frac{P}{-i\omega} + \pi\delta(\omega)\right\} e^{ik\hat{n}\cdot\vec{r}} .$$

---

[*] This well-known result in mathematical physics, sometimes called the Plemelj formula, is derived in many books on quantum mechanics (cf. ref. [4], p. 469, ref. [5], p. 718). It is also derived in some books on applied mathematics (cf. ref. [6], p. 21, ref. [7], p. 190).

APPENDIX B

The Addition Theorems for Cylindrical Waves

We shall derive expansions or addition theorems for the circularly cylindrical wave functions with reference to one origin in terms of those with reference to another origin that is translated, but not rotated, with respect to the first one in the x, y-plane. The z-coordinate is the same in both coordinate systems so that the scalar wave functions can be written as

$$\psi^{(i)}_{\substack{e\\o}m}(\vec{r}) = \psi^{(i)}_{\substack{e\\o}m}(\vec{\rho})e^{ikz\cos\gamma}$$

$$\psi^{(i)}_{\substack{e\\o}m}(\vec{r}\,') = \psi^{(i)}_{\substack{e\\o}m}(\vec{\rho}\,')e^{ikz\cos\gamma}$$

where $\vec{\rho}$ and $\vec{\rho}\,'$ are the position vectors in the x, y-planes, respectively, as shown in Figure B-1.

Addition theorems for scalar cylindrical waves have been given by Weyrich [1] and Stratton [2], but they are not in a form that is useful for the present purposes. Rather than rearrange their forms, we shall derive the required theorems from the beginning.

We take as our point of departure the Sommerfeld integral representation [8]

$$H^{(1)}_m(\lambda\rho) = \frac{1}{\pi}\int_{C_1} e^{i\lambda\rho\cos\alpha + im(\alpha-\frac{\pi}{2})}\,d\alpha$$

$$= \frac{1}{\pi i^m}\int_{C} e^{i\lambda\rho\cos(\alpha-\varphi) + im(\alpha-\varphi)}\,d\alpha \quad \text{(B1)}$$

where C is a path appropriately shifted from the usual Sommerfeld one in the complex α-plane. From Eq. (B1) we have

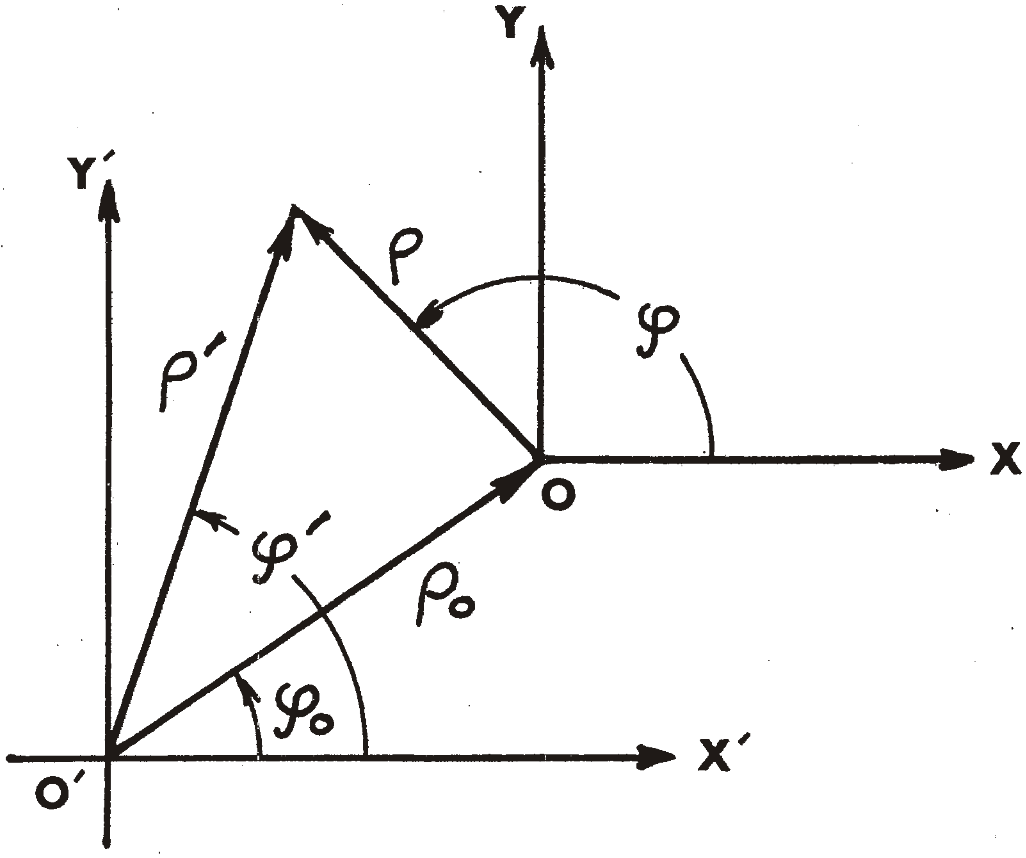

FIGURE B-1 TRANSLATED COORDINATE SYSTEMS USED IN THE DERIVATION OF ADDITION THEOREMS.

$$H_m^{(1)}(\lambda\rho)e^{im\varphi} = \frac{1}{\pi i^m} \int_C e^{i\lambda\rho\cos(\alpha-\varphi) + im\alpha}\, d\alpha \quad . \tag{B2}$$

Letting m → -m, and noting that

$$H_{-m}^{(1)}(Z) = (-1)^m H_m^{(1)}(Z), \tag{B3}$$

we get

$$H_m^{(1)}(\lambda\rho)\, e^{-im\varphi} = \frac{1}{\pi i^m} \int_C e^{i\lambda\rho\cos(\alpha-\varphi) - im\alpha}\, d\alpha \quad . \tag{B4}$$

Now, the plane wave expansion of Eq. (5) can be written in the following form

$$e^{i\vec{k}\cdot\vec{\rho}} = \sum_{n=-\infty}^{\infty} i^n J_n(k\rho\sin\gamma)e^{in(\alpha-\varphi)} \quad . \tag{B5}$$

Thus, for the coordinate systems of Fig. B1 we have from Eqs. (B4) and (B5)

$$\begin{aligned} H_m^{(1)}(k\rho'\sin\gamma)e^{-im\varphi'} &= \frac{1}{\pi i^m} \int_C e^{i\vec{k}\cdot\vec{\rho}' - im\alpha}\, d\alpha \\ &= \frac{1}{\pi i^m} \int_C e^{i\vec{k}\cdot(\vec{\rho}_o + \vec{\rho}) - im\alpha}\, d\alpha \\ &= \frac{1}{\pi i^m} \int_C e^{i\vec{k}\cdot\vec{\rho}_o} \sum_{n=-\infty}^{\infty} i^n J_n(k\rho\sin\gamma)e^{i(n-m)\alpha - in\varphi}\, d\alpha \quad . \end{aligned} \tag{B6}$$

Because of the infinite limits of the contour C, the interchange of the order of summation and integration in Eq. (B6) is not automatically permissible. When $\rho < \rho_o$, however, it can be shown that it is possible to exchange them. Then we get upon noting Eq. (B4)

$$H_m^{(1)}(k\rho'\sin\gamma)e^{-im\varphi'} =$$

$$= \sum_{n=-\infty}^{\infty} J_n(k\rho \sin\gamma)\, H^{(1)}_{m-n}(k\rho_o \sin\gamma)\, e^{i[(n-m)\varphi_o - n\varphi]}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n H^{(1)}_n(k\rho_o \sin\gamma)\, J_{n+m}(k\rho \sin\gamma)\, e^{-i[(n+m)\varphi - n\varphi_o]} \quad . \qquad \text{(B7)}$$

Let m → -m, n → -n in Eq. (B7). We obtain

$$H^{(1)}_m(k\rho' \sin\gamma)\, e^{im\varphi'}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n H^{(1)}_n(k\rho_o \sin\gamma)\, J_{n+m}(k\rho \sin\gamma)\, e^{i[(n+m)\varphi - n\varphi_o]} \quad . \qquad \text{(B8)}$$

Adding Eqs. (B7) and (B8) and then putting the result into summations over only positive values of n, we find that

$$H^{(1)}_m(k\rho' \sin\gamma) \genfrac{}{}{0pt}{}{\cos}{\sin} m\varphi'$$

$$= \tfrac{1}{2} \sum_{n=0}^{\infty} (2-\delta_{on})(-1)^n H^{(1)}_n(k\rho_o \sin\gamma) \; .$$

$$\cdot \left\{ J_{n+m}(k\rho \sin\gamma) \left[\cos n\,\varphi_o \genfrac{}{}{0pt}{}{\cos}{\sin}(n+m)\varphi \pm \sin n\,\varphi_o \genfrac{}{}{0pt}{}{\sin}{\cos}(n+m)\varphi \right] \right.$$

$$\left. +(-1)^m J_{n-m}(k\rho \sin\gamma) \left[\pm \cos n\,\varphi_o \genfrac{}{}{0pt}{}{\cos}{\sin}(n-m)\varphi + \sin n\,\varphi_o \genfrac{}{}{0pt}{}{\sin}{\cos}(n-m)\varphi \right] \right\} ,$$

$$(\rho<\rho_o) . \qquad \text{(B9)}$$

When $\rho'<\rho_o$, we find in the same way that

$$H^{(1)}_m(k\rho \sin\gamma) \genfrac{}{}{0pt}{}{\cos}{\sin} m\varphi$$

$$= \tfrac{1}{2} \sum_{n=0}^{\infty} (2-\delta_{on}) H^{(1)}_n(k\rho_o \sin\gamma) .$$

$$\cdot \left\{ J_{n+m}(k\rho' \sin\gamma) \left[\cos n\,\varphi_o \genfrac{}{}{0pt}{}{\cos}{\sin}(n+m)\varphi' \pm \sin n\varphi_o \genfrac{}{}{0pt}{}{\sin}{\cos}(n+m)\varphi' \right] + \right.$$

$$+ (-1)^m J_{n-m}(k\rho' \sin\gamma)\left[\pm \cos n\varphi_o \begin{matrix}\cos\\ \sin\end{matrix}(n-m)\varphi' + \sin n\varphi_o \begin{matrix}\sin\\ \cos\end{matrix}(n-m)\varphi'\right]\Bigg\}$$

$$(\rho^1<\rho_o). \qquad (B10)$$

The expansions for the cases when $\rho>\rho_o$ or $\rho'>\rho_o$ can be obtained from those given above by interchanging ρ and ρ_o or ρ' and ρ_o in the expressions on the right-hand side.

Let us multiply the expansions of Eqs. (B9) and (B10) by $\exp[ikz\cos\gamma]$ and define

$$A_n(k\rho_o \sin\gamma) = \tfrac{1}{2}(2-\delta_{on})\, H_n^{(1)}(k\rho_o \sin\gamma). \qquad (B11)$$

We thus get the addition theorems for the three-dimensional cylindrical scalar wave functions:

$$\psi^{(3)}_{\substack{e\\o}m}(\vec{r}', \gamma) = \sum_{n=0}^{\infty} (-1)^n A_n(k\rho_o \sin\gamma)\ \cdot$$

$$\cdot \Bigg\{\cos n\varphi_o\, \psi^{(1)}_{\substack{e\\o}n+m}(\vec{r}, \gamma) \pm \sin n\varphi_o\, \psi^{(1)}_{\substack{o\\e}n+m}(\vec{r}, \gamma)$$

$$\pm (-1)^m \cos n\varphi_o\, \psi^{(1)}_{\substack{e\\o}n-m}(\vec{r}, \gamma) + (-1)^m \sin n\varphi_o\, \psi^{(1)}_{\substack{o\\e}n-m}(\vec{r}, \gamma)\Bigg\},$$

$$(\rho<\rho_o), \qquad (B12)$$

$$\psi^{(3)}_{\substack{e\\o}m}(\vec{r}, \gamma) = \sum_{n=0}^{\infty} A_n(k\rho_o \sin\gamma)\ \cdot$$

$$\cdot \Bigg\{\cos n\varphi_o\, \psi^{(1)}_{\substack{e\\o}n+m}(\vec{r}', \gamma) \pm \sin n\varphi_o\, \psi^{(1)}_{\substack{o\\e}n+m}(\vec{r}', \gamma)$$

$$\pm (-1)^m \cos n\varphi_o\, \psi^{(1)}_{\substack{e\\o}n-m}(\vec{r}', \gamma) + (-1)^m \sin n\varphi_o\, \psi^{(1)}_{\substack{o\\e}n+m}(\vec{r}', \gamma)\Bigg\},$$

$$(\rho'<\rho_o). \qquad (B13)$$

Now the curl of a vector point function and the gradient of a scalar point function are invariants under transformations of the coordinate system. We can, therefore, obtain the addition theorems for the cylindrical vector wave functions by inserting the expansions of Eqs. (B12) and (B13) directly into the equations (6) and (7) that define the vector wave functions. In this way we obtain the vector expansions:

$$\vec{M}^{(3)}_{\substack{e\\o}m}(\vec{r}\,', \gamma) = \sum_{n=0}^{\infty} (-1)^n A_n(k\rho_o \sin \gamma) \cdot \left\{ \cos n\, \varphi_o \vec{M}^{(1)}_{\substack{e\\o}n+m}(\vec{r}, \gamma) \pm \sin n\, \varphi_o \vec{M}^{(1)}_{\substack{o\\e}n+m}(\vec{r}, \gamma) \pm (-1)^m \cos n\, \varphi_o \vec{M}^{(1)}_{\substack{e\\o}n-m}(\vec{r}, \gamma) + (-1)^m \sin n\, \varphi_o \vec{M}^{(1)}_{\substack{o\\e}n-m}(\vec{r}, \gamma) \right\}, \quad (\rho<\rho_o), \qquad \text{(B14)}$$

$$\vec{M}^{(3)}_{\substack{e\\o}m}(\vec{r}, \gamma) = \sum_{n=0}^{\infty} A_n(k\rho_o \sin \gamma) \cdot \left\{ \cos n\, \varphi_o \vec{M}^{(1)}_{\substack{e\\o}n+m}(\vec{r}\,', \gamma) \pm \sin n\, \varphi_o \vec{M}^{(1)}_{\substack{o\\e}n+m}(\vec{r}\,', \gamma) \pm (-1)^m \cos n\, \varphi_o \vec{M}^{(1)}_{\substack{e\\o}n-m}(\vec{r}\,', \gamma) + (-1)^m \sin n\, \varphi_o \vec{M}^{(1)}_{\substack{o\\e}n-m}(\vec{r}\,', \gamma) \right\}, \quad (\rho'<\rho_o) . \qquad \text{(B15)}$$

Taking the appropriate curl of Eqs. (B14) and (B15), we see that identical expansions hold for the vector wave functions $\vec{N}^{(3)}_{\substack{e\\o}m}$. We refrain from writing them out explicitly.

It will be useful to have the above vector addition theorems written in a more compact form, namely,

$$\vec{M}^{(3)}_{\substack{e\\o}m}(\vec{r}\,', \gamma) = \sum_{n,\,\mu,\,j} (-1)^n A_n\left(\substack{e\\o}m \,\middle|\, j,\, \mu\right) \vec{M}^{(1)}_{j,\, n+\mu}(\vec{r}, \gamma) , \quad (\rho<\rho_o), \qquad \text{(B16)}$$

$$\vec{M}^{(3)}_{\substack{e\\o}m}(\vec{r}\ ,\ \gamma) = \sum_{n,\ \mu,\ j} A_n\left(\substack{e\\o}m \,\middle|\, j\ ,\ \mu\right) \vec{M}^{(1)}_{j,\ n+\mu}(\vec{r}\,',\ \gamma) \qquad (\rho' < \rho_o) \tag{B17}$$

where

$$
\begin{aligned}
A_n(e,\ m|e,\ m) &= A_n(k\rho_o \sin\gamma) \cos n\,\varphi_o \\
A_n(e,\ m|e,-m) &= (-1)^m A_n(k\rho_o \sin\gamma) \cos n\,\varphi_o \\
A_n(e,\ m|o,\ m) &= A_n(k\rho_o \sin\gamma) \sin n\,\varphi_o \\
A_n(e,\ m|o,-m) &= (-1)^m A_n(k\rho_o \sin\gamma) \sin n\,\varphi_o \\
A_n(o,\ m|o,\ m) &= A_n(k\rho_o \sin\gamma) \cos n\,\varphi_o \\
A_n(o,\ m|o,-m) &= -(-1)^m A_n(k\rho_o \sin\gamma) \cos n\,\varphi_o \\
A_n(o,\ m|e,\ m) &= -A_n(k\rho_o \sin\gamma) \sin n\,\varphi_o \\
A_n(o,\ m|e\ -m) &= (-1)^m A_n(k\rho_o \sin\gamma) \sin n\,\varphi_o
\end{aligned}
\tag{B18}
$$

with $A_n(k\rho_o \sin\gamma)$ given by Eq. (B11). The range of n is all the integral numbers from and including zero to infinity, that of μ is just $-m$ and m, and that of j is e and o.

With the coordinate systems translated as in Fig. 1b

$$\varphi_o = \tfrac{1}{2}\pi, \qquad \rho_o = 2h.$$

Then

$$
\begin{aligned}
A_n(e,\ m|e,\ m) &= A_n(o,\ m|o,\ m) \\
&= \begin{cases} (-1)^{\frac{n}{2}} A_n(2\ k\ h \sin\gamma), & n \text{ even} \\ 0\ , & n \text{ odd} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
A_n(e,\ m|e,\ -m) &= -A_n(o,\ m|o,\ -m) \\
&= \begin{cases} (-1)^{\frac{n}{2}+m} A_n(2\ k\ h \sin\gamma), & n \text{ even} \\ 0\ , & n \text{ odd} \end{cases}
\end{aligned}
$$

$$A_n(e,\ m|o,\ m) = -A_n(o,\ m|e,\ m)$$

$$= \begin{cases} 0 , & n \text{ even} \\ (-1)^{\frac{n-1}{2}} A_n(2\ k\ h \sin\gamma), & n \text{ odd} \end{cases}$$

$$A_n(e,\ m|o,-m) = A_n(o,\ m|e,\ -m)$$

$$= \begin{cases} 0 , & n \text{ even} \\ (-1)^{\frac{n-1}{2}+m} A_n(2\ k\ h \sin\gamma), & n \text{ odd} \end{cases} \qquad \text{(B19)}$$

APPENDIX C

The Natural Resonances

We shall sketch here the derivation of some rough estimates of the roots of

$$H_o^{(1)}(\kappa) - H_o^{(1)}(\beta\kappa)J_o(\kappa) = 0, \qquad \beta \gg 1. \tag{C1}$$

If κ and $\beta\kappa$ are both assumed to be small, there is no solution to the above equation. Let us consider next the case where κ is small but $\beta\kappa$ is not small. We let

$$\kappa = \xi - i\eta, \qquad \xi,\ \eta > 0. \tag{C2}$$

We then have approximately from Eq. (C1)

$$\frac{2i}{\pi} \log \frac{\Gamma(\xi - i\eta)}{2i} = \sqrt{\frac{2}{\pi\beta}}\ \frac{e^{i[\beta(\xi-i\eta)-\frac{\pi}{4}]}}{\sqrt{\xi - i\eta}} \tag{C3}$$

where $\Gamma = 1.781...$. We are only interested in those modes for which the damping is small so we assume that

$$\frac{\eta}{\xi} \ll 1.$$

Keeping only the significant terms in Eq. (C3) we have

$$1 + \frac{2i}{\pi} \log \xi = \frac{(1-i)e^{\beta\eta}}{\sqrt{\beta\pi\xi}} (\cos \beta\xi + i \sin \beta\xi). \tag{C4}$$

This equation suggests that we set

$$\eta = \frac{\log\left[C \frac{\sqrt{2}}{\pi} \log \beta(\beta\pi\xi)^{\frac{1}{2}}\right]}{\beta}, \tag{C5}$$

where C is a constant of order unity. We also set

$$\xi = \frac{b}{\beta}, \qquad 1 < b \ll \beta. \tag{C6}$$

We then get from Eq. (C4)

$$C \cos \beta\xi = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{\log b}{\log \beta} + \frac{\pi}{\sqrt{2}} \frac{1}{\log \beta} \tag{C7}$$

$$C \sin \beta\xi = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{\log b}{\log \beta} + \frac{\pi}{\sqrt{2}} \frac{1}{\log \beta} \tag{C8}$$

We now choose

$$C = \frac{\log \beta - \log b}{\log \beta} \tag{C9}$$

Then

$$\cos \beta\xi = \frac{1}{\sqrt{2}} + \frac{\pi}{\sqrt{2}} \frac{1}{\log \beta - \log b}\ , \tag{C10}$$

$$\sin \beta\xi = -\frac{1}{\sqrt{2}} + \frac{\pi}{\sqrt{2}} \frac{1}{\log \beta - \log b} \tag{C11}$$

Now for small δ

$$\begin{matrix} \cos \\ \sin \end{matrix} \left(-\frac{\pi}{4} + \delta \right) = \pm\frac{1}{\sqrt{2}} + \frac{\delta}{\sqrt{2}} \tag{C12}$$

so we find that

$$\xi_n = \frac{2n\pi - \frac{\pi}{4} + \frac{\pi}{\log \beta - \log 2n\pi}}{\beta}$$

$$\simeq \frac{(2n-\frac{1}{4})\pi}{\beta}\ , \tag{C13}$$

and

$$\eta_n = \frac{\log[2\sqrt{n} \log \frac{\beta}{2n\pi}]}{\beta} \ . \tag{C14}$$

We next examine the case when both $\underline{\varkappa \text{ and } \beta\varkappa \text{ are large}}$. Then we have from Eq. (C1),

$$e^{i\beta\varkappa}\cos(\varkappa-\frac{\pi}{4}) = \sqrt{\frac{\pi}{2}\beta\varkappa} \quad . \tag{C15}$$

or

$$e^{i2\beta\varkappa}[1+\sin 2\varkappa] = \pi\beta\varkappa \quad . \tag{C16}$$

With $\varkappa$ as in Eq. (C2), we obtain from Eq. (C16)

$$e^{2\beta\eta}(1+\sin 2\xi\cosh\eta)$$

$$= \beta\pi(\xi\cos 2\beta\xi - \eta\sin 2\beta\xi), \tag{C17}$$

$$e^{2\beta\eta}\cos 2\xi\sinh\eta$$

$$= \beta\pi(\eta\cos 2\beta\xi + \xi\sin 2\beta\xi) \quad . \tag{C18}$$

These equations may be manipulated to yield

$$e^{4\beta\eta}\{(1+\sin 2\xi\cosh\eta)^2 + (\cos 2\xi\sinh\eta)^2\}$$

$$= (\beta\pi)^2(\xi^2+\eta^2) \quad . \tag{C19}$$

From this equation we find after further manipulation

$$\xi_n \simeq 2n\pi, \tag{C20}$$

$$\eta_n \simeq \frac{\log 2n\beta\pi^2}{2\beta} \quad . \tag{C21}$$

In Eq. (C2) we restricted the roots to the fourth quadrant. These are the ones required in Section VI. We may note that the roots of Eq. (C 1) must lie in the lower half plane. This follows from the physical fact that the waves of the natural modes must be damped because there is no external source of energy. The mathematical analysis in this Appendix is in complete agreement with this.

In a recent report on the transient electromagnetic properties of two parallel wires, Marin [12] has given some numerical results for the roots of the equations $K_o(\gamma a) \pm K_o(\gamma d) = 0$, where K_o is the modified Bessel function of the second kind, a is the radius of the wires, and d their separation. When $\varkappa$ is small and $\beta\varkappa$ is very large, with $\beta = d/a$, the roots of Eq. (C 1) when multiplied by -i should agree with the roots of $K_0(\gamma a) - K_o(\gamma d) = 0$. A comparison of values calculated from Eqs. (C-13) and (C-14) with those computed by Marin shows very good results for the smallest value of a/d, except for the sign of the imaginary part of $-i\varkappa = -\eta - i\xi$. This is quite alright, however, because $\pm\xi_n - i\eta_n$ are both roots of Eq. (C 1) . The choice of $-\xi$ corresponds to the assumption of the time dependence $e^{i\omega t}$ instead of $e^{-i\omega t}$ as used herein.

APPENDIX D

Finitely Conducting Cylindrical Wire

For low frequencies it is not correct to assume that the cylinder is perfectly conducting. It is clear that when the skin depth becomes an appreciable size with respect to the radius of the cylinder, the concept of a perfect conductor is not a physically sound idealization. We must take into account the field within the wire.

Let the conductivity of the wire be σ_c. The propagation constant within the conductor is

$$\begin{aligned} k_c^2 &= \omega^2 \mu_o \epsilon + i\omega\mu_o\sigma_c \\ &\simeq i\omega\mu_o\sigma_c \,. \end{aligned} \tag{D1}$$

We shall discard the negligible term ω^2/c^2 in the expression for k_c^2 throughout this Appendix.

We shall consider only the case of transverse magnetic waves. The magnetic field within the wire can be expanded in the form

$$\vec{H}_c(\vec{r}) = \frac{\sqrt{\epsilon_0/\mu_0}\; E_0}{i\; k \sin\gamma} \sum_{n=0}^{\infty} (2-\delta_{0m})\, i^m \cdot \left\{ a_{em} \cos m\alpha\, \vec{M}_{em}^{(1)}(\vec{r};\, \gamma;\, k_c) + a_{om} \sin m\alpha\, \vec{M}_{om}^{(1)}(\vec{r};\, \gamma;\, k_c) \right\} \tag{D2}$$

where

$$\vec{M}^{(1)}_{\substack{e\\o}m}(\vec{r};\, \gamma;\, k_c) = \nabla\times\hat{e}_z\, \psi^{(1)}_{\substack{e\\o}m}(\vec{r};\, \gamma;\, k_c)\,, \tag{D3}$$

with

$$\psi^{(1)}_{\substack{e\\o}m}(r;\ \gamma;\ k_c) = J_m(k_c\rho)\,{\cos \atop \sin}\, m\varphi\ e^{ikz\cos\gamma} \tag{D4}$$

Now

$$\vec{E}_c = \frac{1}{\sigma_c}\ \nabla\times\vec{H}_c$$

$$= \frac{i\ k\sqrt{\mu_0/\epsilon_0}}{k_c^2}\ \nabla\times\vec{H}_c \tag{D5}$$

and the expansion of the electric field within the conducting cylinder is

$$\vec{E}_c(\vec{r}) = \frac{kE_o}{k_c^2\sin\gamma}\ \sum_{m=0}^{\infty}(2-\delta_{0m})\,i^m\left\{a_{em}\cos m\,\alpha\ \vec{N}^{(1)}_{em}(\vec{r};\ \gamma;\ k_c)\right.$$

$$\left. +\ a_{om}\sin m\,\alpha\ \vec{N}^{(1)}_{om}(\vec{r};\ \gamma;\ k_c)\right\}. \tag{D6}$$

We are interested in generalizing the expression for the current in Eq. (74). We shall keep, therefore, only the terms for m=0. The explicit expressions for the components of the vector wave functions introduced in this Appendix for m=0 are

$$\vec{M}^{(1)}_{e0}(\vec{r};\ \gamma;\ k_c) = -\frac{d}{d\rho}\ J_o(k_c\rho)e^{ikz\cos\gamma}\ \hat{e}_\varphi\ , \tag{D7}$$

$$\vec{N}^{(1)}_{e0}(r;\ \gamma;\ k_c) = i\ \cos\gamma\ \frac{d}{d\rho}\ J_o(k_c\rho)e^{ikz\cos\gamma}\ \hat{e}_\rho$$

$$+\ \frac{k_c^2}{k}\ J_o(k_c\rho)e^{ikz\cos\gamma}\ \hat{e}_z\ . \tag{D8}$$

At the boundary r=a, the tangential components of the electric and magnetic fields must be continuous. Thus we have

$$E_z^{inc}(a, \varphi, z) + E_z^{ref}(a, \varphi, z) + E_z^{scat}(a, \varphi, z)$$

$$+ \left[E_z'^{\ scat}(\vec{r}') \right]_{\rho=a} = E_{cz}(a, \varphi, z) \tag{D9}$$

and

$$H_\varphi^{inc}(a, \varphi, z) + H_\varphi^{ref}(a, \varphi, z) + H_\varphi^{scat}(a, \varphi, z)$$

$$+ [\vec{H}'^{\ scat}(\vec{r}') \cdot \hat{e}_\varphi]_{\rho=a} = H_{c\varphi}(a, \varphi, z) . \tag{D10}$$

The expressions for the field components in Eqs. (D9) and (D10) are

$$E_z^{inc}(a) = E_0 \sin\gamma \, J_o(\lambda a) e^{ikz\cos\gamma}$$

$$E_z^{ref}(a) = -E_0 \sin\gamma \, e^{-i2\lambda h \sin\alpha} J_o(\lambda a) e^{ikz\cos\gamma}$$

$$E_z^{scat}(a) = E_0 \sin\gamma \, c_{e0} H_0^{(1)}(\lambda a) e^{ikz\cos\gamma}$$

$$E_z'^{\ scat}(a) = -E_0 \sin\gamma \, c_{e0} [H_o^{(1)}(\lambda\rho')]_{\rho=a} e^{ikz\cos\gamma}$$

$$E_{cz}(a) = E_0 \frac{1}{\sin\gamma} a_{e0} J_o(k_c a) e^{ikz\cos\gamma} \tag{D11}$$

and

$$H_\varphi^{inc}(a) = -i\sqrt{\frac{\epsilon_0}{\mu_0}} E_0 J_1(\lambda a) e^{ikz\cos\gamma}$$

$$H_\varphi^{ref}(a) = i\sqrt{\frac{\epsilon_0}{\mu_0}} E_0 e^{-i2\lambda h \sin\alpha} J_1(\lambda a) e^{ikz\cos\gamma}$$

$$H_\varphi^{scat}(a) = -i\sqrt{\frac{\epsilon_0}{\mu_0}} E_0 c_{e0} H_1^{(1)}(\lambda a) e^{ikz\cos\gamma}$$

$$H_\varphi'^{\ scat}(a) = i\sqrt{\frac{\epsilon_0}{\mu_0}} E_0 c_{e0} \left[H_1^{(1)}(\lambda\rho')\right]_{\rho=a} \left(\hat{e}_\varphi \cdot \hat{e}_{\varphi'}\right) e^{ikz\cos\gamma}$$

$$H_{c\varphi}(a) = -i\sqrt{\frac{\varepsilon_0}{\mu_0}}\; E_0\, a_{e0} \frac{k_c}{k \sin\gamma} J_1(k_c a) e^{ikz\cos\gamma} \tag{D12}$$

To express $E_z'^{\,scat}(\vec{r}\,')$ and $H_\varphi'^{\,scat}(\vec{r}\,')$ in terms of the unprimed coordinates we need the easily established geometric relation

$$\hat{e}_{\varphi'} = \hat{e}_{\rho'} \sin(\varphi-\varphi') + \hat{e}_\varphi \cos(\varphi-\varphi') \tag{D13}$$

and the following relations from the addition theorem for scalar cylindrical waves that is derived in Appendix B,

$$H_o^{(1)}(\lambda\rho') = H_o^{(1)}(2\lambda h)\, J_o(\lambda\rho) + \varphi\text{-dependent terms} \tag{D14}$$

and

$$\begin{aligned} H_1^{(1)}(\lambda\rho')\hat{e}_{\varphi'} \cdot \hat{e}_\varphi &= H_1^{(1)}(\lambda\rho') \cos(\varphi-\varphi') \\ &= H_o^{(1)}(2\lambda h)\, J_1(\lambda\rho) + \varphi\text{-dependent terms} \quad . \end{aligned} \tag{D15}$$

The boundary condition equations then become

$$\begin{aligned} &J_o(\lambda a) - e^{-i2\lambda h \sin\alpha} J_o(\lambda a) + c_{e0} H_o^{(1)}(\lambda a) \\ &\quad - c_{e0} H_o^{(1)}(2\lambda h)\, J_o(\lambda a) = \frac{a_{e0}}{\sin^2\gamma} J_o(k_c a) \end{aligned} \tag{D16}$$

and

$$\begin{aligned} &-J_1(\lambda a) + e^{-i2\lambda h \sin\alpha} J_1(\lambda a) - c_{e0} H_1^{(1)}(\lambda a) \\ &\quad + c_{e0} H_o^{(1)}(2\lambda h)\, J_1(\lambda a) = \frac{-k_c}{k \sin\gamma}\, a_{e0}\, J_1(k_c a) \;, \end{aligned} \tag{D17}$$

from which we obtain

$$c_{e0} = \frac{\left(1-e^{-i2\lambda h \sin\alpha}\right)\left\{J_o(\lambda a)\, J_1(k_c a)\, \frac{k_c \sin\gamma}{k} - J_1(\lambda a)\, J_o(k_c a)\right\}}{-\left[H_o^{(1)}(\lambda a) - H_o^{(1)}(2\lambda h)\, J_o(\lambda a)\right] J_1(k_c a)\, \frac{k_c \sin\gamma}{k} + \left[H_1^{(1)}(\lambda a) - H_o^{(1)}(2\lambda h)\, J_1(\lambda a)\right] J_o(k_c a)} \,, \tag{D18}$$

$$a_{e0} = \frac{2i \sin^2\gamma \left(1-e^{-i2\lambda h \sin\alpha}\right)}{\pi\left[H_o^{(1)}(\lambda a) - H_o^{(1)}(2\lambda h)\, J_o(\lambda a)\right] J_1(k_c a)\, k_c a \sin^2\gamma - \pi\lambda a \left[H_1^{(1)}(\lambda a) - H_o^{(1)}(2\lambda h)\, J_1(\lambda a)\right] J_o(k_c a)} \tag{D19}$$

The total axial current is

$$I = a \int_0^{2\pi} H_{c\varphi}(a)\, d\varphi \quad . \tag{D20}$$

From Eqs. (D12) and (D20) we get for the total axial current at frequency ω, the expression

$$\sqrt{\frac{\mu_0}{\epsilon_0}}\ \frac{I(\omega)}{2\pi a E_0} =$$

$$= \frac{2 \sin^{2}\gamma\,(1-e^{-i2\lambda h \sin\alpha})\; k_c a\; J_1(k_c a)\, e^{ikz\cos\gamma}}{\pi\lambda a \left\{ \left[H_o^{(1)}(\lambda a) - H_o^{(1)}(2\lambda h)\; J_o(\lambda a) \right] k_c a\; J_1(k_c a)\; \sin^{2}\gamma - \lambda a \left[H_1^{(1)}(\lambda a) - H_o^{(1)}(2\lambda h)\; J_1(\lambda a) \right] J_o(k_c a) \right\}} \qquad \text{(D21)}$$

from which we find

$$I(\omega) \xrightarrow[\omega \to 0]{} 2\, ikh\, \pi a^{2}\, \sigma_c\, E_o\, \sin^{2}\alpha\, \sin\alpha\, e^{ikz\cos\alpha}, \quad (\sigma_c \neq \infty),$$

and clearly then

$$\lim_{\omega \to 0} I(\omega) = 0 \; .$$

But if we let $\sigma_c \to \infty$, Eq. (D21) reduces to Eq. (74).

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