Interaction Notes

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Surface Currents Induced on Structures Attached to an Infinite Elliptic Cylinder
Part I
Detailed Magnetic Field Integral Equation for an Attached Structure Having an Arbitrary Shape

by

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Abstract

The integral equation considered in this note employs the Green's dyadic that causes the integral over the infinite elliptic cylinder to vanish. Some theoretical aspects of this equation are dealt with; however, the major effort is devoted to making this equation amenable to numerical solution. This consists of detailed vector bookkeeping, deciphering the relationship between various Mathieu function notations, and generating Mathieu function expansions to a higher order than could be found in the literature. These expansions are used to identify and remove singularities from the Green's dyadic that have not been previously considered.

cylinders, Green's function, calculations

I. Introduction

The major effort in predicting and understanding the currents induced on aircraft by an incident plane wave or EMP have used models of the aircraft that have not included much of the fine structure. The results of those studies are most meaningful for wavelengths that are long compared to the particular structure on the aircraft that is of interest. There is a class of structures on an aircraft that requires a more accurate knowledge of the current induced on them for wavelengths that are generally of the same size as the typical dimensions of the structure as well as for smaller wavelengths. Some structures of interest are attached to the wings of an aircraft such as gas tanks and missiles.

If one were to use the usual integral equation methods which employ only the free space Green's function, then for wavelengths comparable to the attached structure, the aircraft would have to be so densely zoned so as to lead to a matrix whose size prohibits the inversion by existing computers. This note is the first step in circumventing this problem. By considering the wing of the aircraft to be a perfectly conducting infinite elliptic cylinder we can employ the appropriate dyadic Green's function to derive an integral equation only over the attached structure. The size of the attached structure is such that its zoning leads to a matrix of manageable size. We can justify the infinite nature of the cylinder as well as the omission of the rest of the aircraft by noting that for the range of wavelengths of interest, the induced currents on the attached structure will depend primarily on the geometry of the aircraft near the attached structure.

Even though the Green's dyadic for the elliptic cylinder is known in theory, many points must be clarified before it can be used as part of a numerical procedure. Two different types of singularities of this Green's function are considered in detail. One is the usual singularity that must be considered in the derivation of the integral equation and it comes from letting the observation point off the surface approach the surface. The other type of singularity has not previously been identified and is present even if the observation point is off the surface. We treat the first type of singularity by splitting the Green's dyadic into a free space part and a scattered part.

This leads to a straightforward treatment of the singularity and circumvents the recent controversy concerning L functions in Green's dyadic representations. This splitting is also extremely useful for zoning considerations. The second type of singularity concerns the identification and removal of singularities that occur if we use a real integration path for the integral that occurs in the definition of the Green's dyadic. The identification and removal of these singularities required the generation of Mathieu function expansions to a higher order than could be found in the literature.

In order to generate these expansions we had to decipher and relate various Mathieu function notations. There was another very important reason for being able to relate different Mathieu function notations. The representations of the Green's dyadic that we found in the literature used a different Mathieu function notation than did the program for generating Mathieu functions that was made available to us. To take full advantage of the effort spent in relating Mathieu function notations, we present the definition of all quantities in our integral equation in three common but different Mathieu function notations.

Finally, we should note that the work presented here was found to be necessary in order to obtain numerical results. We have continued this work and have considered a specific class of structures attached to the elliptic cylinder. This problem is now in the stage where a computer program has been completed and is in the process of being debugged.

II. Derivation of the Integral Equation

In this section we derive the integral equation for the surface current density by employing the vector Green's theorem

$$\underline{H}(\underline{r})\cdot\left[\nabla\times\nabla\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}] - k_o^2[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}]\right] - [\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}]\cdot[\nabla\times\nabla\times\underline{H}(\underline{r}) - k_o^2\underline{H}(\underline{r})]$$

$$= -\nabla\cdot\left[(\nabla\times\underline{H}(\underline{r}))\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}] + \underline{H}\times\nabla\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}]\right] \quad (1)$$

as well as the following three equations

$$\nabla\times\nabla\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}] - k_o^2[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}] = \underline{a}\delta(\underline{r}-\underline{r}_o) \quad (2)$$

$$\nabla\times\nabla\times\underline{H}(\underline{r}) - k_o^2\underline{H}(\underline{r}) = 0 \quad (3)$$

$$\nabla\times\underline{H}(\underline{r}) = -i\omega\varepsilon_o\underline{E}(\underline{r}) \quad (4)$$

where $\underline{a}$ is an arbitrary constant vector. Substituting (2), (3), and (4) into (1), we obtain

$$\underline{H}(\underline{r})\cdot\underline{a}\delta(\underline{r}-\underline{r}_o) = -\nabla\cdot\left[-i\omega\varepsilon_o\underline{E}(\underline{r})\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}] - \underline{H}(\underline{r})\times\left(\nabla\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}]\right)\right] \quad (5)$$

We arrive at our first integral expression by integrating (5) over an appropriate volume and using the divergence theorem. This volume is all of space excluding the volume of the elliptic cylinder and the volume of the attached structure. The surfaces enclosing these volumes are S_E and S_A and they are depicted in figure 1. S_E is the surface of the elliptic cylinder with the portion that is covered by the attached structure removed from this infinite elliptic cylindrical surface. Integrating (5) and using the divergence theorem we obtain

$$\underline{H}(\underline{r}_o)\cdot\underline{a} = \int_{S_\infty} -\hat{n}_\infty\cdot\left[-i\omega\varepsilon_o\underline{E}(\underline{r})\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}] + \underline{H}(\underline{r})\times\left(\nabla\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}]\right)\right]dS$$

$$+ \int_{S_E} \hat{n}_E\cdot\left[-i\omega\varepsilon_o\underline{E}(\underline{r})\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}] + \underline{H}(\underline{r})\times\left(\nabla\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}]\right)\right]dS$$

$$+ \int_{S_A} \hat{n}_A\cdot\left[-i\omega\varepsilon_o\underline{E}(\underline{r})\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}] + \underline{H}(\underline{r})\times\left(\nabla\times[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}]\right)\right]dS \quad (6)$$

The signs in the S_E and S_A integrals are different from the S_∞ integral because $\hat{n}_E$ and $\hat{n}_A$ are the outward normals. For completeness, we note all $\hat{n}$'s are $\hat{n}(\underline{r})$'s.

So far we have not considered the metallic nature of S_E and S_A or required $\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}$ to satisfy any boundary conditions. Still not requiring $\underline{\underline{G}}(\underline{r},\underline{r}')\cdot\underline{a}$ to satisfy any boundary conditions but using the fact that $\hat{n}\times\underline{E}(\underline{r}) = 0$ on both S_E and S_A we rewrite (6) as

$$\underline{H}(\underline{r}_o)\cdot\underline{a} = I_\infty - \int_{S_E} \underline{H}(\underline{r})\cdot\left[\hat{n}_E\times\left(\nabla\times\left[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}\right]\right)\right]dS - \int_{S_A} \underline{H}(\underline{r})\cdot\left[\hat{n}_A\times\left(\nabla\times\left[\underline{\underline{G}}(\underline{r},\underline{r}_o)\cdot\underline{a}\right]\right)\right]dS \qquad (7)$$

where I_∞ is the integral over S_∞ and will be evaluated later. We now use the fact that $\underline{a}$ was an arbitrary constant vector and set it equal to the three Cartesian unit vectors (sequentially) to arrive at

$$\underline{H}(\underline{r}_o) = \underline{I}_\infty - \int_{S_E} \underline{H}(\underline{r})\cdot\left[\hat{n}_E\times\left(\nabla\times\underline{\underline{G}}(\underline{r},\underline{r}_o)\right)\right]dS - \int_{S_A} \underline{H}(\underline{r})\cdot\left[\hat{n}_A\times\left(\nabla\times\underline{\underline{G}}(\underline{r},\underline{r}_o)\right)\right]dS \qquad (8)$$

Again we haven't explicitly defined $\underline{I}_\infty$; however, its meaning will shortly be given using an argument that doesn't depend on its detailed structure. We now use the fact that

$$\hat{n}_E\times\left(\nabla\times\underline{\underline{G}}(\underline{r},\underline{r}_o)\right) = 0 \qquad \underline{r} \text{ on } S_E \qquad (9)$$

This is the requirement that makes $\underline{\underline{G}}(\underline{r},\underline{r}_o)$ complicated. Using Tai's notation [1], the $\underline{\underline{G}}(\underline{r},\underline{r}_o)$ that satisfies (9) is referred to using the subscript 2. Using (9) in (8) we obtain

$$\underline{H}(\underline{r}_o) = \underline{I}_\infty - \int_{S_A} \underline{H}(\underline{r})\cdot\left[\hat{n}_A\times\left(\nabla\times\underline{\underline{G}}_2(\underline{r},\underline{r}_o)\right)\right]dS \qquad (10)$$

and we have lost our integral over the infinite elliptic surface. Using (10) we can now give meaning to $\underline{I}_\infty$. First we note that we can prove that $\underline{I}_\infty$ is independent of the shape of the attached structure. Next we imagine S_A to flatten out and approach the surface of the elliptical cylinder, S_E, then the integral over S_A would vanish due to the definition of $\underline{\underline{G}}_2$. This in turn implies that $\underline{I}_\infty$ is the total field, incident plus scattered from the infinite elliptic cylinder, when S_A is not present. We denote this incident plus scattered field as $\underline{H}_T$ and write (10) as

$$\underline{H}(\underline{r}_o) = \underline{H}_T(\underline{r}_o) - \int_{S_A} \underline{H}(\underline{r})\cdot\left[\hat{n}_A\times\left(\nabla\times\underline{\underline{G}}_2(\underline{r},\underline{r}_o)\right)\right]dS \tag{11}$$

Normally (11) would be the appropriate place to stop and derive our surface integral equation; however, we have the problem that $\underline{\underline{G}}_2(\underline{r},\underline{r}_o)$ is not immediately available for an elliptic cylinder. What is available for an elliptic cylinder is $\underline{\underline{G}}_1(\underline{r},\underline{r}_o)$, the electric field dyadic Green's function in Tai's notation [1]. Fortunately there is the following relation given in [1]

$$\nabla\times\underline{\underline{G}}_2(\underline{r},\underline{r}_o) = [\nabla_o\times\underline{\underline{G}}_1(\underline{r}_o,\underline{r})]^* \tag{12}$$

as well as the general dyadic identity

$$\underline{A}\cdot\underline{\underline{B}} = \underline{\underline{B}}^*\cdot\underline{A} \tag{13}$$

where * indicates the transpose of the dyadic. Using these relations and interchanging the $\underline{r}$ and $\underline{r}_o$ notation, (11) becomes

$$\underline{H}(\underline{r}) = \underline{H}_T(\underline{r}) + \int_{S_H} \{[\nabla\times\underline{\underline{G}}_1(\underline{r},\underline{r}_o)]\cdot(\hat{n}_o\times\underline{H}(\underline{r}_o))\}dS_o \tag{14}$$

where we have adopted the notation $\hat{n}_A(\underline{r}_o) = \hat{n}_o$. We can now make direct use of the explicit form of $\underline{\underline{G}}_1(\underline{r},\underline{r}_o)$. Rather than using Tai's representation for $\underline{\underline{G}}_1(\underline{r},\underline{r}_o)$, we use the equivalent representation contained in the book written by his associates [2]. The reason for this is that the notation for the Mathieu functions contained in $\underline{\underline{G}}_1(\underline{r},\underline{r}_o)$ is explicitly related to one set of accepted notation [3]. During the course of this analysis we will have occasion to pay particular attention to Mathieu function notation in order to convert from our original notation to that of McLachlan [4] which proved more convenient. Independent of the Mathieu function notation, the dyadic can be represented as

$$\underline{\underline{G}}_1(\underline{r},\underline{r}_o) = \underline{\underline{G}}_o(\underline{r},\underline{r}_o) + \underline{\underline{G}}_{1S}(\underline{r},\underline{r}_o) \tag{15}$$

where both $\underline{\underline{G}}_o(\underline{r},\underline{r}_o)$, the free space Green's dyadic, and $\underline{\underline{G}}_{1S}(\underline{r},\underline{r}_o)$, the scattered part of the dyadic, are expressed in terms of vector $\underline{M}$ and $\underline{N}$ functions. Subsequent to the publication of [1], Tai discovered the omission of vector $\underline{L}$ functions in his representation of $\underline{\underline{G}}_1$ and corrected this omission in his later work [5]. It is possible to show that the omitted $\underline{L}$ functions are always part of the representation of $\underline{\underline{G}}_o$. We have found that there are a number of good reasons for splitting $\underline{\underline{G}}_1$ as in (15) and then using the ordinary representation of $\underline{\underline{G}}_o$. One reason is that this avoids any questions concerning the addition of $\underline{L}$ functions in the representation of $\underline{\underline{G}}_1$. The other benefits of using the simple representation of $\underline{\underline{G}}_o$ will be mentioned as they are utilized in the presentation of our analysis.

Substituting (15) into (14) and using the fact that

$$\nabla\times\underline{\underline{G}}_o(\underline{r},\underline{r}_o) = \nabla G_o\times\underline{\underline{I}} \tag{16}$$

where $\underline{\underline{I}}$ is the unit dyadic, we obtain

$$\begin{aligned}\underline{H}(\underline{r}) = \underline{H}_T(\underline{r}) &+ \int_{S_A}\left[\nabla G_o\times\left(\hat{n}_o\times\underline{H}(\underline{r}_o)\right)\right]dS_o \\ &+ \int_{S_A}\left[\left[\nabla\times\underline{\underline{G}}_{1S}(\underline{r},\underline{r}_o)\right]\cdot\left(\hat{n}_o\times\underline{H}(\underline{r}_o)\right)\right]dS_o\end{aligned} \tag{17}$$

where

$$G_o = (4\pi|\underline{r}-\underline{r}_o|)^{-1}\exp[ik|\underline{r}-\underline{r}_o|] \tag{18}$$

We now make use of the splitting given in (15) to derive the integral equation corresponding to $\underline{r}$ approaching and subsequently lying on the surface S_A. It is only the integral in (17) containing G_o which contributes to the factor that is removed as $\underline{r}$ approaches $\underline{r}_o$ from off of the surface. The removal of that factor is a well studied problem and can be found in [6]. Letting $\underline{r}$ approach S_A, (17) becomes

$$f(\Omega)\underline{H}(\underline{r}) = H_T(\underline{r}) + \int_{S_A} [\nabla G_o \times (\hat{n}_o \times \underline{H}(\underline{r}_o))] dS_o$$
$$+ \int_{S_A} [\nabla \times \underline{\underline{G}}_{1S}(\underline{r},\underline{r}_o)] \cdot (\hat{n}_o \times \underline{H}(\underline{r}_o)) dS_o \qquad (19)$$

where, from [6],

$$f(\Omega) = 1 - \Omega/4\pi \qquad (20)$$

and Ω is the solid angle subtended by the surface S_A at $\underline{r}$. If we don't choose $\underline{r}$ to approach at a discontinuity in curvature, then $\Omega = 2\pi$ and $f(\Omega)$ assumes the value of ½ which is usually seen in the magnetic field integral equation. We should note that (19) is not valid if we let $\underline{r}$ approach S_A at the junction where it is attached to S_E. An analysis valid for that case would be interesting; however, due to time limitations this question can be avoided by choosing a zoning procedure that does not allow $\underline{r}$ to lie on this junction. A second point concerning (19), but also applicable to the ordinary magnetic field integral equation which contains only the free space Green's function is the omission of a principal value indication on the integral containing G_o. The reason for this omission is subtle and was pointed out by R. Latham* several years ago when considering this integral in the ordinary magnetic field integral equation for points on the surface where $\Omega = 2\pi$. He showed that the integrand was no longer singular when $\underline{r}$ approaches $\underline{r}_o$ along the surface. In the application of the integral equation we are only concerned with this case, even though it was derived by considering $\underline{r}$ off of the surface.

Finally, we obtain the integral equation for the induced surface current density by taking $\hat{n}(\underline{r})\times$ both sides of (19). The resulting integral equation is

$$f(\Omega)\underline{J}(\underline{r}) = \underline{J}_T(\underline{r}) + \int_{S_A} \underline{\underline{K}}(\underline{r},\underline{r}_o) \cdot \underline{J}(\underline{r}_o) dS_o \qquad (21)$$

where we have employed the following definitions

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*personal communication.

$$\underline{J}(\underline{r}) = \hat{n}(\underline{r})\times\underline{H}(\underline{r}) \tag{22a}$$

$$\underline{J}_T(\underline{r}) = \hat{n}(\underline{r})\times\underline{H}_T(\underline{r}) \tag{22b}$$

$$\underline{\underline{K}}(\underline{r},\underline{r}_o) = \underline{\underline{K}}_o(\underline{r},\underline{r}_o) + \underline{\underline{K}}_1(\underline{r},\underline{r}_o) \tag{23a}$$

$$\underline{\underline{K}}_o(\underline{r},\underline{r}_o) = \hat{n}(\underline{r})\times[\nabla G_o\times\underline{\underline{I}}] \tag{23b}$$

$$\underline{\underline{K}}_1(\underline{r},\underline{r}_o) = \hat{n}(\underline{r})\times[\nabla\times\underline{\underline{G}}_{1S}(\underline{r},\underline{r}_o)] \tag{23c}$$

The only new quantity that is associated with our integral equation is $\underline{\underline{K}}_1$ and we shall be primarily concerned with problems related to this quantity in the remaining part of this note.

Before proceeding to the explicit representation of $\underline{\underline{K}}_1$ we introduce

$$\underline{\underline{D}}(\underline{r},\underline{r}_o) = \underline{\underline{D}}_o(\underline{r},\underline{r}_o) + \underline{\underline{D}}_1(\underline{r},\underline{r}_o) \tag{24}$$

where

$$\underline{\underline{K}}(\underline{r},\underline{r}_o) = \hat{n}(\underline{r})\times\underline{\underline{D}}(\underline{r},\underline{r}_o) \tag{25a}$$

$$\underline{\underline{K}}_o(\underline{r},\underline{r}_o) = \hat{n}(\underline{r})\times\underline{\underline{D}}_o(\underline{r},\underline{r}_o) \tag{25b}$$

$$\underline{\underline{K}}_1(\underline{r},\underline{r}_o) = \hat{n}(\underline{r})\times\underline{\underline{D}}_1(\underline{r},\underline{r}_o) \tag{25c}$$

and from (23) we see that

$$\underline{\underline{D}}_o(\underline{r},\underline{r}_o) = \nabla G_o\times\underline{\underline{I}} \tag{26a}$$

$$\underline{\underline{D}}_1(\underline{r},\underline{r}_o) = \nabla\times\underline{\underline{G}}_{1S}(\underline{r},\underline{r}_o) \tag{26b}$$

The reason for introducing $\underline{\underline{D}}$ is that in the application of (21), $\underline{J}(\underline{r})$ will be decomposed into its components along two orthogonal surface tangential directions having unit vectors $\hat{s}$ and $\hat{t}$. Adopting the convention

$$\hat{s}\times\hat{t} = \hat{n} \tag{27}$$

and using the described decomposition

$$\underline{J}(\underline{r}) = J_s(\underline{r})\hat{s} + J_t(\underline{r})\hat{t} \tag{28}$$

as well as (22b), we can write (21) as

$$\begin{aligned}
f(\Omega)J_s(\underline{r}) &= -\hat{t}\cdot\underline{H}_T(\underline{r}) - \int dS_o\{[\hat{t}\cdot\underline{\underline{D}}(\underline{r},\underline{r}_o)\cdot\hat{s}_o]J_s(\underline{r}_o) + [\hat{t}\cdot\underline{\underline{D}}(\underline{r},\underline{r}_o)\cdot\hat{t}_o]J_t(\underline{r}_o)\} \\
f(\Omega)J_t(\underline{r}) &= \hat{s}\cdot\underline{H}_T(\underline{r}) + \int dS_o\{[\hat{s}\cdot\underline{\underline{D}}(\underline{r},\underline{r}_o)\cdot\hat{s}_o]J_s(\underline{r}_o) + [\hat{s}\cdot\underline{\underline{D}}(\underline{r},\underline{r}_o)\cdot\hat{t}_o]J_t(\underline{r}_o)\}
\end{aligned} \tag{29}$$

where the explicit $\underline{r}$ or $\underline{r}_o$ dependence of the unit vectors has been suppressed. The equations given in (29) are a set of two coupled scalar integral equations and they are the ones used to form the eventual matrix equation that leads to the solutions for $J_s(\underline{r})$ and $J_t(\underline{r})$. From (24), (26) and (29) we can see that the only quantities that require further definition are $\underline{H}_T$ and $\underline{\underline{D}}_1$. The study and definition of these quantities will occupy the remaining portion of this note. Both of these quantities contain Mathieu functions and we shall devote the next section to the definition of these functions.

III. Definition of Mathieu Functions and Notation Changes

Our initial notation will be that of Stratton [3] because our initial representations of $\underline{H}_T$ and $\underline{\underline{D}}_1$ taken from [2] are given in that notation. Let us now consider an ellipse having an interfocal distance 2c and having the coordinate system depicted in figure 2 associated with it. In this coordinate system

$$x = c \cosh \xi \cos \eta \tag{30a}$$

$$y = c \sinh \xi \sin \eta \tag{30b}$$

and

$$\xi^{\dagger} = \cosh \xi \tag{31a}$$

$$\eta^{\dagger} = \cos \eta \tag{31b}$$

Equations (30) and (31) already represent a departure from the notation of [2] and [3] in that our $\xi^{\dagger}$ and $\eta^{\dagger}$ correspond to their ξ and η, while our ξ and η correspond to their u and v. Even though our choice of coordinate symbols is different, we will initially use the same symbols for the Mathieu functions. The geometric significance of these coordinates is that they represent the following curves. Constant values of ξ or $\xi^{\dagger}$ represent a family of ellipses having the same foci as depicted in figure 2, while constant values of η or $\eta^{\dagger}$ represent the corresponding set of hyperbolas orthogonal to the ellipses.

From [3], we can present the differential equations satisfied by the Mathieu functions of interest. They are

$$\frac{d^2R}{d\xi^2} + (\lambda^2 \cosh^2\xi - b)R = 0 \tag{32}$$

$$\frac{d^2S}{d\eta^2} + (b - \lambda^2 \cos^2\eta)S = 0 \tag{33}$$

The symbol R is used to denote radial Mathieu functions, while S is used to denote angular Mathieu functions. A reason for this terminology is that R depends on ξ, and increasing ξ corresponds to increasingly larger confocal ellipses, while S depends on η and as can be seen in (33), η corresponds to an angle. It is the fact that η corresponds to an angle that leads to those solutions of (32) and (33) that are of physical interest. These solutions are those having the same value at physical point in space whether that point is represented by the value η or $\eta + 2n\pi$ where n is an integer. This requirement leads to (33) having a non-trivial periodic solution only for b assuming the proper eigenvalue. The sets of eigenvalues assumed by b are denoted $b_m^{(e)}$ or $b_m^{(o)}$ where the superscripts are chosen in accordance to whether the corresponding eigenfunctions are even or odd functions of η. The quantity, λ^2, which is the only quantity left to be defined in (32) and (33) will just be considered to be a real valued parameter in this section. Depending on whether the Mathieu functions are related to $\underline{H}_T$ or $\underline{\underline{D}}_1$, the meaning of λ^2 will be somewhat different.

We are now in a position to present a more detailed representation of the solution to (33). Continuing basically in the notation of [3] we write

$$Se_m(\lambda,\eta) = \sum_n{}' D_n^m(\lambda) \cos n\eta \qquad (m = 0,1,\cdots) \tag{34}$$

and

$$So_m(\lambda,\eta) = \sum_n{}' F_n^m(\lambda) \sin n\eta \qquad (m = 1,2,\cdots) \tag{35}$$

where Se_m represents the eigenfunctions corresponding to $b_m^{(e)}$ and So_m represents the eigenfunctions corresponding to $b_m^{(o)}$. The prime in these summations is used to indicate that n assumes only positive even values if m is even and only positive odd values if m is odd. The normalization of these functions is such that

$$Se_m(\lambda,0) = 1 \tag{36}$$

and

$$\left[\frac{d}{d\eta} So_m(\lambda,\eta)\right]_{\eta=o} = 1 \tag{37}$$

Obtaining the coefficients D_n^m and F_n^m requires an elaborate procedure. We presented the explicit forms of the angular Mathieu functions given in terms of these coefficients in order to be able to show that it is these same coefficients that appear in the definition of the radial Mathieu functions. A method for obtaining these coefficients will be discussed elsewhere.

We now elaborate on the definition of the radial functions. Returning to (32) and (33), we see that the eigenvalues that are determined by the periodicity required for the solutions to (33) also appear in (32). We denote the radial solution associated with $b_m^{(e)}$ as $Re_m^{(i)}$ and the one associated with $b_m^{(o)}$ as $Ro_m^{(i)}$. The superscript i is introduced so that it can take on 2 values, each one corresponding to a linearly independent solution of the second order differential equation. To stay as close as possible to the notation of [2] and [3], the values of this subscript are 1 and 3. We have now presented sufficient background to present the definitions of the radial Mathieu functions. They are

$$Re_m^{(1)}(\lambda,\xi) = \sqrt{\frac{\pi}{2}} \sum_n{}' \, i^{m-n} D_n^m(\lambda) J_n(\lambda \cosh \xi) \tag{38}$$

$$Re_m^{(3)}(\lambda,\xi) = \sqrt{\frac{\pi}{2}} \sum_n{}' \, i^{m-n} D_n^m(\lambda) H_n^{(1)}(\lambda \cosh \xi) \tag{39}$$

$$Ro_m^{(1)}(\lambda,\xi) = \sqrt{\frac{\pi}{2}} \tanh \xi \sum_n{}' \, i^{n-m} n F_n^m(\lambda) J_n(\lambda \cosh \xi) \tag{40}$$

$$Ro_m^{(3)}(\lambda,\xi) = \sqrt{\frac{\pi}{2}} \tanh \xi \sum_n{}' \, i^{n-m} n F_n^m(\lambda) H_n^{(1)}(\lambda \cosh \xi) \tag{41}$$

The prime on the summation has the same meaning as mentioned with regard to the angular Mathieu functions. This completes the definition of the Mathieu functions that are used in [2] to represent $\underline{\underline{D}}_I$ and $\underline{\underline{H}}_T$.

Next we present the previously defined Mathieu functions in McLachlan's notation [4]. As mentioned earlier, we adopt his notation in order to facilitate the use of his book which is an excellent reference. First we must again present Mathieu's differential equations in a slightly different form from (32) and (33). They are

$$\frac{d^2R}{d\xi^2} - (a - 2q \cosh 2\xi)R = 0 \tag{42}$$

and

$$\frac{d^2S}{d\eta^2} + (a - 2q \cos 2\eta)S = 0 \tag{43}$$

Using the relations

$$\cos^2\eta = \frac{1}{2}(1 + \cos 2\eta) \tag{44a}$$

and

$$\cosh^2\xi = \frac{1}{2}(1 + \cosh 2\xi) \tag{44b}$$

we can write (32) and (33) as

$$\frac{d^2R}{d\xi^2} - (b - \lambda^2/2 - \lambda^2/2 \cosh 2\xi)R = 0 \tag{45}$$

and

$$\frac{d^2S}{d\eta^2} + (b - \lambda^2/2 - \lambda^2/2 \cos 2\eta)S = 0 \tag{46}$$

Comparing (42) and (43) to (45) and (46) we see that the eigenvalues, omitting subscripts and superscripts, are related by

$$a = b - \lambda^2/2 \tag{47}$$

while the parameter λ^2 is related to the parameter q by

$$q = \lambda^2/4 \tag{48}$$

The independent variables ξ and η have the same meaning for both sets of Mathieu equations. We now see that the solutions to each set of equations can be chosen so that they are proportional. In McLachlan's notation, the even S functions are denoted ce and the odd S functions are denoted se, while the R functions are denoted Ce, Se, $Me^{(1)}$, and $Ne^{(1)}$. In the remaining part of this section we will seek the proportionality factors corresponding to the following proportionalities where McLachlan's notation appears to the left and Stratton's to the right

$$ce_m(q,\eta) \propto Se_m(\lambda,\eta) \tag{49a}$$

$$se_m(q,\eta) \propto So_m(\lambda,\eta) \tag{49b}$$

$$Ce_m(q,\xi) \propto Re_m^{(1)}(\lambda,\xi) \tag{49c}$$

$$Se_m(q,\xi) \propto Ro_m^{(1)}(\lambda,\xi) \tag{49d}$$

$$Me_m^{(1)}(q,\xi) \propto Re_m^{(3)}(\lambda,\xi) \tag{49e}$$

$$Ne_m^{(1)}(q,\xi) \propto Ro_m^{(3)}(\lambda.\xi) \tag{49f}$$

First we will obtain the proportionality constants for the angular Mathieu functions. We now write (49a) as

$$ce_m(q,\eta) = K_m(q)Se_m(\lambda,\eta) \tag{50}$$

and using (36) we see that $K_m(q) = ce_m(q,o)$, so that

$$ce_m(q,\eta) = ce_m(q,o)Se_m(\lambda,\eta) \tag{51}$$

The odd angular Mathieu function is treated by writing (49b) as

$$se_m(q,\eta) = K_m(q)So_m(\lambda,\eta) \tag{52}$$

where we have used the same symbol $K_m(q)$ as in (50) although it will be shown to have a different meaning. We do this to avoid unnecessary notation problems since K_m will now be evaluated and not used again. Taking $d/d\eta$ of both sides of (52) and using (37) we find that $K_m = se'_m(q,o)$ where the prime indicates the η derivative. Using this result in (52) we have

$$se_m(q,\eta) = se'_m(q,o)So_m(\lambda,\eta) \tag{53}$$

We now devote our analysis toward obtaining the proportionality constants for the radial Mathieu functions. From [3] we have the following asymptotic formulas for

$$Re_n^{(3)} \sim Ro_n^{(3)} \sim v^{-\frac{1}{2}} e^{i(v-(2n+1)\pi/4)} = (-i)^n v^{-\frac{1}{2}} e^{i(v-\pi/4)} \tag{54}$$

while from [4] we have

$$Me_{2m}^{(1)} \sim p_{2m} \left(\frac{2}{\pi v}\right)^{\frac{1}{2}} e^{i(v-\pi/4)} \tag{55}$$

$$Me_{2m+1}^{(1)} \sim p_{2m+1} \left(\frac{2}{\pi v}\right)^{\frac{1}{2}} e^{i(v-3\pi/4)} = -ip_{2m+1} \left(\frac{2}{\pi v}\right)^{\frac{1}{2}} e^{i(v-\pi/4)} \tag{56}$$

$$Ne_{2m+1}^{(1)} \sim s_{2m+1} \left(\frac{2}{\pi v}\right)^{\frac{1}{2}} e^{i(v-3\pi/4)} = -is_{2m+1} \left(\frac{2}{\pi v}\right)^{\frac{1}{2}} e^{i(v-\pi/4)} \tag{57}$$

$$Ne_{2m+2}^{(1)} \sim s_{2m+2} \left(\frac{2}{\pi v}\right)^{\frac{1}{2}} e^{i(v-\pi/4)} \tag{58}$$

where

$$v = 2q^{\frac{1}{2}} \cosh \xi = \lambda \cosh \xi \tag{59}$$

and (54) through (58) are valid for $|v| >> n,m$. Letting the n that appears in (54) take on the values 2m, 2m+1, and 2m+2 and then equating asymptotic forms we obtain

$$Me_{2m}^{(1)} = \sqrt{\frac{2}{\pi}}\, p'_{2m} Re_{2m}^{(3)} \tag{60}$$

$$Me_{2m+1}^{(1)} = \sqrt{\frac{2}{\pi}}\, p'_{2m+1} Re_{2m+1}^{(3)} \tag{61}$$

$$Ne_{2m+1}^{(1)} = \sqrt{\frac{2}{\pi}}\, s'_{2m+1} Ro_{2m+1}^{(3)} \tag{62}$$

$$Ne_{2m+2}^{(1)} = \sqrt{\frac{2}{\pi}}\, s'_{2m+2} Ro_{2m+2}^{(3)} \tag{63}$$

where

$$p'_{2m} = (-1)^m p_{2m} = (-1)^m ce_{2m}(o,q) ce_{2m}(\pi/2,q)/A_o^{(2m)} \tag{64}$$

$$p'_{2m+1} = (-1)^m p_{2m+1} = (-1)^{m+1} ce_{2m+1}(o,q) ce'_{2m+1}(\pi/2,q)/q^{\frac{1}{2}} A_1^{(2m+1)} \tag{65}$$

$$s'_{2m+1} = (-1)^m s_{2m+1} = (-1)^m se'_{2m+1}(o,q) se_{2m+1}(\pi/2,q)/q^{\frac{1}{2}} B_1^{(2m+1)} \tag{66}$$

$$s'_{2m+2} = (-1)^{m+1} s_{2m+2} = (-1)^{m+1} se'_{2m+2}(o,q) se'_{2m+2}(\pi/2,q)/q B_2^{(2m+2)} \tag{67}$$

and the primes on the angular Mathieu functions indicate $d/d\eta$. The A's that appear in (64) and (65) are analogous to the D's that are used in the definition (34) and the B's that appear in (66) and (67) are analogous to the F's that appear in the definition (35). The exact meaning of the p_n's and s_n's is not of immediate concern at this time, what is important is that all of the conversion factors are in McLachlan's notation.

To obtain the remaining conversion factors we note, for real q, that

$$Re_n^1 = Re\{Re_n^{(3)}\} \tag{68}$$

$$Ro_n^1 = Re\{Ro_n^{(3)}\} \tag{69}$$

$$Ce_n = \mathrm{Re}\{Me_n^{(1)}\} \tag{70}$$

$$Se_n(\xi) = \mathrm{Re}\{Ne_n^{(1)}\} \tag{71}$$

where the notation Re{ } notes the real part of the quantity in brackets. We include the argument ξ in (71) to emphasize that the quantity $Se_n(\xi)$ is a radial Mathieu function in McLachlan's notation rather than the even angular function $Se_n(\eta)$ in Stratton's notation. The proportionality factors between the bracketed quantities are necessarily the same as for the real parts. For completeness we list the relationships

$$Ce_{2m} = \sqrt{\frac{2}{\pi}}\, p'_{2m} Re^{(1)}_{2m} \tag{72}$$

$$Ce_{2m+1} = \sqrt{\frac{2}{\pi}}\, p'_{2m+1} Re^{(1)}_{2m+1} \tag{73}$$

$$Se_{2m+1}(\xi) = \sqrt{\frac{2}{\pi}}\, s'_{2m+1} Ro^{(1)}_{2m+1} \tag{74}$$

$$Se_{2m+2}(\xi) = \sqrt{\frac{2}{\pi}}\, s'_{2m+2} Ro_{2m+2} \tag{75}$$

Finally we present the relationship between McLachlan's notation and the notation used by Blanch in the often used handbook published by the National Bureau of Standards [7]. We have two reasons for refering to Blanch's notation. One is that the NBS handbook is a good and a readily available source and the second is that both $\underline{H}_T$ and $\underline{\underline{D}}_1$ assume their simplest form in Blanch's notation. The notation for angular Mathieu functions of McLachlan and Blanch is identical, while Blanch presents the following relations for radial Mathieu functions

$$Ce_n = p'_n Mc_n^{(1)} \tag{76}$$

$$Se_n(\xi) = s'_n Ms_n^{(1)} \tag{77}$$

where the M's represent Blanch's radial Mathieu functions and the p_n' and s_n' are those given in (64) through (67). The following relations are also given by Blanch for real q

$$Mc_n^{(1)} = \mathrm{Re}\{Mc_n^{(3)}\} . \tag{78}$$

and

$$Ms_n^{(1)} = \mathrm{Re}\{Ms_n^{(3)}\} \tag{79}$$

From (70), (71) and (76) through (79) we obtain the following relations

$$Me_n^{(1)} = p_n' Mc_n^{(3)} \tag{80}$$

$$Ne_n^{(1)} = s_n' Ms_n^{(3)} \tag{81}$$

This completes all necessary notation relationships needed for the presentation of $\underline{H}_T$ and $\underline{\underline{D}}_1$.

IV. Representation of $\underline{H}_T$

A general linearly polarized plane wave, depicted in figure 3, has the form

$$\underline{E}_i = E_o[\cos \phi_p \hat{x}' + \sin \phi_p \hat{y}']e^{ik_o \hat{k}\cdot\underline{r}} \tag{82}$$

where we are free to choose the direction of the wave normal $\hat{k}$ in any convenient manner. Once the direction $\hat{k}$ is chosen we can impose the requirement that $\underline{E}_i$ lies in a plane perpendicular to $\hat{k}$ as the following

$$\hat{k}\cdot\hat{x}' = 0 \tag{83a}$$

and

$$\hat{k}\cdot\hat{y}' = 0 \tag{83b}$$

We can satisfy (83) and set up a convenient right handed coordinate system $(\hat{x}',\hat{y}',\hat{k})$ by choosing

$$\hat{x}' = -\frac{\hat{k}\times(\hat{k}\times\hat{z})}{|\hat{k}\times(\hat{k}\times\hat{z})|} \tag{84}$$

and

$$\hat{y}' = \frac{\hat{k}\times\hat{z}}{|\hat{k}\times\hat{z}|} \tag{85}$$

Noting (82) we see that ϕ_p is the angle that $\underline{E}_i$ makes with the x' axis and it is referred to as the polarization angle.

From Maxwell's equation

$$\nabla\times\underline{E}_i = i\omega\mu_o\underline{H}_i \tag{86}$$

and the right hand nature of $(\hat{x}',\hat{y}',\hat{k})$, it follows that the incident magnetic field is given by

$$\underline{H}_i = YE_o[-\sin \phi_p \hat{x}' + \cos \phi_p \hat{y}']e^{ik_o \hat{k}\cdot\underline{r}} \tag{87}$$

where Y is the intrinsic admittance of free space

$$Y = \sqrt{\varepsilon_o/\mu_o} \tag{88}$$

In order to be able to use the results in [2], we choose $\hat{k}$ as

$$\hat{k} = -\cos\phi_o \sin\theta_o \hat{x} - \sin\phi_o \sin\theta_o \hat{y} - \cos\theta_o \hat{z} \tag{89}$$

and the (x,y,z) coordinate system is the one depicted in figure 2. The results of reference [2] that we wish to use are now described. For an E-polarized incident plane wave, $H_z = 0$, described by

$$\underline{E}^{(1)} = -\sin\theta_o(\cos\phi_o \cos\theta_o\hat{x} + \sin\phi_o \cos\theta_o\hat{y} - \sin\theta_o\hat{z})e^{ik_o\hat{k}\cdot\underline{r}} \tag{90}$$

and

$$\underline{H}^{(1)} = -Y\sin\theta_o(\sin\phi_o\hat{x} - \cos\phi_o\hat{y})e^{ik_o\hat{k}\cdot\underline{r}} \tag{91}$$

the scattered fields from an infinite perfectly conducting cylinder of arbitrary cross-section are given by

$$\underline{E}_{sc}^{(1)} = -\frac{i}{k_o}\cos\theta_o\left(\frac{\partial V_1}{\partial x}\hat{x} + \frac{\partial V_1}{\partial y}\hat{y}\right) + \sin^2\theta_o V_1\hat{z} \tag{92}$$

and

$$\underline{H}_{sc}^{(1)} = -\frac{iY}{k_o}\left(\frac{\partial V_1}{\partial y}\hat{x} - \frac{\partial V_1}{\partial x}\hat{y}\right) \tag{93}$$

where

$$V_1 = e^{-ik_o z\cos\theta_o} E_{z_{sc}}(\theta_o=\pi/2,\ k_o \to k_o\sin\theta_o) \tag{94}$$

Similarly for an H-polarized incident plane wave, $E_z = 0$, described by

$$\underline{E}^{(2)} = -\frac{1}{Y}\sin\theta_o(\sin\phi_o\hat{x} - \cos\phi_o\hat{y})e^{ik_o\hat{k}\cdot\underline{r}} \tag{95}$$

and

$$\underline{H}^{(2)} = \sin\theta_o(\cos\phi_o\cos\theta_o\hat{x} + \sin\phi_o\cos\theta_o\hat{y} - \sin\theta_o\hat{z})e^{ik_o\hat{k}\cdot\underline{r}} \tag{96}$$

the scattered fields from an infinite perfectly conducting cylinder of arbitrary cross-section are given by

$$\underline{E}_{sc}^{(2)} = -\frac{i}{k_o Y}\left(\frac{\partial V_2}{\partial y}\hat{x} - \frac{\partial V_2}{\partial x}\hat{y}\right) \tag{97}$$

and

$$\underline{H}_{sc}^{(2)} = \frac{i}{k_o}\cos\theta_o\left(\frac{\partial V_2}{\partial x}\hat{x} + \frac{\partial V_2}{\partial y}\hat{y}\right) - \sin^2\theta_o V_2\hat{z} \tag{98}$$

where

$$V_2 = e^{-ik_o z\cos\theta_o} H_{z_{sc}}\cdot(\theta_o = \pi/2,\ k_o \rightarrow k_o\sin\theta_o) \tag{99}$$

We now write our incident field given by (82) and (87) as the linear combination

$$\underline{E}_i = \alpha\underline{E}^{(1)} + \beta\underline{E}^{(2)} \tag{100}$$

$$\underline{H}_i = \alpha\underline{H}^{(1)} + \beta\underline{H}^{(2)} \tag{101}$$

where α and β will now be determined. The determination of these quantities is facilitated by noting that $\underline{E}^{(1)}$ has a z component while $\underline{E}^{(2)}$ does not and $\underline{H}^{(2)}$ has a z component while $\underline{H}^{(1)}$ does not. Using this, we take the dot product of both sides of (100) and (101) with $\hat{z}$ and cancelling the exponential, we obtain

$$E_o \cos \phi_p \hat{z}\cdot\hat{x}' = \alpha\hat{z}\cdot\underline{E}^{(1)} = \alpha \sin^2\theta_o \tag{102}$$

and

$$-E_o Y \sin \phi_p \hat{z}\cdot\hat{x}' = \beta\hat{z}\cdot\underline{H}^{(2)} = -\beta \sin^2\theta_o \tag{103}$$

Substituting (89) into (84) and performing the algebra we can show

$$\hat{z}\cdot\hat{x}' = \sin \theta_o \tag{104}$$

and using this result in the previous two equations we find that

$$\alpha = \frac{E_o \cos \phi_p}{\sin \theta_o} \tag{105}$$

and

$$\beta = \frac{E_o Y \sin \phi_p}{\sin \theta_o} \tag{106}$$

Recalling that $\underline{H}_T$ is given by

$$\underline{H}_T = \underline{H}_i + \underline{H}_{sc} = \underline{H}_i + \alpha\underline{H}_{sc}^{(1)} + \beta\underline{H}_{sc}^{(2)} \tag{107}$$

we combine the results of this section to write

$$\underline{H}_T = (-\sin \phi_p\hat{x}' + \cos \phi_p\hat{y}')e^{ik_o\underline{k}\cdot\underline{r}} - \frac{i \cos \phi_p}{k_o \sin \theta_o}\left(\frac{\partial V_1}{\partial x}\hat{x} - \frac{\partial V_1}{\partial y}\hat{y}\right)$$

$$+ \frac{i \sin \phi_p \cos \theta_o}{k_o \sin \theta_o}\left(\frac{\partial V_2}{\partial x}\hat{x} + \frac{\partial V_2}{\partial y}\hat{y}\right) - \sin \theta_o \sin \phi_p V_2\hat{z} \tag{108}$$

where the incident field has been normalized with respect to E_oY. For completeness we note that the substitution of (89) into (84) and (85) leads to

$$x' = -\cos \phi_o \cos \theta_o\hat{x} - \sin \phi_o \cos \theta_o\hat{y} + \sin \theta_o\hat{z} \tag{109}$$

and

$$\hat{y}' = -\sin\phi_o \hat{x} + \cos\phi_o \hat{y} \tag{110}$$

We now see that all that remains for the complete definition of $\underline{H}_T$ are the explicit representations of $\partial V_1/\partial x$, $\partial V_1/\partial y$, V_2, $\partial V_2/\partial x$, and $\partial V_2/\partial y$. Using (94) and (99) together with the explicit representations in [2], we can write

$$V_1 = -\left\{\sum_{m=o}^{\infty} (-i)^m \left[(N_m^{(e)}(\lambda'))^{-1} \frac{Re_m^{(1)}(\lambda',\xi_1^*)}{Re_m^{(3)}(\lambda',\xi_1^*)} Re_m^{(3)}(\lambda',\xi) Se_m(\lambda',\phi_o) Se_m(\lambda',\eta) \right.\right.$$
$$\left.\left. + (N_m^{(o)}(\lambda'))^{-1} \frac{Ro_m^{(1)}(\lambda',\xi_1^*)}{Ro_m^{(3)}(\lambda',\xi_1^*)} Ro_m^{(3)}(\lambda',\xi) So_m(\lambda',\phi_o) So_m(\lambda',\eta) \right]\right\} (8\pi)^{\frac{1}{2}} e^{-ik_o z\cos\theta_o} \tag{111}$$

and

$$V_2 = -\left\{\sum_{m=o}^{\infty} (-i)^m \left[(N_m^{(e)}(\lambda'))^{-1} \frac{Re_m^{(1)'}(\lambda',\xi_1^*)}{Re_m^{(3)'}(\lambda',\xi_1^*)} Re_m^{(3)}(\lambda',\xi) Se_m(\lambda',\phi_o) Se_m(\lambda',\eta) \right.\right.$$
$$\left.\left. + (N_m^{(o)}(\lambda'))^{-1} \frac{Ro_m^{(1)'}(\lambda',\xi_1^*)}{Ro_m^{(3)'}(\lambda',\xi_1^*)} Ro_m^{(3)}(\lambda',\xi) So_m(\lambda',\phi_o) So_m(\lambda',\eta) \right]\right\} (8\pi)^{\frac{1}{2}} e^{-ik_o z\cos\theta_o} \tag{112}$$

where

$$\lambda' = k_o c \sin\theta_o \tag{113}$$

$$N_m^{(e)}(\lambda') = \int_o^{2\pi} d\eta [Se_m(\lambda',\eta)]^2 \tag{114}$$

$$N_m^{(o)}(\lambda') = \int_o^{2\pi} d\eta [So_m(\lambda',\eta)]^2 \tag{115}$$

The c that appears in (113) is the semi-focal distance of the system of ellipses depicted in figure 2 while ξ_1^* is the value of ξ corresponding to the surface of the perfectly conducting elliptical cylinder. The R's and S's are Mathieu functions given in Stratton's notation as discussed in the previous section and the prime associated with the R' appearing in (112) represents $d/d\xi$.

Before proceeding further we will convert the representations of V_1 and V_2 to McLachlan's notation [4] and to Blanch's notation [7] by using the results presented in the previous section. Multiplying the even terms in (111) and (112) by 1 expressed as

$$1 = \frac{[(2/\pi)^{\frac{1}{2}} p_m']^2}{[(2/\pi)^{\frac{1}{2}} p_m']^2} \frac{[(ce_m(q,o)]^2}{[(ce_m(q,o)]^2} \tag{116}$$

and the odd terms by 1 expressed as

$$1 = \frac{[(2/\pi)^{\frac{1}{2}} s_m']^2}{[(2/\pi)^{\frac{1}{2}} s_m']^2} \frac{[se_m'(q,o)]^2}{[se_m'(q,o)]^2} \tag{117}$$

and using (51), (53), (60) through (63), and (72) through (75) we can write

$$V_1 = -\left\{ \sum_{m=o}^{\infty} (-i)^m \left[(N_m^{(e)*})^{-1} \frac{Ce_m(q',\xi_1^*)}{Me_m^{(1)}(q',\xi_1^*)} \frac{Me_m^{(1)}(q',\xi)}{(2/\pi)^{\frac{1}{2}} p_m'} ce_m(q',\phi_o) ce_m(q',\eta) \right.\right.$$
$$\left.\left. + (N_m^{(o)*})^{-1} \frac{Se_m(q',\xi_1^*)}{Ne_m^{(1)}(q',\xi_1^*)} \frac{Ne_m^{(1)}(q',\xi)}{(2/\pi)^{\frac{1}{2}} s_m'} se_m(q',\phi_o) se_m(q',\eta) \right] \right\} (8\pi)^{\frac{1}{2}} e^{-ik_o \cos \theta_o} \tag{118}$$

$$V_2 = -\left\{ \sum_{m=o}^{\infty} (-i)^m \left[(N_m^{(e)*})^{-1} \frac{Ce_m'(q',\xi_1^*)}{Me_m^{(1)'}(q',\xi_1^*)} \frac{Me_m^{(1)}(q',\xi)}{(2/\pi)^{\frac{1}{2}} p_m'} ce_m(q',\phi_o) ce_m(q',\eta) \right.\right.$$
$$\left.\left. + (N_m^{(o)*})^{-1} \frac{Se_m'(q',\xi_1^*)}{Ne_m^{(1)'}(q',\xi_1^*)} \frac{Ne_m^{(1)}(q',\xi)}{(2/\pi)^{\frac{1}{2}} s_m'} se_m(q',\phi_o) se_m(q',\eta) \right] \right\} (8\pi)^{\frac{1}{2}} e^{-ik_o \cos \theta_o} \tag{119}$$

where

$$q' = \frac{(\lambda')^2}{4} = \left(\frac{k_o c \sin \theta_o}{2}\right)^2 \tag{120}$$

$$N_m^{(e)*} = \int_0^{2\pi} d\eta[ce_m(q',\eta)]^2 = \pi \tag{121}$$

$$N_m^{(o)*} = \int_0^{2\pi} d\eta[se_m(q',\eta)]^2 = \pi \tag{122}$$

and the Mathieu functions are given in McLachlan's notation. The second equality in (121) and (122) expresses results from the normalization employed by McLachlan for his angular Mathieu functions. Rewriting $(-i)^m$ and using (64) through (67) as well as (121) and (122) we can write

$$V_2 = -2e^{-ik_o z \cos\theta_o} S_1 \tag{123}$$

and

$$V_1 = -2e^{-ik_o z \cos\theta_o} S_1^* \tag{124}$$

where

$$S_1 = \sum_{m=o}^{\infty} \Bigg\{ \Big[D_{2m}^{(e)}(q',\xi_1^*) p_{2m}^{-1} Me_{2m}^{(1)}(q',\xi) ce_{2m}(q',\phi_o) ce_{2m}(q',\eta)$$

$$+ D_{2m+2}^{(o)}(q',\xi_1^*) s_{2m+2}^{-1} Ne_{2m+2}^{(1)}(q',\xi) se_{2m+2}(q',\phi_o) se_{2m+2}(q',\eta) \Big]$$

$$- i \Big[D_{2m+1}^{(e)} p_{2m+1}^{-1} Me_{2m+1}^{(1)}(q',\xi) ce_{2m+1}(q',\phi_o) ce_{2m+1}(q',\eta)$$

$$+ D_{2m+1}^{(o)} s_{2m+1}^{-1} Ne_{2m+1}^{(1)}(q',\xi) se_{2m+1}(q',\phi_o) se_{2m+1}(q',\eta) \Big] \Bigg\} \tag{125}$$

and

$$S_1^* = \sum_{m=o}^{\infty} \left\{ \left[C_{2m}^{(e)}(q',\xi_1^*) p_{2m}^{-1} Me_{2m}^{(1)}(q',\xi) ce_{2m}(q',\phi_o) ce_{2m}(q',\eta) \right. \right.$$

$$\left. + C_{2m+2}^{(o)}(q',\xi_1^*) s_{2m+2}^{-1} Ne_{2m+2}^{(1)}(q',\xi) se_{2m+2}(q',\phi_o) se_{2m+2}(q',\eta) \right]$$

$$- i \left[C_{2m+1}^{(e)}(q',\xi_1^*) p_{2m+1}^{-1} Me_{2m+1}^{(1)}(q',\xi) ce_{2m+1}(q',\phi_o) ce_{2m+1}(q',\eta) \right.$$

$$\left. \left. + C_{2m+1}^{(o)}(q',\xi_1^*) s_{2m+1}^{-1} Ne_{2m+1}^{(1)}(q',\xi) se_{2m+1}(q',\phi_o) se_{2m+1}(q',\eta) \right] \right\} \qquad (126)$$

In (125) and (126) we have used the fact that

$$s_o(q',\eta) \equiv 0 \qquad (127)$$

as well as the definitions

$$D_m^{(e)}(q',\xi_1^*) = \frac{Ce_m'(q',\xi_1^*)}{Me_m^{(1)'}(q',\xi_1^*)} \qquad (128)$$

$$D_m^{(o)}(q',\xi_1^*) = \frac{Se_m'(q',\xi_1^*)}{Ne_m^{(1)'}(q',\xi_1^*)} \qquad (129)$$

$$C_m^{(e)}(q',\xi_1^*) = \frac{Ce_m(q',\xi_1^*)}{Me_m^{(1)}(q',\xi_1^*)} \qquad (130)$$

$$C_m^{(o)}(q',\xi_1^*) = \frac{Se_m(q',\xi_1^*)}{Ne_m^{(1)}(q',\xi_1^*)} \qquad (131)$$

Returning to (108) and using (109), (110) and the relationships

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial V}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad (132)$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial V}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{133}$$

as well as (123) and (124) we can express $\underline{H}_T$ as

$$\begin{aligned}
\underline{H}_T = \hat{x}\Bigg[&(\sin\phi_p \cos\phi_o \cos\theta_o - \cos\phi_p \sin\phi_o)e^{ik_o\hat{k}\cdot\underline{r}} \\
&+ \frac{2ie^{-ik_o z\cos\theta_o}}{k_o \sin\theta_o}\left(\cos\phi_p\left(S_2\frac{\partial\xi}{\partial x} + S_3\frac{\partial\eta}{\partial x}\right)\right. \\
&\qquad\left. - \sin\phi_p\cos\theta_o\left(S_4\frac{\partial\xi}{\partial x} + S_5\frac{\partial\eta}{\partial x}\right)\right)\Bigg] \\
+ \hat{y}\Bigg[&(\sin\phi_p \sin\phi_o \cos\theta_o + \cos\phi_p \cos\phi_o)e^{ik_o\hat{k}\cdot\underline{r}} \\
&- \frac{2ie^{-ik_o z\cos\theta_o}}{k_o \sin\theta_o}\left(\cos\phi_p\left(S_2\frac{\partial\xi}{\partial y} + S_3\frac{\partial\eta}{\partial y}\right)\right. \\
&\qquad\left. + \sin\phi_p\cos\theta_o\left(S_4\frac{\partial\xi}{\partial y} + S_5\frac{\partial\eta}{\partial y}\right)\right)\Bigg] \\
+ \hat{z}\Big[&-\sin\phi_p \sin\theta_o e^{ik_o\hat{k}\cdot\underline{r}} + 2e^{-ik_o z\cos\theta_o}\sin\theta_o \sin\phi_p S_1\Big]
\end{aligned} \tag{134}$$

where

$$\begin{aligned}
S_2 = \frac{\partial S_1^*}{\partial\xi} = \sum_{m=o}^{\infty}\Bigg\{&\Big[C_{2m}^{(e)}(q',\xi_1^*)p_{2m}^{-1}Me_{2m}^{(1)'}(q',\xi)ce_{2m}(q',\phi_o)ce_{2m}(q',\eta) \\
&+ C_{2m+2}^{(o)}(q',\xi_1^*)s_{2m+2}^{-1}Ne_{2m+2}^{(1)'}(q',\xi)se_{2m+2}(q',\phi_o)se_{2m+2}(q',\eta)\Big] \\
&- i\Big[C_{2m+1}^{(e)}(q',\xi_1^*)p_{2m+1}^{-1}Me_{2m+1}^{(1)'}(q',\xi)ce_{2m+1}(q',\phi_o)ce_{2m+1}(q',\eta) \\
&+ C_{2m+1}^{(o)}(q',\xi_1^*)s_{2m+1}^{-1}Ne_{2m+1}^{(1)'}(q',\xi)se_{2m+1}(q',\phi_o)se_{2m+1}(q',\eta)\Big]\Bigg\}
\end{aligned} \tag{135}$$

$$S_3 = \frac{\partial S_1^*}{\partial \eta} = \sum_{m=o}^{\infty} \left\{ \left[C_{2m}^{(e)}(q',\xi_1^*) p_{2m}^{-1} Me_{2m}^{(1)}(q',\xi) ce_{2m}(q',\phi_o) ce'_{2m}(q',\eta) \right. \right.$$

$$\left. + C_{2m+2}^{(o)}(q',\xi_1^*) s_{2m+2}^{-1} Ne_{2m+2}^{(1)}(q',\xi) se_{2m+2}(q',\phi_o) se'_{2m+2}(q',\eta) \right]$$

$$- i \left[C_{2m+1}^{(e)}(q',\xi_1^*) p_{2m+1}^{-1} Me_{2m+1}^{(1)}(q',\xi) ce_{2m+1}(q',\phi_o) ce'_{2m+1}(q',\eta) \right.$$

$$\left. \left. + C_{2m+1}^{(o)}(q',\xi_1^*) s_{2m+1}^{-1} Ne_{2m+1}^{(1)}(q',\xi) se_{2m+1}(q',\phi_o) se'_{2m+1}(q',\eta) \right] \right\} \quad (136)$$

$$S_4 = \frac{\partial S_1}{\partial \xi} = \sum_{m=o}^{\infty} \left\{ \left[D_{2m}^{(e)}(q',\xi_1^*) p_{2m}^{-1} Me_{2m}^{(1)'}(q',\xi) ce_{2m}(q',\phi_o) ce_{2m}(q',\eta) \right. \right.$$

$$\left. + D_{2m+2}^{(o)}(q',\xi_1^*) s_{2m+2}^{-1} Ne_{2m+2}^{(1)'}(q',\xi) se_{2m+2}(q',\phi_o) se_{2m+2}(q',\eta) \right]$$

$$- i \left[D_{2m+1}^{(e)}(q',\xi_1^*) p_{2m+1}^{-1} Me_{2m+1}^{(1)'}(q',\xi) ce_{2m+1}(q',\phi_o) ce_{2m+1}(q',\eta) \right.$$

$$\left. \left. + D_{2m+1}^{(o)}(q',\xi_1^*) s_{2m+1}^{-1} Ne_{2m+1}^{(1)'}(q',\xi) se_{2m+1}(q',\phi_o) se_{2m+1}(q',\eta) \right] \right\} \quad (137)$$

$$S_5 = \frac{\partial S_1}{\partial \xi} = \sum_{m=o}^{\infty} \left\{ \left[D_{2m}^{(e)}(q',\xi_1^*) p_{2m}^{-1} Me_{2m}^{(1)}(q',\xi) ce_{2m}(q',\phi_o) ce'_{2m}(q',\eta) \right. \right.$$

$$\left. + D_{2m+2}^{(o)}(q',\xi_1^*) s_{2m+2}^{-1} Ne_{2m+2}^{(1)}(q',\xi) se_{2m+2}(q',\phi_o) se'_{2m+2}(q',\eta) \right]$$

$$- i \left[D_{2m+1}^{(e)}(q',\xi_1^*) p_{2m+1}^{-1} Me_{2m+1}^{(1)}(q',\xi) ce_{2m+1}(q',\phi_o) ce'_{2m+1}(q',\eta) \right.$$

$$\left. \left. + D_{2m+1}^{(o)}(q',\xi_1^*) s_{2m+1}^{-1} Ne_{2m+1}^{(1)}(q',\xi) se_{2m+1}(q',\phi_o) se'_{2m+1}(q',\eta) \right] \right\} \quad (138)$$

The primes appearing in the preceding equations indicate the appropriate $d/d\xi$ or $d/d\eta$. All that remains for the complete representation of $\underline{H}_T$ is the explicit representation of the derivatives $\partial\xi/\partial x$, $\partial\eta/\partial x$, $\partial\xi/\partial y$, and $\partial\eta/\partial y$. These derivatives are found from taking $\partial/\partial x$ and $\partial/\partial y$ of (30a) and (30b) and then solving the system of linear equations. The results are

$$\frac{\partial\xi}{\partial x} = -\frac{\partial\eta}{\partial y} = \frac{a}{c\beta} \tag{139}$$

$$\frac{\partial\eta}{\partial x} = -\frac{\partial\xi}{\partial y} = -\frac{b}{c\beta} \tag{140}$$

where

$$a = \frac{\sinh\xi\cos\eta}{\beta} \tag{141}$$

$$b = \frac{\cosh\xi\sin\eta}{\beta} \tag{142}$$

$$\beta = (\cosh^2\xi - \cos^2\eta)^{\frac{1}{2}} \tag{143}$$

and c is the previously mentioned semi-focal distance. This completes the explicit representation of $\underline{H}_T$ in McLachlan's Mathieu function notation.

The representation of $\underline{H}_T$ in Blanch's notation is now readily determined. Equation (134) and the auxiliary definitions are still applicable with the Mathieu functions in S_1 through S_5 appropriately converted. Using (64) through (67), (76), (77), (80), (81), and the fact that the remaining notation is the same as McLachlan's we can write these S's as

$$\begin{aligned} S_1 = \sum_{m=o}^{\infty} (-1)^m \Bigg\{ & \Big[D_{2m}^{(e)}(q',\xi_1^*) Mc_{2m}^{(3)}(q',\xi) ce_{2m}(q',\phi_o) ce_{2m}(q',\eta) \\ & - D_{2m+2}^{(o)}(q',\xi_1^*) Ms_{2m+2}^{(3)}(q',\xi) se_{2m+2}(q',\phi_o) se_{2m+2}(q',\eta) \Big] \\ & - i\Big[D_{2m+1}^{(e)}(q',\xi_1^*) Mc_{2m+1}^{(3)}(q',\xi) ce_{2m+1}(q',\phi_o) ce_{2m+1}(q',\eta) \\ & + D_{2m+1}^{(o)}(q',\xi_1^*) Ms_{2m+1}^{(3)}(q',\xi) se_{2m+1}(q',\phi_o) ce_{2m+1}(q',\eta) \Big] \Bigg\} \end{aligned} \tag{144}$$

$$S_2 = \sum_{m=o}^{\infty} (-1)^m \Bigg\{ \Big[C_{2m}^{(e)}(q',\xi_1^*) Mc_{2m}^{(3)'}(q',\xi) ce_{2m}(q',\phi_o) ce_{2m}(q',\eta)$$

$$- C_{2m+2}^{(o)}(q',\xi_1^*) Ms_{2m+2}^{(3)'}(q',\xi) se_{2m+2}(q',\phi_o) se_{2m+2}(q',\eta) \Big]$$

$$- i \Big[C_{2m+1}^{(e)}(q',\xi_1^*) Mc_{2m+1}^{(3)}(q',\xi) ce_{2m+1}(q',\phi_o) ce_{2m+1}(q',\eta)$$

$$+ C_{2m+1}^{(o)}(q',\xi_1^*) Ms_{2m+1}^{(3)'}(q',\xi) se_{2m+1}(q',\phi_o) se_{2m+1}(q',\eta) \Big] \Bigg\} \quad (145)$$

$$S_3 = \sum_{m=o}^{\infty} (-1)^m \Bigg\{ \Big[C_{2m}^{(e)}(q',\xi_1^*) Mc_{2m}^{(3)}(q',\xi) ce_{2m}(q',\phi_o) ce'_{2m}(q',\eta)$$

$$- C_{2m+2}^{(o)}(q',\xi_1^*) Ms_{2m+2}^{(3)}(q',\xi) se_{2m+2}(q',\phi_o) se'_{2m+2}(q',\eta) \Big]$$

$$- i \Big[C_{2m+1}^{(e)}(q',\xi_1^*) Mc_{2m+1}^{(3)}(q',\xi) ce_{2m+1}(q',\phi_o) ce'_{2m+1}(q',\eta)$$

$$+ C_{2m+1}^{(o)}(q',\xi_1^*) Ms_{2m+1}^{(3)}(q',\xi) se_{2m+1}(q',\phi_o) se'_{2m+1}(q',\eta) \Big] \Bigg\} \quad (146)$$

$$S_4 = \sum_{m=o}^{\infty} (-1)^m \Bigg\{ \Big[D_{2m}^{(e)}(q',\xi_1^*) Mc_{2m}^{(3)'}(q',\xi) ce_{2m}(q',\phi_o) ce_{2m}(q',\eta)$$

$$- D_{2m+2}^{(o)}(q',\xi_1^*) Ms_{2m+2}^{(3)'}(q',\xi) se_{2m+2}(q',\phi_o) se_{2m+2}(q',\eta) \Big]$$

$$- i \Big[D_{2m+1}^{(e)}(q',\xi_1^*) Mc_{2m+1}^{(3)'}(q',\xi) ce_{2m+1}(q',\phi_o) ce_{2m+1}(q',\eta)$$

$$+ D_{2m+1}^{(o)}(q',\xi_1^*) Ms_{2m+1}^{(3)'}(q',\xi) se_{2m+1}(q',\phi_o) se_{2m+1}(q',\eta) \Big] \Bigg\} \quad (147)$$

$$S_5 = \sum_{m=0}^{\infty} (-1)^m \Bigg\{ \Big[D_{2m}^{(e)}(q',\xi_1^*) Mc_{2m}^{(3)}(q',\xi) ce_{2m}(q',\phi_o) ce'_{2m}(q',\eta)$$

$$- D_{2m+2}^{(o)}(q',\xi_1^*) Ms_{2m+2}^{(3)}(q',\xi) se_{2m+2}(q',\phi_o) se'_{2m+2}(q',\eta) \Big]$$

$$- i \Big[D_{2m+1}^{(e)}(q',\xi_1^*) Mc_{2m+1}^{(3)}(q',\xi) ce_{2m+1}(q',\phi_o) ce'_{2m+1}(q',\eta)$$

$$+ D_{2m+1}^{(o)}(q',\xi_1^*) Ms_{2m+1}^{(3)}(q',\xi) se_{2m+1}(q',\phi_o) se'_{2m+1}(q',\eta) \Big] \Bigg\} \qquad (148)$$

where

$$C_m^{(e)}(q',\xi_1^*) = \frac{Mc_m^{(1)}(q',\xi_1^*)}{Mc_m^{(3)}(q',\xi_1^*)} \qquad (149)$$

$$C_m^{(o)}(q',\xi_1^*) = \frac{Ms_m^{(1)}(q',\xi_1^*)}{Ms_m^{(3)}(q',\xi_1^*)} \qquad (150)$$

$$D_m^{(e)}(q',\xi_1^*) = \frac{Mc_m^{(1)'}(q',\xi_1^*)}{Mc_m^{(3)'}(q',\xi_1^*)} \qquad (151)$$

$$D_m^{(o)}(q',\xi_1^*) = \frac{Ms_m^{(1)'}(q',\xi_1^*)}{Ms_m^{(3)'}(q',\xi_1^*)} \qquad (152)$$

This completes the specification of $\underline{H}_T$ in all notations of interest.

V. First Representation of $\underline{\underline{D}}_1$

Recalling the definition of $\underline{\underline{D}}_1$ given in (26b) as

$$\underline{\underline{D}}_1(\underline{r},\underline{r}_o) = \nabla\times\underline{\underline{G}}_{1S}(\underline{r},\underline{r}_o) \tag{153}$$

we see that we require a representation of G_{1S}. We obtain this representation from [2] where it is given as

$$\begin{aligned} G_{1S}(\underline{r},\underline{r}_o) = -\frac{i}{2\pi}\int_{C_h}\frac{dh}{k_o^2-h^2}\sum_{m=o}^{\infty}\Big\{ & (N_m^{(e)}(\lambda))^{-1}\Big[C_m^{(e)}(\lambda,\xi_1^*)\underline{Me}_m^{(3)}(h,\underline{r})\underline{M}^{(3)}(-h,\underline{r}_o) \\ & + D_m^{(e)}(\lambda,\xi_1^*)\underline{Ne}_m^{(3)}(h,\underline{r})\underline{N}(-h,\underline{r}_o)\Big] \\ & + (N_m^{(o)}(\lambda))^{-1}\Big[C_m^{(o)}(\lambda,\xi_1^*)\underline{Mo}_m^{(3)}(h,\underline{r})\underline{Mo}_m^{(3)}(-h,\underline{r}_o) \\ & + D_m^{(o)}(\lambda,\xi_1^*)\underline{No}_m^{(3)}(h,\underline{r})\underline{No}_m^{(3)}(-h,\underline{r}_o)\Big]\Big\} \end{aligned} \tag{154}$$

where $N_m^{(e)}$ and $N_m^{(o)}$ have been defined in (114) and (115) and the contour of integration is depicted in figure 4.

$$\lambda^2 = c^2(k_o^2 - h^2) \tag{155}$$

$$C_m^{(e)}(\lambda,\xi_1^*) = \frac{Re_m^{(1)}(\lambda,\xi_1^*)}{Re_m^{(3)}(\lambda,\xi_1^*)} \tag{156}$$

$$C_m^{(o)}(\lambda,\xi_1^*) = \frac{Ro_m^{(1)}(\lambda,\xi_1^*)}{Ro_m^{(3)}(\lambda,\xi_1^*)} \tag{157}$$

$$D_m^{(e)}(\lambda,\xi_1^*) = \frac{Re_m^{(1)'}(\lambda,\xi_1^*)}{Re_m^{(3)'}(\lambda,\xi_1^*)} \tag{158}$$

$$D_m^{(o)}(\lambda,\xi_1^*) = \frac{Ro_m^{(1)'}(\lambda,\xi_1^*)}{Ro_m^{(3)'}(\lambda,\xi_1^*)} \tag{159}$$

$$\underline{M}\alpha_m^{(3)}(h,\underline{r}) = \frac{e^{ihz}}{c\beta}\left[\hat{u}R\alpha_m^{(3)}(\lambda,\xi)S\alpha_m'(\lambda,\eta) - \hat{v}R\alpha_m^{(3)'}(\lambda,\xi)S\alpha_m(\lambda,\eta)\right] \qquad (\alpha = e,o) \quad (160)$$

$$\underline{N}\alpha_m(h,\underline{r}) = \frac{ihe^{ihz}}{k_o c\beta}\left[\hat{u}R\alpha_m^{(3)'}(\lambda,\xi)S\alpha_m(\lambda,\eta) + \hat{v}R\alpha_m^{(3)}(\lambda,\xi)S\alpha_m'(\lambda,\eta)\right]$$

$$+ \frac{\lambda^2}{c^2 k_o} e^{ihz}\hat{z}R\alpha_m^{(3)}(\lambda,\xi)S\alpha_m(\lambda,\eta) \qquad (\alpha = e,o) \qquad (161)$$

$$\hat{u} = \beta^{-1}(a\hat{x} + b\hat{y}) \qquad (162)$$

$$\hat{v} = \beta^{-1}(-b\hat{x} + a\hat{y}) \qquad (163)$$

and a, b, and β are defined in (141), (142), and (143). The primes that appear in (158) through (161) indicate the appropriate derivative $d/d\xi$ or $d/d\eta$ and the R's and S's are Mathieu functions given in Stratton's notation [3]. The coordinate system corresponding to the $\hat{x}$, $\hat{y}$, and $\hat{z}$ is, as before, depicted in figure 2.

Using the relationships

$$\nabla\times\underline{M} = k_o\underline{N} \qquad (164)$$

and

$$\nabla\times\underline{N} = k_o\underline{M} \qquad (165)$$

together with (153), (154) and (155) we can write D_1 as

$$\underline{\underline{D}}_1(\underline{r},\underline{r}_o) = -\frac{ic^2}{2\pi}\int_{C_h}\frac{dh}{\lambda^2}\sum_{m=o}^{\infty}\left\{(N_m^{(e)}(\lambda))^{-1}\left[C_m^{(e)}\underline{Ne}_m^{(3)}(h,\underline{r})\underline{Me}_m^{(3)}(-h,\underline{r}_o)\right.\right.$$

$$\left. + D_m^{(e)}\underline{Me}_m^{(3)}(h,\underline{r})\underline{Ne}_m^{(3)}(-h,\underline{r}_o)\right] \qquad (166)$$

$$\left. + (N_m^{(o)}(\lambda))^{-1}\left[C_m^{(o)}\underline{No}_m^{(3)}(h,\underline{r})\underline{Mo}_m^{(3)}(-h,\underline{r}_o) + D_m^{(o)}\underline{Mo}_m^{(3)}(h,\underline{r})\underline{No}_m^{(3)}(-h,\underline{r}_o)\right]\right\}$$

The preceding equation is essentially the complete representation of $\underline{\underline{D}}_1$ in Stratton's notation. The remaining portion of this section will be devoted to: (1) presenting $\underline{\underline{D}}_1$ in Cartesian components; (2) converting to McLachlan's notation; (3) converting to Blanch's notation; (4) changing the dummy integration variable, h, to a more convenient integration variable.

Simply expanding (166) we obtain the Cartesian representation of $\underline{\underline{D}}_1$ which is

$$\underline{\underline{D}}_1(\underline{r},\underline{r}_o) = D_{xx_o}\hat{x}\hat{x} + D_{xy_o}\hat{x}\hat{y} + D_{yx_o}\hat{y}\hat{x} + D_{yy_o}\hat{y}\hat{y}$$
$$+ D_{zx_o}\hat{z}\hat{x} + D_{zy_o}\hat{z}\hat{y} + D_{xz_o}\hat{x}\hat{z} + D_{yz_o}\hat{y}\hat{z} \tag{167}$$

where

$$D_{xx_o} = (2\pi\beta\beta_o)^{-1}(K_1(aa_o) + K_2(ab_o) - K_3(ba_o) - K_4(bb_o)) \tag{168}$$

$$D_{xy_o} = (2\pi\beta\beta_o)^{-1}(K_1(ab_o) - K_2(aa_o) - K_3(bb_o) + K_4(ba_o)) \tag{169}$$

$$D_{yx_o} = (2\pi\beta\beta_o)^{-1}(K_1(ba_o) + K_2(bb_o) + K_3(aa_o) + K_4(ab_o)) \tag{170}$$

$$D_{yy_o} = (2\pi\beta\beta_o)^{-1}(K_1(bb_o) - K_2(ba_o) + K_3(ab_o) - K_4(aa_o)) \tag{171}$$

$$D_{zx_o} = -i\beta/(2\pi c\beta\beta_o)(K_5a_o + K_6b_o) \tag{172}$$

$$D_{zy_o} = -i\beta/(2\pi c\beta\beta_o)(K_5b_o - K_6a_o) \tag{173}$$

$$D_{xz_o} = -i\beta_o/(2\pi c\beta\beta_o)(K_7a + K_8b) \tag{174}$$

$$D_{yz_o} = -i\beta_o/(2\pi c\beta\beta_o)(K_7 b - K_8 a) \quad (175)$$

$$K_1 = \int_{C_h} (\Sigma_{(1)}(C) - \Sigma^\dagger_{(1)}(D))h\lambda^{-2} e^{ih(z-z_o)} dh \quad (176)$$

$$K_2 = \int_{C_h} (\Sigma_{(2)}(C) + \Sigma_{(3)}(D))h\lambda^{-2} e^{ih(z-z_o)} dh \quad (177)$$

$$K_3 = \int_{C_h} (\Sigma_{(3)}(C) + \Sigma_{(2)}(D))h\lambda^{-2} e^{ih(z-z_o)} dh \quad (178)$$

$$K_4 = \int_{C_h} (\Sigma^\dagger_{(1)}(C) - \Sigma_{(1)}(D))h\lambda^{-2} e^{ih(z-z_o)} dh \quad (179)$$

$$K_5 = \int_{C_h} (\Sigma^\dagger_{(4)}(C)) e^{ih(z-z_o)} dh \quad (180)$$

$$K_6 = \int_{C_h} (\Sigma^\dagger_{(5)}(C)) e^{ih(z-z_o)} dh \quad (181)$$

$$K_7 = \int_{C_h} (\Sigma_{(4)}(D)) e^{ih(z-z_o)} dh \quad (182)$$

$$K_8 = \int_{C_h} (\Sigma_{(5)}(D)) e^{ih(z-z_o)} dh \quad (183)$$

In the preceding equations a, b, and β are the quantities given in (141), (142), and (143) while

$$a_o = a(\underline{r}_o) = \frac{\sinh \xi_o \cos \eta_o}{\beta_o} \quad (184)$$

$$b_o = b(\underline{r}_o) = \frac{\cosh \xi_o \sin \eta_o}{\beta_o} \quad (185)$$

$$\beta_o = \beta(\underline{r}_o) = (\cosh^2\xi_o - \cos^2\eta_o)^{\frac{1}{2}} \tag{186}$$

The explicit representations of the remaining terms are

$$\Sigma_{(1)}(\Gamma) = \sum_{m=o}^{\infty} \Big\{\Gamma_m^{(e)}(\lambda,\xi_1^*)(N_m^{(e)}(\lambda))^{-1}Se_m(\lambda,\eta)Re_m^{(3)'}(\lambda,\xi)Se_m'(\lambda,\eta_o)Re_m^{(3)}(\lambda,\xi_o)$$

$$+ \Gamma_m^{(o)}(\lambda,\xi_1^*)(N_m^{(o)}(\lambda))^{-1}So_m(\lambda,\eta)Ro_m^{(3)'}(\lambda,\xi)So_m'(\lambda,\eta_o)Ro_m^{(3)}(\lambda,\xi_o)\Big\} \tag{187}$$

$$\Sigma_{(2)}(\Gamma) = \sum_{m=o}^{\infty} \Big\{\Gamma_m^{(e)}(\lambda,\xi_1^*)(N_m^{(e)}(\lambda))^{-1}Se_m(\lambda,\eta)Re_m^{(3)'}(\lambda,\xi)Se_m(\lambda,\eta_o)Re_m^{(3)'}(\lambda,\xi_o)$$

$$+ \Gamma_m^{(o)}(\lambda,\xi_1^*)(N_m^{(o)}(\lambda))^{-1}So_m(\lambda,\eta)Ro_m^{(3)'}(\lambda,\xi)So_m(\lambda,\eta_o)Ro_m^{(3)'}(\lambda,\xi_o)\Big\} \tag{188}$$

$$\Sigma_{(3)}(\Gamma) = \sum_{m=o}^{\infty} \Big\{\Gamma_m^{(e)}(\lambda,\xi_1^*)(N_m^{(e)}(\lambda))^{-1}Se_m'(\lambda,\eta)Re_m^{(3)}(\lambda,\xi)Se_m'(\lambda,\eta_o)Re_m^{(3)}(\lambda,\xi_o)$$

$$+ \Gamma_m^{(o)}(\lambda,\xi_1^*)(N_m^{(o)}(\lambda))^{-1}So_m'(\lambda,\eta)Ro_m^{(3)}(\lambda,\xi)So_m'(\lambda,\eta_o)Ro_m^{(3)}(\lambda,\xi_o)\Big\} \tag{189}$$

$$\Sigma_{(4)}(\Gamma) = \sum_{m=o}^{\infty} \Big\{\Gamma_m^{(e)}(\lambda,\xi_1^*)(N_m^{(e)}(\lambda))^{-1}Se_m'(\lambda,\eta)Re_m^{(3)}(\lambda,\xi)Se_m(\lambda,\eta_o)Re_m^{(3)}(\lambda,\xi_o)$$

$$+ \Gamma_m^{(o)}(\lambda,\xi_1^*)(N_m^{(o)}(\lambda))^{-1}So_m'(\lambda,\eta)Ro_m^{(3)}(\lambda,\xi)So_m(\lambda,\eta_o)Ro_m^{(3)}(\lambda,\xi_o)\Big\} \tag{190}$$

$$\Sigma_{(5)}(\Gamma) = \sum_{m=o}^{\infty} \Big\{\Gamma_m^{(e)}(\lambda,\xi_1^*)(N_m^{(e)}(\lambda))^{-1}Se_m(\lambda,\eta)Re_m^{(3)'}(\lambda,\xi)Se_m(\lambda,\eta_o)Re_m^{(3)}(\lambda,\xi_o)$$

$$+ \Gamma_m^{(o)}(\lambda,\xi_1^*)(N_m^{(o)}(\lambda))^{-1}So_m(\lambda,\eta)Ro_m^{(3)'}(\lambda,\xi)So_m(\lambda,\eta_o)Ro_m^{(3)}(\lambda,\xi_o)\Big\} \tag{191}$$

where Γ equals $C(\lambda,\xi_1^*)$ or $D(\lambda,\xi_1^*)$. The $\dagger$ superscript attached to the Σ's appearing in (176), (179), (180), and (181) indicate an interchange of the $\underline{r}$ and $\underline{r}_o$ dependence. For example

$$\Sigma^{\dagger}_{(1)}(D) = \sum_{m=o}^{\infty} \Big\{ D_m^{(e)}(\lambda,\xi_1^*)(N_m^{(e)}(\lambda))^{-1} Se_m(\lambda,\eta_o) Re_m^{(3)'}(\lambda,\xi_o) Se'_m(\lambda,\eta) Re_m^{(3)}(\lambda,\xi)$$

$$+ D_m^{(o)}(\lambda,\xi_1^*)(N_m^{(o)}(\lambda))^{-1} So_m(\lambda,\eta_o) Ro_m^{(3)'}(\lambda,\xi_o) So'_m(\lambda,\eta) Ro_m^{(3)}(\lambda,\xi) \Big\} \quad (192)$$

In order to present $\underline{\underline{D}}_1$ in McLachlan's and Blanch's notation it is only necessary to convert the Mathieu functions appearing in (187) through (191) to the appropriate notation. In either of these notations, the natural integration variable is q rather than h. According to (48) and (155), the relationship between these variables is

$$q = c^2(k_o^2 - h^2)/4 \quad (193)$$

Making this change of variables and using (51), (53), (60) through (63), (72) through (75), (121), (122) and writing the exponential in terms of sine and cosine as well as using the symmetry properties of these trigonometric functions, we can write

$$K_1 = \frac{i}{c^2} \int_{C_q} dqq^{-1} \sin(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)[\Sigma_{(1)}(C,q) - \Sigma^{\dagger}_{(1)}(D,q)] \quad (194)$$

$$K_2 = \frac{i}{c^2} \int_{C_q} dqq^{-1} \sin(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)[\Sigma_{(2)}(C,q) + \Sigma_{(3)}(D,q)] \quad (195)$$

$$K_3 = \frac{i}{c^2} \int_{C_q} dqq^{-1} \sin(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)[\Sigma_{(3)}(C,q) + \Sigma_{(2)}(D,q)] \quad (196)$$

$$K_4 = \frac{i}{c^2} \int_{C_q} dqq^{-1} \sin(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)[\Sigma^{\dagger}_{(1)}(C,q) - \Sigma_{(1)}(D,q)] \quad (197)$$

$$K_5 = \frac{2}{c} \int_{C_q} dq \Big\{ (q_u - q)^{-\frac{1}{2}} \cos(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)\Sigma^{\dagger}_{(4)}(C,q)$$

$$- U(q)(q_u - q)^{-\frac{1}{2}}\Sigma^{\dagger}_{(4)}(C,q_u)\Big\} + \frac{4}{c} q_u^{\frac{1}{2}}\Sigma^{\dagger}_{(4)}(C,q_u) \quad (198)$$

$$K_6 = \frac{2}{c}\int_{C_q} dq\Big\{(q_u - q)^{-\frac{1}{2}} \cos(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)\Sigma^{\dagger}_{(5)}(C,q)$$

$$- U(q)(q_u - q)^{-\frac{1}{2}}\Sigma^{\dagger}_{(5)}(C,q_u)\Big\} + \frac{4}{c}\, q_u^{\frac{1}{2}}\Sigma^{\dagger}_{(5)}(C,q_u) \qquad (199)$$

$$K_7 = \frac{2}{c}\int_{C_q} dq\Big\{(q_u - q)^{-\frac{1}{2}} \cos(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)\Sigma_{(4)}(D,q)$$

$$- U(q)(q_u - q)^{-\frac{1}{2}}\Sigma_{(4)}(D,q_u)\Big\} + \frac{4}{c}\, q_u^{\frac{1}{2}}\Sigma_{(4)}(D,q_u) \qquad (200)$$

$$K_8 = \frac{2}{c}\int_{C_q} dq\Big\{(q_u - q)^{-\frac{1}{2}} \cos(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)\Sigma_{(5)}(D,q)$$

$$- U(q)(q_u - q)^{-\frac{1}{2}}\Sigma_{(5)}(D,q_u)\Big\} + \frac{4}{c}\, q_u^{\frac{1}{2}}\Sigma_{(5)}(D,q_u) \qquad (201)$$

where

$$q_u = \frac{k_o^2 c^2}{4} \qquad (202)$$

and C_q is depicted in figure 5. The additonal terms in (198) through (201) result from the removal of the artifically induced singularity at q_u which resulted from the change of integration variables. The U(q) that appears in these equations is the Heaviside step function. The $\Sigma(\Gamma)$'s are exactly the same quantities defined in (187) through (189); however, they are expressed in McLachlan's notation as

$$\Sigma_{(1)}(\Gamma,q) = \frac{1}{2}\sum_{m=o}^{\infty}\Big\{\Gamma^{(e)}_{2m}(q,\xi_1^*)(p_{2m}(q))^{-2}ce_{2m}(q,\eta)Me^{(1)'}_{2m}(q,\xi)ce'_{2m}(q,\eta_o)Me^{(1)}_{2m}(q,\xi_o)$$

$$+ \Gamma^{(e)}_{2m+1}(q,\xi_1^*)(p_{2m+1}(q))^{-2}ce_{2m+1}(q,\eta)Me^{(1)'}_{2m+1}(q,\xi)ce'_{2m+1}(q,\eta_o)Me^{(1)}_{2m+1}(q,\xi_o)$$

(203)

$$+ \Gamma^{(o)}_{2m+1}(q,\xi_1^*)(s_{2m+1}(q))^{-2}se_{2m+1}(q,\eta)Ne^{(1)'}_{2m+1}(q,\xi)se'_{2m+1}(q,\eta_o)Ne^{(1)}_{2m+1}(q,\xi_o)$$

$$+ \Gamma^{(o)}_{2m+2}(q,\xi_1^*)(s_{2m+2}(q))^{-2}se_{2m+2}(q,\eta)Ne^{(1)'}_{2m+2}(q,\xi)se'_{2m+2}(q,\eta_o)Ne^{(1)}_{2m+2}(q,\xi_o)\Big\}$$

$$\Sigma_{(2)}(\Gamma,q) = \frac{1}{2}\sum_{m=o}^{\infty}\Big\{\Gamma_{2m}^{(e)}(q,\xi_1^*)(p_{2m}(q))^{-2}ce_{2m}(q,\eta)Me_{2m}^{(1)'}(q,\xi)ce_{2m}(q,\eta_o)Me_{2m}^{(1)'}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(e)}(q,\xi_1^*)(p_{2m+1}(q))^{-2}ce_{2m+1}(q,\eta)Me_{2m+1}^{(1)'}(q,\xi)ce_{2m+1}(q,\eta_o)Me_{2m+1}^{(1)'}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(o)}(q,\xi_1^*)(s_{2m+1}(q))^{-2}se_{2m+1}(q,\eta)Ne_{2m+1}^{(1)'}(q,\xi)se_{2m+1}(q,\eta_o)Ne_{2m+1}^{(1)'}(q,\xi_o) \qquad (204)$$

$$+\Gamma_{2m+2}^{(o)}(q,\xi_1^*)(s_{2m+2}(q))^{-2}se_{2m+2}(q,\eta)Ne_{2m+2}^{(1)'}(q,\xi)se_{2m+2}(q,\eta_o)Ne_{2m+2}^{(1)'}(q,\xi_o)\Big\}$$

$$\Sigma_{(3)}(\Gamma,q) = \frac{1}{2}\sum_{m=o}^{\infty}\Big\{\Gamma_{2m}^{(e)}(q,\xi_1^*)(p_{2m}(q))^{-2}ce'_{2m}(q,\eta)Me_{2m}^{(1)}(q,\xi)ce'_{2m}(q,\eta_o)Me_{2m}^{(1)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(e)}(q,\xi_1^*)(p_{2m+1}(q))^{-2}ce'_{2m+1}(q,\eta)Me_{2m+1}^{(1)}(q,\xi)ce'_{2m+1}(q,\eta_o)Me_{2m+1}^{(1)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(o)}(q,\xi_1^*)(s_{2m+1}(q))^{-2}se'_{2m+1}(q,\eta)Ne_{2m+1}^{(1)}(q,\xi)se'_{2m+1}(q,\eta_o)Ne_{2m+1}^{(1)}(q,\xi_o) \qquad (205)$$

$$+\Gamma_{2m+2}^{(o)}(q,\xi_1^*)(s_{2m+2}(q))^{-2}se'_{2m+2}(q,\eta)Ne_{2m+2}^{(1)}(q,\xi)se'_{2m+2}(q,\eta_o)Ne_{2m+2}^{(1)}(q,\xi_o)\Big\}$$

$$\Sigma_{(4)}(\Gamma,q) = \frac{1}{2}\sum_{m=o}\Big\{\Gamma_{2m}^{(e)}(q,\xi_1^*)(p_{2m}(q))^{-2}ce'_{2m}(q,\eta)Me_{2m}^{(1)}(q,\xi)ce_{2m}(q,\eta_o)Me_{2m}^{(1)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(e)}(q,\xi_1^*)(p_{2m+1}(q))^{-2}ce'_{2m+1}(q,\eta)Me_{2m+1}^{(1)}(q,\xi)ce_{2m+1}(q,\eta_o)Me_{2m+1}^{(1)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(o)}(q,\xi_1^*)(s_{2m+1}(q))^{-2}se'_{2m+1}(q,\eta)Ne_{2m+1}^{(1)}(q,\xi)se_{2m+1}(q,\eta_o)Ne_{2m+1}^{(1)}(q,\xi_o) \qquad (206)$$

$$+\Gamma_{2m+2}^{(o)}(q,\xi_1^*)(s_{2m+2}(q))^{-2}se'_{2m+2}(q,\eta)Ne_{2m+2}^{(1)}(q,\xi)se_{2m+2}(q,\eta_o)Ne_{2m+2}^{(1)}(q,\xi_o)\Big\}$$

$$\Sigma_{(5)}(\Gamma,q) = \frac{1}{2}\sum_{m=o}^{\infty}\Big\{\Gamma_{2m}^{(e)}(q,\xi_1^*)(p_{2m}(q))^{-2}ce_{2m}(q,\eta)Me_{2m}^{(1)'}(q,\xi)ce_{2m}(q,\eta_o)Me_{2m}^{(1)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(e)}(q,\xi_1^*)(p_{2m+1}(q))^{-2}ce_{2m+1}(q,\eta)Me_{2m+1}^{(1)'}(q,\xi)ce_{2m+1}(q,\eta_o)Me_{2m+1}^{(1)}(q,\xi_o)$$

(207)

$$+\Gamma_{2m+1}^{(o)}(q,\xi_1^*)(s_{2m+1}(q))^{-2}se_{2m+1}(q,\eta)Ne_{2m+1}^{(1)'}(q,\xi)se_{2m+1}(q,\eta_o)Ne_{2m+1}^{(1)}(q,\xi_o)$$

$$+\Gamma_{2m+2}^{(o)}(q,\xi_1^*)(s_{2m+2}(q))^{-2}se_{2m+2}(q,\eta)Ne_{2m+2}^{(1)'}(q,\xi)se_{2m+2}(q,\eta_o)Ne_{2m+2}^{(1)}(q,\xi_o)\Big\}$$

where $\Gamma = C$ corresponds to

$$\Gamma_m^{(e)}(q,\xi_1^*) = C_m^{(e)}(q,\xi_1^*) = Ce_m(q,\xi_1^*)/Me_m^{(1)}(q,\xi_1^*) \tag{208}$$

$$\Gamma_m^{(o)}(q,\xi_1^*) = C_m^{(o)}(q,\xi_1^*) = Se_m(q,\xi_1^*)/Ne_m^{(1)}(q,\xi_1^*) \tag{209}$$

and where $\Gamma = D$ corresponds to

$$\Gamma_m^{(e)}(q,\xi_1^*) = D_m^{(e)}(q,\xi_1^*) = Ce'_m(q,\xi_1^*)/Me_m^{(1)'}(q,\xi_1^*) \tag{210}$$

$$\Gamma_m^{(o)}(q,\xi_1^*) = D_m^{(o)}(q,\xi_1^*) = Se'_m(q,\xi_1^*)/Ne_m^{(1)'}(q,\xi_1^*) \tag{211}$$

Retaining the same meaning for $\dagger$, we see that (167) through (175) together with (194) through (211) complete the representation of $\underline{\underline{D}}_1$ in McLachlan's notation.

Using the fact that McLachlan's and Blanch's notation for angular Mathieu functions is the same as well as (76), (77), (80), and (81) we can write the preceding equations as

$$\Sigma_{(1)}(\Gamma,q) = \frac{1}{2}\sum_{m=0}^{\infty}\Big\{\Gamma_{2m}^{(e)}(q,\xi_1^*)ce_{2m}(q,\eta)Mc_{2m}^{(3)'}(q,\xi)ce'_{2m}(q,\eta_o)Mc_{2m}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(e)}(q,\xi_1^*)ce_{2m+1}(q,\eta)Mc_{2m+1}^{(3)'}(q,\xi)ce'_{2m+1}(q,\eta_o)Mc_{2m+1}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(o)}(q,\xi_1^*)se_{2m+1}(q,\eta)Ms_{2m+1}^{(3)'}(q,\xi)se'_{2m+1}(q,\eta_o)Ms_{2m+1}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+2}^{(o)}(q,\xi_1^*)se_{2m+2}(q,\eta)Ms_{2m+2}^{(3)'}(q,\xi)se'_{2m+2}(q,\eta_o)Ms_{2m+2}^{(3)}(q,\xi_o)\Big\} \qquad (212)$$

$$\Sigma_{(2)}(\Gamma,q) = \frac{1}{2}\sum_{m=0}^{\infty}\Big\{\Gamma_{2m}^{(e)}(q,\xi_1^*)ce_{2m}(q,\eta)Mc_{2m}^{(3)'}(q,\xi)ce_{2m}(q,\eta_o)Mc_{2m}^{(3)'}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(e)}(q,\xi_1^*)ce_{2m+1}(q,\eta)Mc_{2m+1}^{(3)'}(q,\xi)ce_{2m+1}(q,\eta_o)Mc_{2m+1}^{(3)'}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(o)}(q,\xi_1^*)se_{2m+1}(q,\eta)Ms_{2m+1}^{(3)'}(q,\xi)se_{2m+1}(q,\eta_o)Ms_{2m+1}^{(3)'}(q,\xi_o)$$

$$+\Gamma_{2m+2}^{(o)}(q,\xi_1^*)se_{2m+2}(q,\eta)Ms_{2m+2}^{(3)'}(q,\xi)se_{2m+2}(q,\eta_o)Ms_{2m+2}^{(3)'}(q,\xi_o)\Big\} \qquad (213)$$

$$\Sigma_{(3)}(\Gamma,q) = \frac{1}{2}\sum_{m=0}^{\infty}\Big\{\Gamma_{2m}^{(e)}(q,\xi_1^*)ce'_{2m}(q,\eta)Mc_{2m}^{(3)}(q,\xi)ce'_{2m}(q,\eta_o)Mc_{2m}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(e)}(q,\xi_1^*)ce'_{2m+1}(q,\eta)Mc_{2m+1}^{(3)}(q,\xi)ce'_{2m+1}(q,\eta_o)Mc_{2m+1}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(o)}(q,\xi_1^*)se'_{2m+1}(q,\eta)Ms_{2m+1}^{(3)}(q,\xi)se'_{2m+1}(q,\eta_o)Ms_{2m+1}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+2}^{(o)}(q,\xi_1^*)se'_{2m+2}(q,\eta)Ms_{2m+2}^{(3)}(q,\xi)se'_{2m+2}(q,\eta_o)Ms_{2m+2}^{(3)}(q,\xi_o)\Big\} \qquad (214)$$

$$\Sigma_{(4)}(\Gamma,q) = \frac{1}{2}\sum_{m=o}^{\infty}\Big\{\Gamma_{2m}^{(e)}(q,\xi_1^*)ce'_{2m}(q,\eta)Mc_{2m}^{(3)}(q,\xi)ce_{2m}(q,\eta_o)Mc_{2m}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(e)}(q,\xi_1^*)ce'_{2m+1}(q,\eta)Mc_{2m+1}^{(3)}(q,\xi)ce_{2m+1}(q,\eta_o)Mc_{2m+1}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(o)}(q,\xi_1^*)se'_{2m+1}(q,\eta)Ms_{2m+1}^{(3)}(q,\xi)se_{2m+1}(q,\eta_o)Ms_{2m+1}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+2}^{(o)}(q,\xi_1^*)se'_{2m+2}(q,\eta)Ms_{2m+2}^{(3)}(q,\xi)se_{2m+2}(q,\eta_o)Ms_{2m+2}^{(3)}(q,\xi_o)\Big\} \quad (215)$$

$$\Sigma_{(5)}(\Gamma,q) = \frac{1}{2}\sum_{m=o}^{\infty}\Big\{\Gamma_{2m}^{(e)}(q,\xi_1^*)ce_{2m}(q,\eta)Mc_{2m}^{(3)'}(q,\xi)ce_{2m}(q,\eta_o)Mc_{2m}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(e)}(q,\xi_1^*)ce_{2m+1}(q,\eta)Mc_{2m+1}^{(3)'}(q,\xi)ce_{2m+1}(q,\eta_o)Mc_{2m+1}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+1}^{(o)}(q,\xi_1^*)se_{2m+1}(q,\eta)Ms_{2m+1}^{(3)'}(q,\xi)se_{2m+1}(q,\eta_o)Ms_{2m+1}^{(3)}(q,\xi_o)$$

$$+\Gamma_{2m+2}^{(o)}(q,\xi_1^*)se_{2m+2}(q,\eta)Ms_{2m+2}^{(3)'}(q,\xi)se_{2m+2}(q,\eta_o)Ms_{2m+2}^{(3)}(q,\xi_o)\Big\} \quad (216)$$

where $\Gamma = C$ corresponds to

$$\Gamma_m^{(e)}(q,\xi_1^*) = C_m^{(e)}(q,\xi_1^*) = Mc_m^{(1)}(q,\xi_1^*)/Mc_m^{(3)}(q,\xi_1^*) \quad (217)$$

$$\Gamma_m^{(o)}(q,\xi_1^*) = C_m^{(o)}(q,\xi_1^*) = Ms_m^{(1)}(q,\xi_1^*)/Ms_m^{(3)}(q,\xi_1^*) \quad (218)$$

and where $\Gamma = D$ corresponds to

$$\Gamma_m^{(e)}(q,\xi_1^*) = D_m^{(e)}(q,\xi_1^*) = Mc_m^{(1)'}(q,\xi_1^*)/Mc_m^{(3)'}(q,\xi_1^*) \quad (219)$$

$$\Gamma_m^{(o)}(q,\xi_1^*) = D_m^{(o)}(q,\xi_1^*) = Ms_m^{(1)'}(q,\xi_1^*)/Ms_m^{(3)'}(q,\xi_1^*) \quad (220)$$

Equations (167) through (175), (194) through (202), and (212) through (220) complete the definition of $\underline{\underline{D}}_1$ in Blanch's notation.

We complete this section by noting that the integrals appearing in (194) through (197) have an apparent non-integrable singularity at q = 0. This singularity will be the subject of the next section.

VI. Small q Behavior of Terms Necessary for $\underline{\underline{D}}_1$ Singularity Removal

In this section we will examine the small q behavior of the sums and combination of sums that appear in (194) through (201). We have presented the explicit representation of these sums in both McLachlan's and Blanch's notation; however, in this section we will only consider McLachlan's notation. We choose this notation because we found many useful small q expansions in McLachlan's book. Even though we obtain the small q expansions of the sums using McLachlan's notation, the ultimate value of these sums is independent of the notation and consequently the small q behavior of these sums which will be explicitly represented is directly applicable when the constituent terms of the sums are expressed in Blanch's notation.

In addition to the explicit small q behavior of certain functions, all other relationships concerning Mathieu functions and presented in this section can be found in McLachlan's book, unless otherwise noted. Until stated otherwise we will present results valid for small positive q. The small q behavior for angular Mathieu functions is given by

$$ce_o(q,\eta) = 2^{-\frac{1}{2}}[1 - q/2 \cos 2\eta + O(q^2)] \tag{221}$$

$$ce_m(q,\eta) = \cos m\eta + O(q) \qquad m \geq 1 \tag{222}$$

$$se_m(q,\eta) = \sin m\eta + O(q) \qquad m \geq 1 \tag{223}$$

The behavior of the derivatives of the preceding quantities can be found by differentiating the previous relationships and using the fact that a multiplicative function of η accompanies the order symbol. For completeness the behavior of their derivatives is

$$ce'_o(q,\eta) = q2^{-\frac{1}{2}} \sin 2\eta[1 + O(q)] \tag{224}$$

$$ce'_m(q,\eta) = -m \sin m\eta + O(q) \tag{225}$$

$$se'_m(q,\eta) = m \cos m\eta + O(q) \quad (226)$$

where ' indicates $d/d\eta$.

The small q behavior of certain radial Mathieu functions is determined by using the relations

$$Ce_n(q,\xi) = ce_n(q,i\xi) \quad (227)$$

$$Se_n(q,\xi) = -ise_n(q,i\xi) \quad (228)$$

Using (227) and (228) together with (221) through (226), we have

$$Ce_o(q,\xi) = 2^{-\frac{1}{2}}[1 - q/2 \cosh 2\xi + O(q^2)] \quad (229)$$

$$Ce_m(q,\xi) = \cosh m\xi + O(q) \quad (230)$$

$$Se_m(q,\xi) = \sinh m\xi + O(q) \quad (231)$$

$$Ce'_o(q,\xi) = -q2^{-\frac{1}{2}} \sinh 2\xi[1 + O(q)] \quad (232)$$

$$Ce'_m(q,\xi) = m \sinh m\xi + O(q) \quad (233)$$

$$Se'_m(q,\xi) = m \cosh m\xi + O(q) \quad (234)$$

where ' indicates $d/d\xi$.

The remaining functions whose small q behavior is required before the small q behavior of the sums appearing in (194) through (201) are $Me_m^{(1)}(q,\xi)$, $Me_m^{(1)'}(q,\xi)$, $Ne_m^{(1)}(q,\xi)$, and $Ne_m^{(1)'}(q,\xi)$. We were not able to find the small q behavior of these functions expressed to a high enough order for our purposes. We will now present a derivation of the necessary small q behavior. Still using McLachlan's book and notation we start with the Bessel function product series representation of these radial Mathieu functions. These product series representations are

$$Me_{2n}^{(1)}(q,\xi) = \frac{p_{2n}(q)}{A_o^{(2n)}(q)} M_{2n}(q,\xi) \tag{235}$$

$$Me_{2n+1}^{(1)}(q,\xi) = \frac{p_{2n+1}(q)}{A_1^{(2n+1)}(q)} M_{2n+1}(q,\xi) \tag{236}$$

$$Ne_{2n+1}^{(1)}(q,\xi) = \frac{s_{2n+1}(q)}{B_1^{(2n+1)}(q)} N_{2n+1}(q,\xi) \tag{237}$$

$$Ne_{2n+2}^{(1)}(q,\xi) = -\frac{s_{2n+2}(q)}{B_2^{(2n+2)}(q)} N_{2n+2}(q,\xi) \tag{238}$$

where

$$M_{2n}(q,\xi) = \sum_{r=o}^{\infty} (-1)^r A_{2r}^{(2n)}(q) A_r(q,\xi) \tag{239}$$

$$M_{2n+1}(q,\xi) = \sum_{r=o}^{\infty} (-1)^r A_{2r+1}^{(2n+1)}(q) B_r(q,\xi) \tag{240}$$

$$N_{2n+1}(q,\xi) = \sum_{r=o}^{\infty} (-1)^r B_{2r+1}^{(2n+1)}(q) F_r(q,\xi) \tag{241}$$

$$N_{2n+2}(q,\xi) = \sum_{r=o}^{\infty} (-1)^r B_{2r+2}^{(2n+2)}(q) G_r(q,\xi) \tag{242}$$

with

$$A_r(q,\xi) = J_r(v_1) H_r^{(1)}(v_2) \tag{243}$$

$$B_r(q,\xi) = J_r(v_1)H^{(1)}_{r+1}(v_2) + J_{r+1}(v_1)H^{(1)}_r(v_2) \tag{244}$$

$$F_r(q,\xi) = J_r(v_1)H^{(1)}_{r+1}(v_2) - J_{r+1}(v_1)H^{(1)}_r(v_2) \tag{245}$$

$$G_r(q,\xi) = J_r(v_1)H^{(1)}_{r+2}(v_2) - J_{r+2}(v_1)H^{(1)}_r(v_2) \tag{246}$$

$$v_1 = q^{\frac{1}{2}}e^{-\xi} \tag{247}$$

$$v_2 = q^{\frac{1}{2}}e^{\xi} \tag{248}$$

The quantities p_{2m}, p_{2m+1}, s_{2m+1}, and s_{2m+2} have been defined in (64) through (67), while the $A\begin{Bmatrix}p\\q\end{Bmatrix}$'s and $B\begin{Bmatrix}p\\q\end{Bmatrix}$'s were also discussed with regard to these equations. These $A\begin{Bmatrix}p\\q\end{Bmatrix}$'s and $B\begin{Bmatrix}p\\q\end{Bmatrix}$'s are discussed throughout McLachlan's book and the pertinent aspects of the small q behavior of these quantities will be utilized as needed without further reference to McLachlan. We now proceed by presenting the small q behavior of A_r, B_r, F_r, and G_r using a standard reference on Bessel functions [7]. The small q behavior of these quantities is readily determined to be

$$A_o(q,\xi) = 1 + \frac{2i(\gamma - \ln 2)}{\pi} + \frac{i}{\pi}\ln q + \frac{2i}{\pi}\xi + O(q \ln q) \tag{249}$$

$$A_1(q,\xi) = -\frac{i}{\pi}e^{-2\xi}[1 - \frac{q}{4}e^{2\xi}\ln q + O(q)] \tag{250}$$

$$A_r(q,\xi) = -\frac{i}{\pi r}e^{-2r\xi}[1 + O(q)] \qquad r \geq 2 \tag{251}$$

$$B_o(q,\xi) \sim -\frac{2i}{\pi}q^{-\frac{1}{2}}e^{-\xi}[1 - \frac{q}{2}e^{\xi}\cosh\xi \ln q] \tag{252}$$

$$B_r(q,\xi) = -\frac{2i}{\pi} q^{-\frac{1}{2}} e^{-(2r+1)\xi}[1 + O(q)] \qquad r \geq 1 \tag{253}$$

$$F_o(q,\xi) \sim -\frac{2i}{\pi} q^{-\frac{1}{2}} e^{-\xi}[1 - \frac{q}{2} e^{\xi} \sinh \xi \ln q] \tag{254}$$

$$F_r(q,\xi) = -\frac{2i}{\pi} q^{-\frac{1}{2}} e^{-(2r+1)\xi}[1 + O(q)] \qquad r \geq 1 \tag{255}$$

$$G_r(q,\xi) = -\frac{4i(r+1)}{\pi} q^{-1} e^{-(2r+2)\xi}[1 + O(q)] \qquad r \geq 0 \tag{256}$$

where γ is Euler's constant given by

$$\gamma = .5772157\cdots \tag{257}$$

Using (249) through (256) together with the relations

$$A^{(m)}_{(m\pm 2r)}(q) \sim B^{(m)}_{(m\pm 2r)}(q) = O(q^r) \tag{258}$$

$$A^{(m)}_{(m)}(q) \sim B^{(m)}_{(m)}(q) = 1 + O(q^2) \qquad m \geq 1 \tag{259}$$

we see that the sums appearing in (239) through (242) reduce, for sufficiently small q, to considering at most the three terms corresponding to r equal to n - 1, n, and n + 1. With the exception of the calculation of $M_2(q,\xi)$, (239) through (242), and (249) through (259) are sufficient to derive the following results

$$M_o(q,\xi) = 2^{-\frac{1}{2}}[1 + \frac{2i(\gamma - \ln 2)}{\pi} + \frac{i}{\pi} \ln q + \frac{2i}{\pi} \xi + O(q \ln q)] \tag{260}$$

$$M_{2n}(q,\xi) = -\frac{i}{\pi n} (-1)^n e^{-2n\xi}[1 + O(q)] \qquad n \geq 1 \tag{261}$$

$$M_1(q,\xi) \sim -\frac{2i}{\pi} q^{-\frac{1}{2}} e^{-\xi}[1 - \frac{q}{2} e^{\xi} \cosh \xi \ln q] \tag{262}$$

$$M_{2n+1}(q,\xi) = -\frac{2i}{\pi} q^{-\frac{1}{2}}(-1)^n e^{-(2n+1)\xi}[1 + O(q)] \qquad n \geq 1 \tag{263}$$

$$N_1(q,\xi) \sim -\frac{2i}{\pi} q^{-\frac{1}{2}} e^{-\xi}[1 - \frac{q}{2} e^{\xi} \sinh \xi \ln q] \tag{264}$$

$$N_{2n+1}(q,\xi) \sim M_{2n+1}(q,\xi) \qquad n \geq 1 \tag{265}$$

$$N_{2n+2}(q,\xi) = -\frac{4i(n+1)}{\pi} q^{-1}(-1)^n e^{-(2n+2)\xi}[1 + O(q)] \qquad n \geq 0 \tag{266}$$

In order to obtain Me_2 corresponding to $n = 1$ in (261), it was necessary to use the relation

$$A_o^{(2)}(q) \sim \frac{q}{4} \tag{267}$$

so that the $q \ln q$ terms in (249) and (250) cancel. For the derivation of the remaining quantities in (260) through (266), the omitted coefficient in (258) for $r = 1$ would be of consequence only if we wished to present the explicit form of the $O(q)$ term.

In order to complete the specification of the small positive q behavior of the radial Mathieu functions defined in (235) through (238) we must specify the small q behavior of the coefficients. First we present

$$A_o^{(o)} = 2^{-\frac{1}{2}}[1 + O(q^2)] \tag{268}$$

$$A_o^{(2n)} = \frac{q^n}{2^{2n-1}(2n)!}[1 + O(q^2)] \tag{269}$$

$$A_1^{(2n+1)} \sim B_1^{(2n+1)} = \frac{q^n}{2^{2n}(2n)!}[1 + O(q^2)] \tag{270}$$

$$B_2^{(2n+2)} \sim \frac{(n+1)q^n}{2^{2n}(2n+1)!}[1 + O(q^2)] \tag{271}$$

Combining the preceding equations with the definitions (64) through (67) and (221) through (226) we have

$$p_o \sim 2^{-\frac{1}{2}} \tag{272}$$

$$p_{2n} \sim (-1)^n 2^{2n-1}(2n)!\, q^{-n} \tag{273}$$

$$p_{2n+1} \sim (-1)^n 2^{2n}(2n + 1)!\; q^{-(n+\frac{1}{2})} \tag{274}$$

$$s_{2n+1} \sim (-1)^n 2^{2n}(2n + 1)!\; q^{-(n+\frac{1}{2})} \tag{275}$$

$$s_{2n+2} \sim (-1)^{n+1} 2^{2n+1}(2n + 2)!\; q^{-(n+1)} \tag{276}$$

and the next terms are at least $O(q)$. Combining (235) through (238), (260) through (266), and (268) through (276) we have our desired results which are expressed as

$$Me_o^{(1)}(q,\xi) = 2^{-\frac{1}{2}}[1 + \frac{2i(\gamma - \ln 2)}{\pi} + \frac{i}{\pi}\ln q + \frac{2i}{\pi}\xi + O(q \ln q)] \tag{277}$$

$$Me_{2n}^{(1)}(q,\xi) = -\frac{i}{\pi n}(2^{2n-1}(2n)!)^2 e^{-2n\xi} q^{-2n}[1 + O(q)] \qquad n \geq 1 \tag{278}$$

$$Me_1^{(1)}(q,\xi) \sim -\frac{2i}{\pi} e^{-\xi} q^{-1}[1 - \frac{q}{2} e^{\xi} \cosh \xi \ln q] \tag{279}$$

$$Me^{(1)}_{2n+1}(q,\xi) = -\frac{2i}{\pi(2n+1)}\left(2^{2n}(2n+1)!\right)^2 e^{-(2n+1)\xi} q^{-(2n+1)}[1 + O(q)] \quad n \geq 1 \tag{280}$$

$$Ne^{(1)}_{1}(q,\xi) \sim -\frac{2i}{\pi} e^{-\xi} q^{-1}[1 - \frac{q}{2} e^{\xi} \sinh \xi \ln q] \tag{281}$$

$$Ne_{2n+1}(q,\xi) \sim Me_{2n+1}(q,\xi) \qquad n \geq 1 \tag{282}$$

$$Ne_{2n+2}(q,\xi) = -\frac{2i}{\pi(2n+2)}\left(2^{2n+1}(2n+2)!\right)^2 e^{-(2n+2)\xi} q^{-(2n+2)}[1 + O(q)] \quad n \geq 0 \tag{283}$$

We also need the small q behavior of the ξ derivative of the preceding Mathieu functions. This can be found by differentiating the preceding relationships and using the fact that a multiplicative function of ξ accompanies the order symbol. The results are

$$Me^{(1)'}_{o}(q,\xi) = \frac{i2^{\frac{1}{2}}}{\pi} + O(q \ln q) \tag{284}$$

$$Me^{(1)'}_{2n}(q,\xi) = \frac{2i}{\pi}\left(2^{2n-1}(2n)!\right)^2 e^{-2n} q^{-2n}[1 + O(q)] \qquad n \geq 1 \tag{285}$$

$$Me^{(1)'}_{1}(q,\xi) \sim \frac{2i}{\pi} e^{-\xi} q^{-1}[1 + \frac{q}{2} e^{\xi} \sinh \xi \ln q] \tag{286}$$

$$Me^{(1)'}_{2n+1}(q,\xi) = \frac{2i}{\pi}\left(2^{2n}(2n+1)!\right)^2 e^{-(2n+1)\xi} q^{-(2n+1)}[1 + O(q)] \qquad n \geq 1 \tag{287}$$

$$Ne^{(1)'}_{1}(q,\xi) \sim \frac{2i}{\pi} e^{-\xi} q^{-1}[1 + \frac{q}{2} e^{\xi} \cosh \xi \ln q] \tag{288}$$

$$Ne^{(1)'}_{2n+1}(q,\xi) \sim Me^{(1)'}_{2n+1}(q,\xi) \qquad n \geq 1 \tag{289}$$

$$Ne^{(1)'}_{2n+2}(q,\xi) = \frac{2i}{\pi}(2^{2n+1}(2n+2)!)^2 e^{-(2n+2)\xi} q^{-(2n+2)}[1+O(q)] \qquad n \geq 0 \qquad (290)$$

In (221) through (226), (229) through (234), and (272) through (290) we have presented the small q behavior of all quantities necessary to study the singularities that appear in (194) through (201) for positive q.

We now rewrite the sums appearing in (194) as

$$2\Sigma_{(1)}(C,q) = \sum_{m=o}^{\infty} \alpha^{(1)}_{2m}(C) + \alpha^{(1)}_{2m+1}(C) + \beta^{(1)}_{2m+1}(C) + \beta^{(1)}_{2m+2}(C) \qquad (291)$$

where

$$\alpha^{(1)}_{2m}(C) = C^{(e)}_{2m}(q,\xi^*_1)(p_{2m}(q))^{-2} ce_{2m}(q,\eta) Me^{(1)'}_{2m}(q,\xi) ce'_{2m}(q,\eta_o) Me^{(1)}_{2m}(q,\xi_o) \qquad (292)$$

$$\alpha^{(1)}_{2m+1}(C) = C^{(e)}_{2m+1}(q,\xi^*_1)(p_{2m+1}(q))^{-2} ce_{2m+1}(q,\eta) Me^{(1)'}_{2m+1}(q,\xi) ce'_{2m+1}(q,\eta_o) Me^{(1)}_{2m+1}(q,\xi_o) \qquad (293)$$

$$\beta^{(1)}_{2m+1}(C) = C^{(o)}_{2m+1}(q,\xi^*_1)(s_{2m+1}(q))^{-2} se_{2m+1}(q,\eta) Ne^{(1)'}_{2m+1}(q,\xi) se'_{2m+1}(q,\eta_o) Ne^{(1)}_{2m+1}(q,\xi_o) \qquad (294)$$

$$\beta^{(1)}_{2m+2}(C) = C^{(o)}_{2m+2}(q,\xi^*_1)(s_{2m+2}(q))^{-2} se_{2m+2}(q,\eta) Ne^{(1)'}_{2m+2}(q,\xi) se'_{2m+2}(q,\eta_o) Ne^{(1)}_{2m+2}(q,\xi_o) \qquad (295)$$

and

$$2\Sigma^{\dagger}_{(1)}(D,q) = \sum_{m=o}^{\infty} \alpha^{(1)\dagger}_{2m}(D) + \alpha^{(1)\dagger}_{2m+1}(D) + \beta^{(1)\dagger}_{2m+1}(D) + \beta^{(1)\dagger}_{2m+2}(D) \qquad (296)$$

where, as before $\dagger$ indicates an interchange of $\underline{r}$ and $\underline{r}_o$, so that

$$\alpha^{(1)\dagger}_{2m}(D) = D^{(e)}_{2m}(q,\xi^*_1)(p_{2m}(q))^{-2} ce_{2m}(q,\eta_o) Me^{(1)'}_{2m}(q,\xi_o) ce'_{2m}(q,\eta) Me^{(1)}_{2m}(q,\xi) \qquad (297)$$

$$\alpha^{(1)\dagger}_{2m+1}(D) = D^{(e)}_{2m+1}(q,\xi^*_1)(p_{2m+1}(q))^{-2} ce_{2m+1}(q,\eta_o) Me^{(1)'}_{2m+1}(q,\xi_o) ce'_{2m+1}(q,\eta) Me^{(1)}_{2m+1)}(q,\xi) \qquad (298)$$

$$\beta_{2m+1}^{(1)\dagger}(D) = D_{2m+1}^{(o)}(q,\xi_1^*)(s_{2m+1}(q))^{-2} se_{2m+1}(q,\eta_o) Ne_{2m+1}^{(1)'}(q,\xi_o) se'_{2m+1}(q,\eta) Ne_{2m+1}^{(1)}(q,\xi) \tag{299}$$

$$\beta_{2m+2}^{(1)\dagger}(D) = D_{2m+2}^{(o)}(q,\xi_1^*)(s_{2m+2}(q))^{-2} se_{2m+2}(q,\eta_o) Ne_{2m+2}^{(1)'}(q,\xi_o) se'_{2m+2}(q,\eta) Ne_{2m+2}^{(1)}(q,\xi) \tag{300}$$

with the C's and D's defined in (208) through (211). Using the small positive q results presented in this section we can express the α's and β's as

$$\alpha_o^{(1)}(C) = O(q) \tag{301}$$

$$\alpha_1^{(1)}(C) \sim -\frac{2i}{\pi} \cosh \xi_1^* \cos \eta \sin \eta_o e^{-(\xi+\xi_o-\xi_1^*)} [1+\tfrac{1}{2}Q_s(\xi)-\tfrac{1}{2}Q_c(\xi_o)+\tfrac{1}{2}Q_c(\xi_1^*)] \tag{302}$$

$$\alpha_{2m}^{(1)}(C) = -\frac{2i(2m)}{\pi} \cosh 2m\xi_1^* \cos 2m\eta \sin 2m\eta_o e^{-2m(\xi+\xi_o-\xi_1^*)} + O(q) \quad m \geq 1 \tag{303}$$

$$\alpha_{2m+1}^{(1)}(C) = -\frac{2i(2m+1)}{\pi} \cosh(2m+1)\xi_1^* \cos(2m+1)\eta \sin(2m+1)\eta_o e^{-(2m+1)(\xi+\xi_o-\xi_1^*)} + O(q) \quad m \geq 1 \tag{304}$$

$$\beta_1^{(1)}(C) \sim \frac{2i}{\pi} \sinh \xi_1^* \sin \eta \cos \eta_o e^{-(\xi+\xi_o-\xi_1^*)} [1+\tfrac{1}{2}Q_c(\xi)-\tfrac{1}{2}Q_s(\xi_o)+\tfrac{1}{2}Q_s(\xi_1^*)] \tag{305}$$

$$\beta_{2m+1}^{(1)}(C) = \frac{2i(2m+1)}{\pi} \sinh(2m+1)\xi_1^* \sin(2m+1)\eta \cos(2m+1)\eta_o e^{-(2m+1)(\xi+\xi_o-\xi_1^*)} + O(q) \quad m \geq 1 \tag{306}$$

$$\beta_{2m+2}^{(1)}(C) = \frac{2i(2m+2)}{\pi} \sinh(2m+2)\xi_1^* \sin(2m+2)\eta \cos(2m+2)\eta_o e^{-(2m+2)(\xi+\xi_o-\xi_1^*)} + O(q) \quad m \geq 0 \tag{307}$$

$$\alpha_o^{(1)\dagger}(D) = O(q^2 \ln q) \tag{308}$$

$$\alpha_1^{(1)\dagger}(D) \sim \frac{2i}{\pi} \sinh \xi_1^* \sin \eta \cos \eta_o e^{-(\xi+\xi_o-\xi_1^*)} [1-\tfrac{1}{2}Q_c(\xi)+\tfrac{1}{2}Q_s(\xi_o)-\tfrac{1}{2}Q_s(\xi_1^*)] \tag{309}$$

$$\alpha^{(1)\dagger}_{2m}(D) = \frac{2i(2m)}{\pi} \sinh 2m\xi^*_1 \sin 2m\eta \cos 2m\eta_o e^{-2m(\xi+\xi_o-\xi^*_1)} + O(q) \quad m \geq 1 \tag{310}$$

$$\alpha^{(1)\dagger}_{2m+1}(D) = \frac{2i(2m+1)}{\pi} \sinh(2m+1)\xi^*_1 \sin(2m+1)\eta \cos(2m+1)\eta_o e^{-(2m+1)(\xi+\xi_o-\xi^*_1)} + O(q) \quad m \geq 1 \tag{311}$$

$$\beta^{(1)\dagger}_{1}(D) \sim -\frac{2i}{\pi} \cosh \xi^*_1 \cos \eta \sin \eta_o e^{-(\xi+\xi_o-\xi^*_1)} [1-\tfrac{1}{2}Q_s(\xi)+\tfrac{1}{2}Q_c(\xi_o)-\tfrac{1}{2}Q_c(\xi^*_1)] \tag{312}$$

$$\beta^{(1)\dagger}_{2m+1}(D) = -\frac{2i(2m+1)}{\pi} \cosh(2m+1)\xi^*_1 \cos(2m+1)\eta \sin(2m+1)\eta_o e^{-(2m+1)(\xi+\xi_o-\xi^*_1)} + O(q) \quad m \geq 1 \tag{313}$$

$$\beta^{(1)\dagger}_{2m+2}(1) = -\frac{2i(2m+2)}{\pi} \cosh(2m+2)\xi^*_1 \cos(2m+2)\eta \sin(2m+2)\eta_o e^{-(2m+2)(\xi+\xi_o-\xi^*_1)} + O(q) \quad m \geq 0 \tag{314}$$

where

$$Q_c(\xi) = qe^{\xi} \cosh \xi \ln q \tag{315}$$

and

$$Q_s(\xi) = qe^{\xi} \sinh \xi \ln q \tag{316}$$

Combining the small q behavior of the two sums, we have

$$2[\Sigma_{(1)}(C,q) - \Sigma^{\dagger}_{(1)}(D,q)] = a^*_1 q \ln q + O(q) \tag{317}$$

where

$$a^*_1 = \frac{2i}{\pi} e^{-(\xi+\xi_o-\xi^*_1)} [\sinh \xi^*_1 \sin \eta \cos \eta_o (Q_c(\xi) - Q_s(\xi_o) + Q_s(\xi^*_1))$$

$$- \cosh \xi^*_1 \cos \eta \sin \eta_o (Q_s(\xi) - Q_c(\xi_o) + Q_c(\xi^*_1))] \tag{318}$$

It is significant to note that the $O(q^o)$ behavior of one sum exactly cancelled the $O(q^0)$ term of the other sum. This is important because of the q^{-1} factor multiplying the difference between the sums which appear in the integral contained in (194). Performing the same analysis on the sums contained in (196) and (197) we also find a cancellation of the $O(q^0)$ terms and find

$$2[\Sigma_{(3)}(C,q) + \Sigma_{(2)}(D,q)] = a_3^* q \ln q + O(q) \tag{319}$$

where

$$a_3^* = \frac{2i}{\pi} e^{-(\xi+\xi_o-\xi_1^*)}[\sinh \xi_1^* \cos \eta \cos \eta_o (Q_s(\xi) + Q_s(\xi_o) - Q_s(\xi_1^*))$$

$$+ \cosh \xi_1^* \sin \eta \sin \eta_o (Q_c(\xi) + Q_c(\xi_o) - Q_c(\xi_1^*))] \tag{320}$$

and

$$2[\Sigma^{\dagger}_{(1)}(C,q) - \Sigma_{(1)}(D,q)] = a_4^* q \ln q + O(q) \tag{321}$$

where

$$a_4^* = -\frac{2i}{\pi} e^{-(\xi+\xi_o-\xi_1^*)}[\sinh \xi_1^* \cos \eta \sin \eta_o (Q_s(\xi) - Q_c(\xi_o) - Q_s(\xi_1^*))$$

$$- \cosh \xi_1^* \sin \eta \cos \eta_o (Q_c(\xi) - Q_s(\xi_o) - Q_c(\xi_1^*))] \tag{322}$$

The small positive q behavior of the combination of sums appearing in (195) requires further discussion before it is presented. In the determination of (317) through (322) the $O(q^0)$ term was eliminated in a term by term subtraction for summing indices two or larger. The terms corresponding to an index equal to zero were $O(q)$ or smaller while the actual behavior exhibited in (318), (320), and (322) correspond to the index one. The combination of sums appearing in (195) behaves differently in that one term with zero summing index is not negligible; however, all of the other terms behave as those in the other combination of sums just described. We now present the small positive q behavior of the combination of sums in (195). It is

$$2[\Sigma_{(2)}(C,q) + \Sigma_{(3)}(D,q)] = a_2^* q \ln q + \alpha_o^{(2)}(C) + O(q) \tag{323}$$

where

$$a_2^* = -\frac{2i}{\pi} e^{-(\xi+\xi_o-\xi_1^*)}[\sinh \xi_1^* \sin \eta \sin \eta_o (Q_c(\xi) + Q_c(\xi_o) + Q_s(\xi_1^*))$$

$$+ \cosh \xi_1^* \cos \eta \cos \eta_o (Q_s(\xi) + Q_s(\xi_o) + Q_c(\xi_1^*))] \tag{324}$$

$$\alpha_o^{(2)}(C) = C_o^{(e)}(q,\xi_1^*)(p_o(q))^{-2} ce_o(q,\eta) Me_o^{(1)'}(q,\xi) ce_o(q,\eta_o) Me_o^{(1)'}(q,\xi_o) \tag{325}$$

The small q behavior of $\alpha_o^{(2)}$ is found to be

$$\alpha_o^{(2)}(C) \sim \frac{2i}{\pi \ln q} \tag{326}$$

It should be mentioned that the absence of the q^{-1} factor in (198) through (201) can be shown to be sufficient for the q = 0 behavior of the sums included within the integrals to be of no concern.

Equations (317) through (326) complete the description of the small positive q behavior of the sum combinations of interest. To obtain the small negative q behavior of these sums we use the following relations

$$ce_{2n}(\eta,-q) = (-1)^n ce_{2n}(\pi/2 - \eta,q) \tag{327}$$

$$ce_{2n+1}(\eta,-q) = (-1)^n se_{2n+1}(\pi/2 - \eta,q) \tag{328}$$

$$se_{2n+1}(\eta,-q) = (-1)^n ce_{2n+1}(\pi/2 - \eta,q) \tag{329}$$

$$se_{2n+2}(\eta,-q) = (-1)^n se_{2n+2}(\pi/2 - \eta,q) \tag{330}$$

$$Ce_n(\xi,-q) = ce_n(i\xi,-q) \tag{331}$$

$$Se_n(\xi,-q) = -ise_n(i\xi,-q) \tag{332}$$

$$Me_{2n}^{(1)}(\xi,-q) = (-1)^n Me_{2n}^{(1)}(\xi + i\pi/2,q) \tag{333}$$

$$Me_{2n+1}^{(1)}(\xi,-q) = i(-1)^{n+1} Ne_{2n+1}^{(1)}(\xi + i\pi/2,q) \tag{334}$$

$$Ne_{2n+1}^{(1)}(\xi,-q) = i(-1)^{n+1} Me_{2n+1}^{(1)}(\xi + i\pi/2,q) \tag{335}$$

$$Ne_{2n+2}^{(1)}(\xi,-q) = (-1)^{n+1} Ne_{2n+2}^{(1)}(\xi + i\pi/2,q) \tag{336}$$

We can use the analysis presented in this section to give the small positive q behavior of the right hand side of each of the preceding equations. We thus obtain the small negative q behavior of all of the Mathieu functions that appear in the sums of interest. The small negative q behavior of the necessary derivatives can be found by differentiating the appropriate expressions with, as in the positive q case, special attention to the zero order term. Following this procedure we still have the quantities $(p_m(-q))^{-2}$ and $(s_m(-q))^{-2}$ to be defined before the sums can be evaluated. The following relations are readily derived

$$(p_{2n}(-q))^2 = (p_{2n}(q))^2 \tag{337}$$

$$(p_{2n+1}(-q))^2 = -(s_{2n+1}(q))^2 \tag{338}$$

$$(s_{2n+1}(-q))^2 = -(p_{2n+1}(q))^2 \tag{339}$$

$$(s_{2n+2}(-q))^2 = (s_{2n+2}(q))^2 \tag{340}$$

Using (327) through (340) together with the procedure that has been described we can show that the only modification needed in (317) through (326) for small negative q is the replacement of ln q by ln|q|.

VII. Description of $\underline{\underline{D}}_1$ With Singularities Removed

Using the results of the previous section we can add and subtract the same term in the expressions contained in (194) through (197) to obtain

$$K_1 = \frac{i}{c^2}\int_{-q_L}^{q_u} dq\Big\{q^{-1}\sin(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)[\Sigma_{(1)}(C,q) - \Sigma^{\dagger}_{(1)}(D,q)]$$

$$- \tfrac{1}{2}\sin(2q_u^{\frac{1}{2}}(z - z_o)/c)a_1^* \ln|q|\Big\}$$

$$+ \frac{i}{2c^2} a_1^* \sin(2q_u^{\frac{1}{2}}(z - z_o)/c)\Big\{q_u(\ln q_u - 1) + q_L(\ln q_L - 1)\Big\} \qquad (341)$$

$$K_2 = \frac{i}{c^2}\int_{-q_L}^{q_u} dq\Big\{q^{-1}\sin(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)[\Sigma_{(2)}(C,q) + \Sigma_{(3)}(D,q)]$$

$$- \tfrac{1}{2}\sin(2q_u^{\frac{1}{2}}(z - z_o)/c)\Big[a_2^* \ln|q| + \frac{2i}{\pi q \ln|q|}\{U(q + b) - U(q - a)\}\Big]\Big\}$$

$$+ \frac{i}{2c^2}\sin(2q_u^{\frac{1}{2}}(z - z_o)/c\Big[a_2^*\Big\{q_u(\ln q_u - 1) + q_L(\ln q_L - 1)\Big\}$$

$$+ \frac{2i}{\pi}\Big\{\ln|\ln a| - \ln|\ln b|\Big\}\Big] \qquad 0 < a,b < 1 \qquad (342)$$

$$K_3 = \frac{i}{c^2}\int_{-q_L}^{q_u} dq\Big\{q^{-1}\sin(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)[\Sigma_{(3)}(C,q) + \Sigma_{(2)}(D,q)]$$

$$- \tfrac{1}{2}\sin(2\, q_u^{\frac{1}{2}}(z - z_o)/c)a_3^* \ln|q|\Big\}$$

$$+ \frac{i}{2c^2} a_3^* \sin(2q_u^{\frac{1}{2}}(z - z_o)/c)\Big\{q_u(\ln q_u - 1) + q_L(\ln q_L - 1)\Big\} \qquad (343)$$

$$K_4 = \frac{i}{c^2}\int_{-q_L}^{q_u} dq\Big\{q^{-1}\sin(2(q_u - q)^{\frac{1}{2}}(z - z_o)/c)[\Sigma^{\dagger}_{(1)}(C,q) - \Sigma_{(1)}(D,q)]$$

$$- \tfrac{1}{2}\sin(2q_u^{\frac{1}{2}}(z - z_o)/c)a_4^* \ln|q|\Big\}$$

$$+ \frac{i}{2c^2} a_4^* \sin(2q_u^{\frac{1}{2}}(z - z_o)/c)\Big\{q_u(\ln q_u - 1) + q_L(\ln q_L - 1)\Big\} \qquad (344)$$

where a_1^* through a_4^* are defined in (318), (320), (322), and (324). Aside from the straightforward algebra and integration of the subtracted singularity the following more subtle considerations are related to the previous equations.

1. Now that the singularity at $q = 0$ has been studied we can choose the integration path along the real axis in the q plane in (194) through (201) with the exception of (195). The $O((q \ln|q|)^{-1})$ behavior associated with the sum combination appearing in K_2 and given explicitly in (323), (325), and (326) requires further discussion. Because of this behavior the integral describing K_2 given in (195) would diverge if the integration path were along the real axis rather than the contour C_q. The representation of K_2 given in (177) in terms of a contour integral in the h plane would also diverge if the integration path were along the real axis rather than C_h. Working in the more familiar h plane, we can define a single branch cut at both $\pm k_o$, each of which is sufficient to make both log and square root functions analytic. The radius of the semicircles can then approach zero without C_h crossing either cut. This results in K_2 being defined as a principal value integral in both (177) and (195). The half residue contribution which comes from the semicircles approaching zero is zero in this case because $(\ln|q|)^{-1}$ approaches zero as q approaches zero. In order to obtain the analytic representation of the integral of the subtracted term we had to make use of the principal value definition. That is we subtracted, outside of the obvious constant, the integral now defined and evaluated

$$\int_{-b}^{a} \frac{dq}{q \ln|q|} = \ln|\ln a| - \ln|\ln b| \qquad 0 < a,b < 1 \tag{345}$$

The constants a and b are left arbitrary at this stage of the analysis. We introduce these constants to avoid introducing an artifical singularity $(\ln|q|)^{-1}$ at $q = 1$.

Now that the singularity has been removed from K_2, the integrals in (198) through (201) and (341) through (344) have integration paths along the real q axis and there is no need to take the principal value of any of the integrals.

2. The lower limit of integration in (198) through (201) should be changed to $-q_L$ rather than $-\infty$ to be consistent with (341) through (344). Because the lower limit appearing in these equations is $-q_L$ rather than $-\infty$, every integral

corresponding to an integration between $-\infty$ and $-q_L$ has been completely discarded. This was done due to the fact that the angular Mathieu functions contained in these integrals do not have large negative q asymptotic representations that allow analytic evaluations for these parts of the integrals. Even though we do not have a convenient representation of all Mathieu functions, we can still demonstrate the decay of all sums for large negative q so that depending on the criterion chosen, we can find a value of q_L which makes the discarded integral negligible.

3. The sums contained in the preceding equations as well as those contained in K_5 through K_8 contain an infinite number of terms. These sums must be terminated according to some prescribed smallness criterion.

The final representation of $\underline{D}_1$ applicable for either McLachlan's or Blanch's notation is found in (167) through (175), (198) through (201) with the integration path modified as described in this section, and (194) through (197) replaced by (341) through (344). In McLachlan's notation the appropriate sums are defined in (203) through (211), while in Blanch's notation they are defined in (212) through (220).

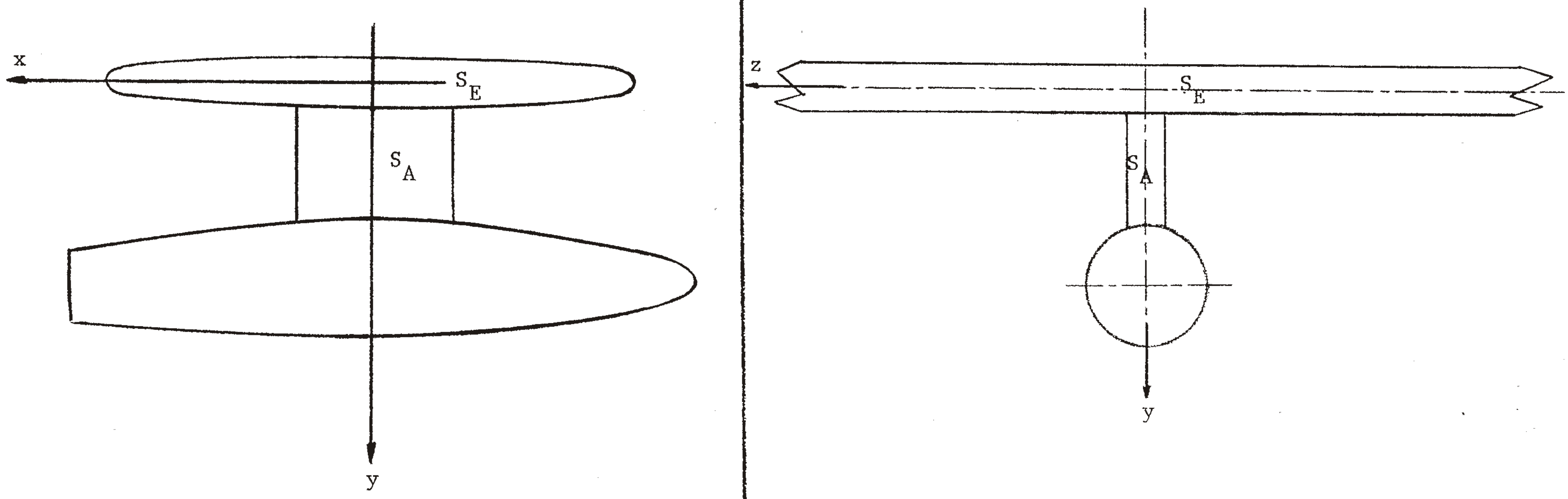

Figure 1. Arbitrary Surface Attached to an Infinite Elliptic Surface.

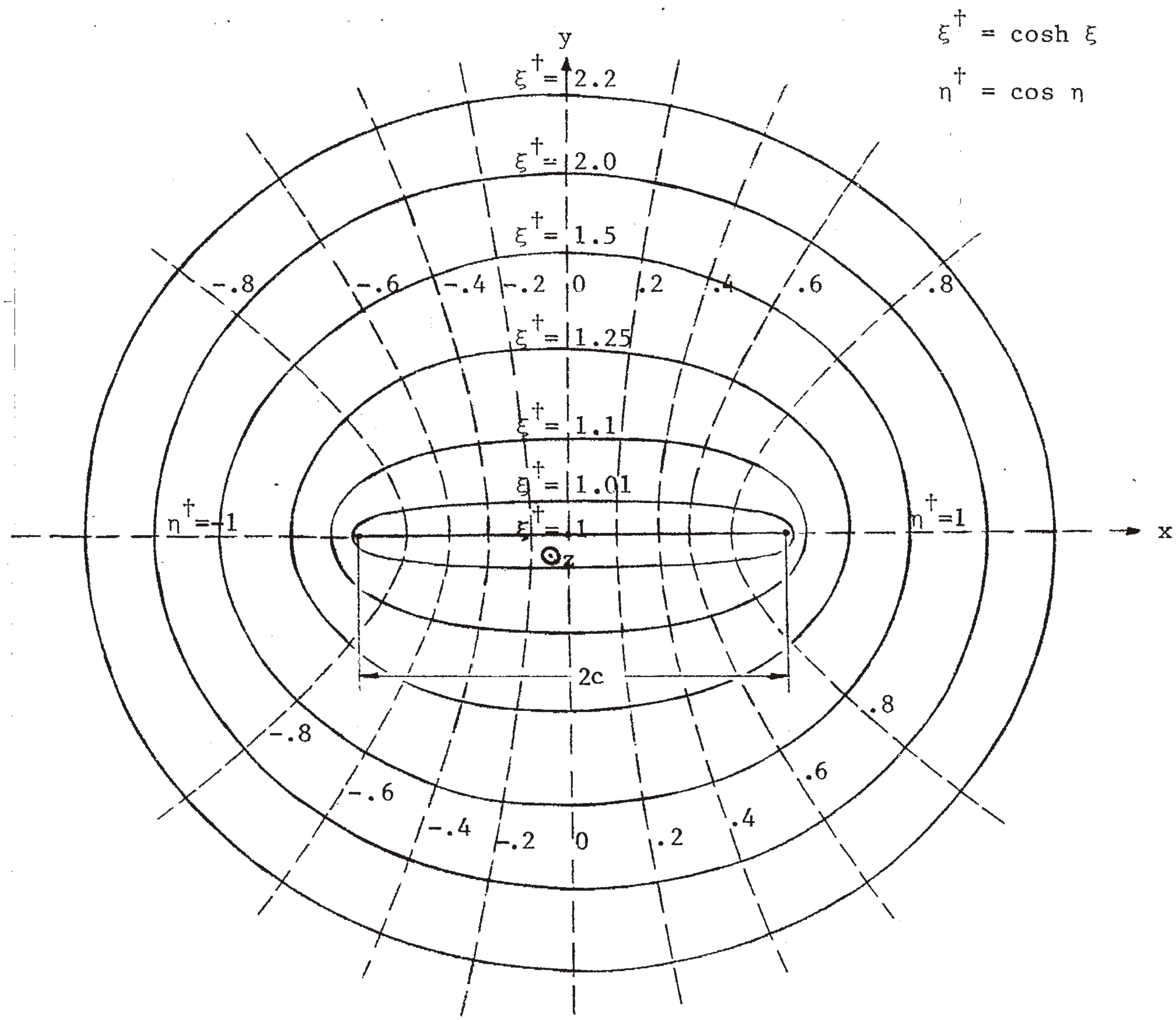

Figure 2. Relevant Cartesian and Elliptic Coordinate Systems.

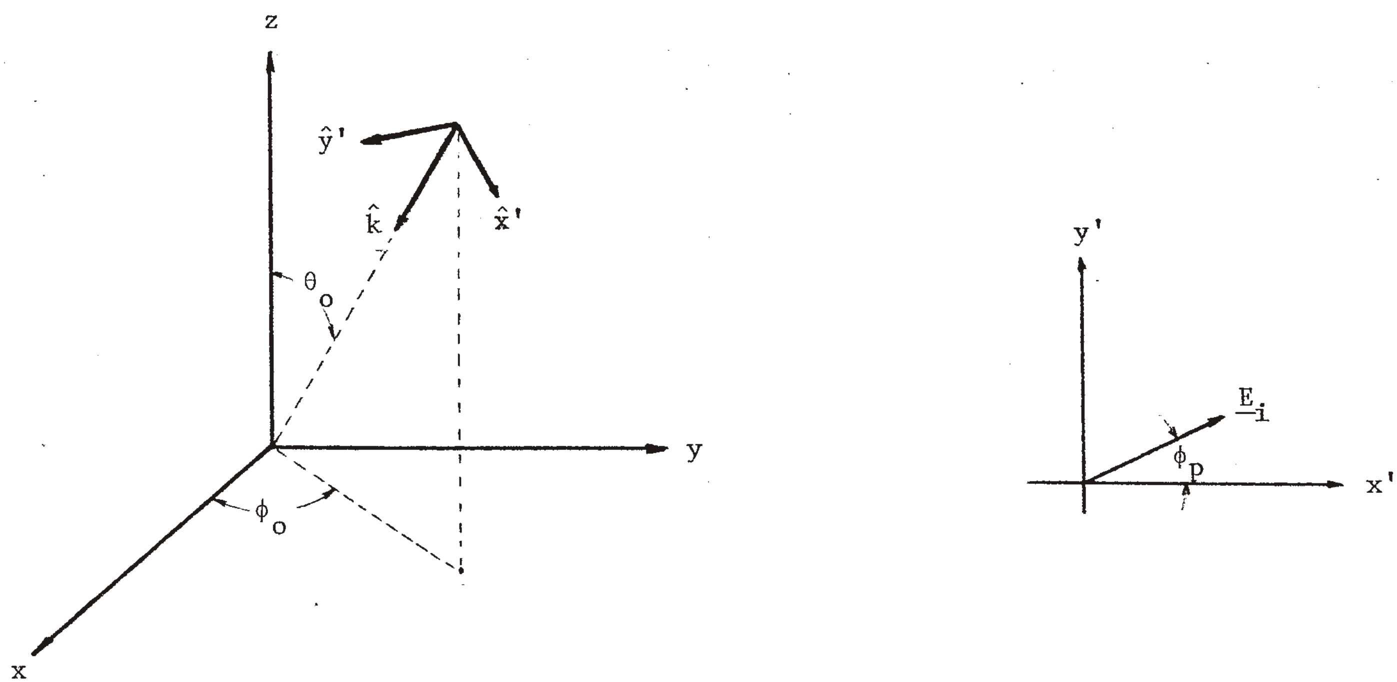

Figure 3. Incident Plane Wave Description.

h plane

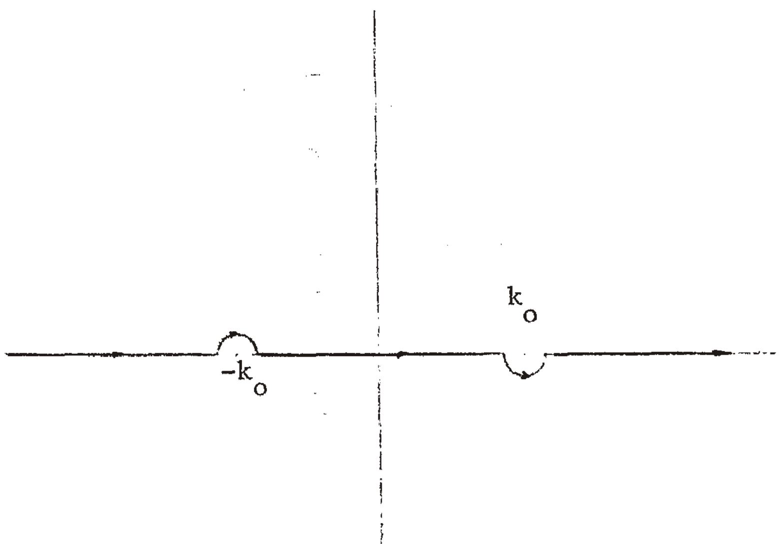

Figure 4. Integration Contour C_h.

q plane

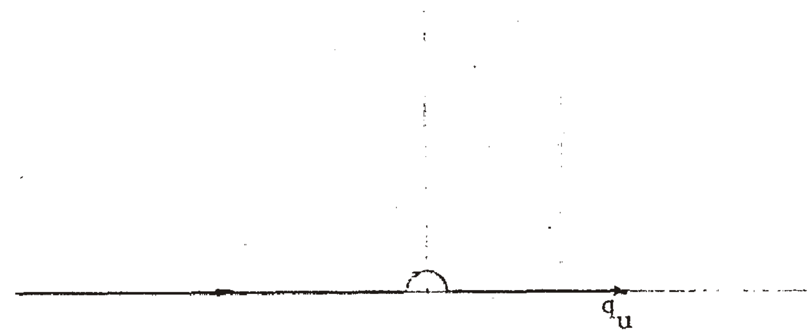

Figure 4. Integration Contour C_q.

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References

1. Chen-To Tai, Dyadic Green's Functions in Electromagnetic Theory, Intext Educational Publishers, 1971.

2. J. J. Bowman, T. B. A. Senior and P. L. E. Uslenghi, Electromagnetic and Acoustic Scattering by Simple Shapes, Wiley Interscience, 1969.

3. J. A. Stratton, Electromagnetic Theory, McGraw-Hill, 1941.

4. N. W. McLachlan, Theory and Application of Mathieu Functions, Dover, 1964.

5. Chen-To Tai, "Eigen-Function Expansion of Dyadic Green's Functions," Mathematics Note 28, July 1973.

6. W. A. Davis, T. Itoh and R. Mittra, "Electromagnetic Pulse Scattering from Hollow Cylinders and Half-Planes," Interaction Note 97, March 1972.

7. M. Abramowitz and I. A. Stegun, Handbook of Mathematic Functions, National Bureau of Standards, 1965.