Interaction Notes

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On EMP Excitations of Cavities with Small Openings

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Abstract

The problem of EMP excitation of cavities through small apertures is discussed. Of particular importance are the study of radiation damping and the resonant frequency shift of an open cavity. Some fallacies in the literature are corrected.

1. Introduction

The main purpose of the theoretical study of cavity excitations through small apertures is to understand the shielding of a hollow conducting box, inside of which are some electronic equipments, against the external electromagnetic radiation. Small apertures are holes from which these equipments are connected to other systems or antennas. It is important that the transient response of the cavity field be known, when the aperture is illuminated by an electromagnetic pulse.

In general, the transient response of a cavity coupled through a small aperture on a conducting body can be studied in two stages in time: (1) The first stage reflects the structure of the scatterer. The induced surface charge and current densities decay according to the natural modes of the scattering body.[1] (2) In the second stage these surface charge and current densities drive the cavity fields. Afterwards, the resonant modes of the cavity fields dominate. They decay according to the conductivity of the cavity wall and the aperture size. To study such a complex problem, it is convenient to simplify it into three separate problems, just as in the study of braided shield cables given in [2]. They are (a) the exterior EMP scattering problem of determining the induced surface charge and current densities of the scatterer with the aperture short circuited, (b) the problem of calculating the equivalent static electric and magnetic polarizabilities α_e and $\overline{\overline{\alpha}}_M$ of the apertures, and (c) the interior problem of determining the resonant modes of the cavity.

In this note we treat problem (c), assuming problems (a) and (b) to have been solved. To study such a problem in the most expedient manner, we make extensive use of references. Among them are Stratton's famous book,[3] Bethe's original paper,[4] and tables of mathematical functions.[5] Throughout the analysis, known results are not derived rigorously, but presented in an easily acceptable form. For instance, the well-known integral equations for the fictitious magnetic charges and currents for a circular aperture are not solved, only the solutions are quoted. In Section 2 we briefly review Bethe's diffraction theory by small circular apertures in a conducting plane. Equivalent electric and magnetic polarizabilities α_e and $\overline{\overline{\alpha}}_M$ for a circular aperture are given. Stress has been given on how a magnetic dipole in the plane of the aperture produces fields satisfying the boundary condition on the conducting plane. Similarly, fields of an electric dipole normal to the plane satisfy the boundary conditions on it. Electric and magnetic fields on and near the aperture are given. However, Boukamp's improved result[6] of the fields near the aperture up to the first order in k is not reproduced here.

In Section 3 we give a self-contained analysis of the cavity field. Special emphasis is placed on the solenoidal part of the fields. First, the coefficients of the normal modes expansion are obtained through a self-consistent scheme. That is, in obtaining the equivalent dipole moments, we assume the aperture field as the difference of the external short-circuited fields and the cavity field. This leads to the shifting of the resonant frequencies from a corresponding closed cavity. However, this does not give rise to the radiation damping of the energy inside the cavity. In order to calculate the damping due to the aperture radiation, we obtain the total power of the radiated field, from which the radiation damping is derived.

The general theory is then applied to a hemispherical cavity with an opening on the plane wall in Section 4. Detailed calculations for the expansion coefficients of the solenoidal fields are carried out. Finally, in Section 5 we study the remote sensing of small cavity backed apertures. Transfer functions are discussed in a block diagram.

Although the problem of an open cavity has been treated in most textbooks on EM theory, some confusions arise due to their mistreatment. In Appendix A, we point out the most common algebraic fallacy in the literature. For completeness, we include a treatment of cavity fields via scalar and vector potentials. A rectangular cavity field is included as an example. Appendix C treats some improvement on the derivation of radiation damping via energy conservation.

2. Bethe's diffraction theory by small apertures in a conducting plane

The diffraction problem of electromagnetic waves by a circular aperture small compared to the wavelength in a conducting plane is solved by H. A. Bethe.[4] Based on the use of fictitious magnetic charges and currents in the apertures, which induce fields satisfying the boundary conditions on the conducting plane, he adjusted the charges and currents to give the correct tangential magnetic, and normal electric fields in the aperture. These fictitious sources are then approximated by a magnetic moment in the plane of the aperture, and an electric dipole perpendicular to it. His theory has been popularized by various textbook authors.[7,8,9] The readers are advised to read Bethe's original article and these fine textbooks. For completeness and convenience, a brief discussion of Bethe's theory is given.

Consider a conducting plane under the illumination of electromagnetic radiation. Then surface charge density, ρ_s, and current density, $\vec{J}_s$, are induced on the conducting plane. If a circular aperture is cut on the conducting plane, the field is perturbed so that the incident normal electric and tangential magnetic fields do not terminate as $\rho_s/2\varepsilon$ and $1/2(\vec{u}_n \times \vec{J}_s)$, but enter through the aperture. For an aperture small compared to the wavelength, the fields in the aperture can be considered as essentially constant. Thus, a quasi-static approach may be adapted. Further relationship between the incident plane wave and short circuited currents and charges are discussed later in the section.

Since magnetic surface charges and currents are the sources, it is expedient to use the magnetic scalar and vector potentials ϕ^* and $\vec{A}^*$ described in Appendix B. Thus, the electric and magnetic fields in the cavity can be expressed in terms of

$$\vec{E} = -\frac{1}{\varepsilon}\vec{\nabla} \times \vec{A}^*$$
$$\vec{H} = -\vec{\nabla}\phi^* - s\vec{A}^* \tag{2.1}$$

Since ρ_s^* and $\vec{J}_s^*$ are the only sources, one can have[4]

$$\vec{A}^* = \frac{1}{4\pi}\int \vec{J}_s^*(\vec{r}\,')\, \frac{e^{-\frac{s}{c}|\vec{r}-\vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|}\, da' \tag{2.2}$$

$$\phi^* = \frac{1}{4\pi}\int \rho_s^*(r') \frac{e^{-\frac{s}{c}|\vec{r}-\vec{r}'|}}{|\vec{r} - \vec{r}'|} da' \tag{2.3}$$

Furthermore, the quasi-static assumption enables us to let s = 0 near the aperture. Therefore, from the second equation of (2.1) we have near the aperture

$$\vec{H} = -\vec{\nabla}\phi^* \tag{2.4}$$

Tangential $\vec{H}$ and normal $\vec{E}$ can be shown to be approximately constant in the plane of the aperture, based on the usual tangential magnetic fields associated with the Babinet decomposition for symmetric and antisymmetric parts with respect to an infinite plane. Since tangential $\vec{H}$ is assumed to be constant, ϕ^* can be expressed as

$$\phi^* = -\int_o^r \vec{H} \cdot d\vec{r} = -\vec{H}_{t_a} \cdot \vec{r}$$

$$\vec{H}_{t_a} = \frac{1}{2} \vec{u}_n \times \vec{J}_s \tag{2.5}$$

Here $\vec{H}_{ta}$ is the tangential incident magnetic field at the aperture based on a Babinet planar approximation.

Note that $\vec{H}_{ta}$ and $\vec{E}_{na}$ are strictly speaking half the resultant fields at the aperture; for an infinite plane (in general only so) they are half the incident fields. Remembering that s = 0, (2.3) and (2.5) give

$$-\vec{H}_{t_a} \cdot \vec{r} = \frac{1}{4\pi\mu}\int \rho_s^*(r') \frac{da'}{|\vec{r} - \vec{r}'|} \tag{2.6}$$

On the other hand, the magnetic vector potential is given by

$$\vec{A}^*(\vec{r}) = \frac{1}{4\pi}\int \vec{J}_s^*(\vec{r}') \frac{da}{|\vec{r} - \vec{r}'|} \tag{2.7}$$

Here, the constancy of E_{normal} gives

$$\vec{A}^*(r) = \frac{1}{2}\vec{E}_{n_a} \times \vec{r}\ , \qquad E_{n_a} = \frac{\rho_s}{2\varepsilon} \tag{2.8}$$

E_{n_a} is the Babinet normal incident electric field at the aperture. Bethe solved the integral equations (2.6) and (2.7) with (2.8) to give

$$\rho_s^* = -\frac{4\mu}{\pi(a^2 - r'^2)^{1/2}}\vec{r}' \cdot \vec{H}_{t_a} \tag{2.9}$$

$$\vec{J}_s^* = \frac{2}{\pi(a^2 - r'^2)^{1/2}}\vec{r}' \times \vec{E}_{n_a} \tag{2.10}$$

Here a and r' are the radius and the distance from the center of the circular aperture.

Alternatively, it is a fairly well known result from electrostatics[3] that a constant inside field is produced by a uniform distribution of dipoles in an ellipsoid, the dipoles having the direction as the field. A uniform dipole ellipsoid of dimensions a, a, h with h → 0 gives rise to the fields which we are solving. Assuming the aperture is small, we can approximate the field due to a uniformly distributed dipole ellipsoid by a point dipole at the center of the aperture.

Since the magnetic moment and electric dipole are, by definition, given by

$$\vec{m} = \frac{1}{\mu}\int \vec{r}\rho_s^* da \tag{2.11}$$

$$\vec{p} = \frac{1}{2}\int \vec{r} \times \vec{J}_s^* da \tag{2.12}$$

It is easy to substitute (2.9) into (2.11), and (2.10) into (2.12) to yield

$$\vec{m} = -\frac{16}{3}a^3\vec{H}_{t_a} = -\frac{8}{3}a^3\vec{u}_n \times \vec{J}_s \tag{2.13}$$

$$\vec{p} = \frac{8\varepsilon}{3}a^3 E_{n_a}\vec{u}_n = \frac{4}{3}a^3 \rho_s \vec{u}_n \tag{2.14}$$

Therefore, we can express (2.13) and (2.14) as

$$\vec{m} = \bar{\bar{\alpha}}_m \cdot \vec{H}_{t_a} \tag{2.15}$$

$$\vec{p} = \alpha_e \vec{E}_{n_a} \tag{2.16}$$

Here $\bar{\bar{\alpha}}_m$ and α_e are magnetic and electric polarizabilities. For a circular aperture

$$\bar{\bar{\alpha}}_m = -\frac{16}{3} a^3 \vec{u}_t \vec{u}_t \tag{2.17}$$

$$\alpha_e = \frac{8\varepsilon}{3} a^3 \tag{2.18}$$

with $\vec{u}_t$ as the unit vector along $\vec{H}_t$.

In order to see the fields due to the magnetic moment $\vec{m}$ and electric dipole $\vec{p}$ satisfying the boundary condition, they are given as[3]

$$\vec{H}_b = \frac{1}{4\pi} \vec{\nabla}[\vec{m} \cdot \vec{\nabla}(\frac{1}{r})] \tag{2.19}$$

$$\vec{E}_a = \frac{1}{4\pi\varepsilon} \vec{\nabla}[\vec{p} \cdot \vec{\nabla}(\frac{1}{r})] \tag{2.20}$$

The directions of the fields of (2.19) and (2.20) are shown in Figs. (2.1) and (2.2). We can see in Fig. (2.1) that the magnetic field is constant on the circular aperture. The field outside the aperture is the same as that of a magnetic moment $\vec{m}$. Since $\vec{H}_b$ lies in the conducting plane, it is easy to see $\vec{u}_n \cdot \vec{H}_b = 0$. In Fig. (2.2) the electric field of an electric dipole in a meridian plane through an axis of a dipole $\vec{p}$ is shown. Furthermore, we approximate the fictitious magnetic current, $\vec{J}_s^*$, on the aperture by an electric dipole, $\vec{p}$, perpendicular to the aperture. Therefore, Fig. (2.2) can be considered as a static electric field due to an aperture in a plane with its normal along $\vec{p}$.

It is noteworthy that eqns. (2.19) and (2.20), and therefore Figs. (2.1) and (2.2), do not give accurate results for fields near the aperture. Perhaps the most important discrepancy is that the approximate field does not exhibit a singularity at the edge. This can be seen from a uniform distribution

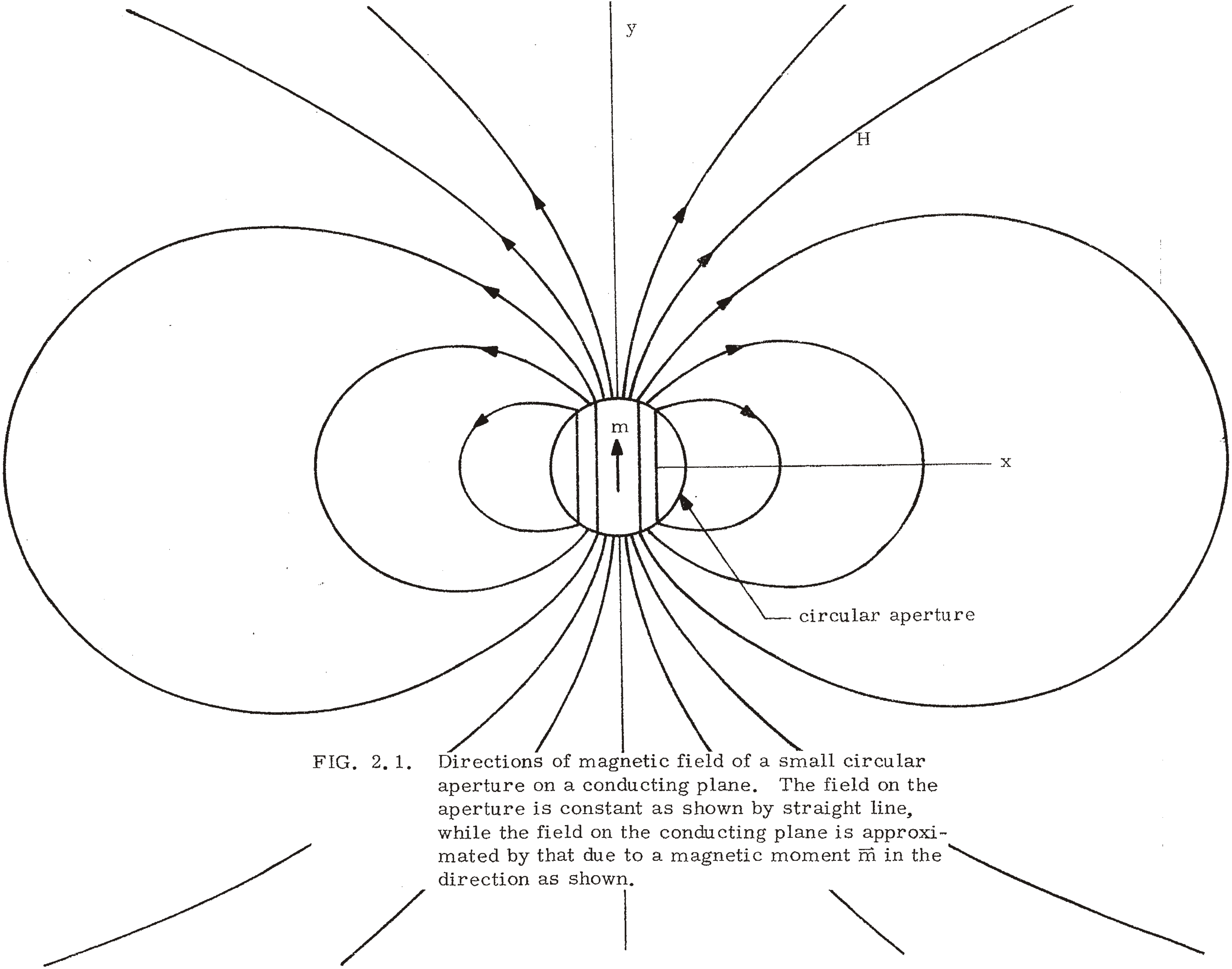


FIG. 2.1. Directions of magnetic field of a small circular aperture on a conducting plane. The field on the aperture is constant as shown by straight line, while the field on the conducting plane is approximated by that due to a magnetic moment $\vec{m}$ in the direction as shown.

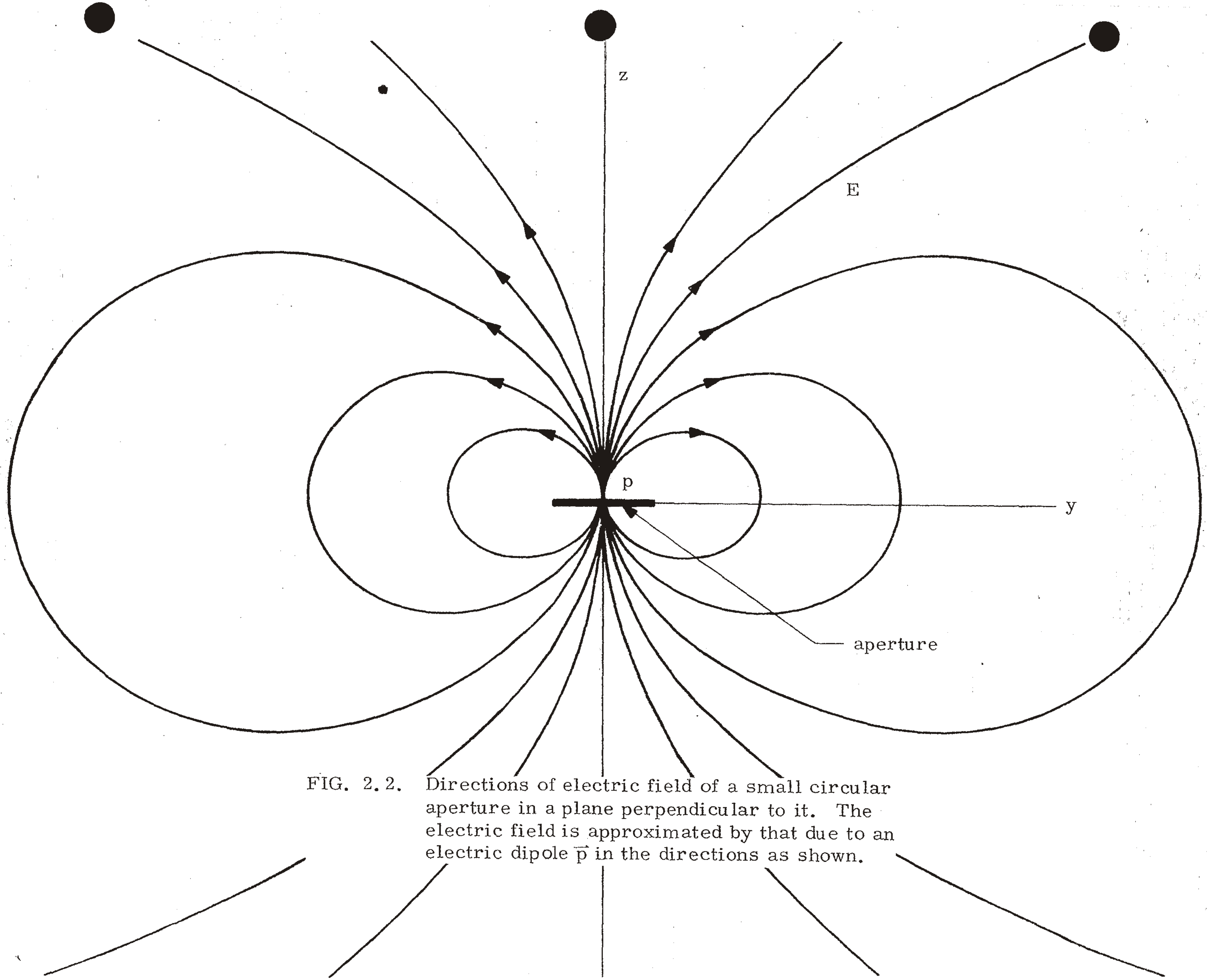


FIG. 2.2. Directions of electric field of a small circular aperture in a plane perpendicular to it. The electric field is approximated by that due to an electric dipole $\vec{p}$ in the directions as shown.

of dipoles inside an ellipsoidal model discussed earlier. Using a well known solution in electrostatics[3] mentioned earlier, we can obtain ϕ^* and $H = \nabla\phi^*$ from its electrostatic counterpart. They are given in the ellipsoidal coordinates as

$$\phi^*(\xi) = \frac{q^*}{8\pi} \int_{\xi}^{\infty} \frac{d\xi}{R_\xi} \tag{2.21}$$

$$|\vec{H}| = \frac{\partial\phi^*}{\partial x_n} = \frac{1}{h}\frac{\partial\phi^*}{\partial x_n} = \frac{q^*}{4\pi(a^2+\xi)(h^2+\xi)^{1/2}} \frac{1}{\sqrt{\frac{r^2}{(a^2+\xi)^2} + \frac{z^2}{(h^2+\xi)^2}}} \tag{2.22}$$

To find the tangential magnetic field near the edge, we let $z = 0$, and make use of the coordinate surface

$$\frac{r^2}{a^2+\xi} + \frac{z^2}{\xi} = 1 \tag{2.23}$$

Thus, for small ξ we have

$$H_{tan} \sim \frac{q^*}{4\pi\, a\sqrt{\xi}} = \frac{q^*}{4\pi\, a\sqrt{r^2 - a^2}} \tag{2.24}$$

Before expressing q^* in terms of short circuited $\vec{J}_S$, we also obtain normal magnetic field on the aperture. They are obtained by letting $\xi = 0$, and $h \to 0$ as follows:

$$|\vec{H}| = \frac{q^*}{4\pi\, a^2 h} \frac{1}{\sqrt{\frac{r^2}{a^4} + \frac{z^2}{h^4}}} \tag{2.25}$$

and

$$\frac{z^2}{h^2} = 1 - \frac{r^2}{a^2} \tag{2.26}$$

The substitution of (2.26) into (2.25) and limit of $h \to 0$ gives

$$|\vec{H}| = \frac{q^*}{4\pi a \sqrt{a^2 - r^2}} \tag{2.27}$$

However, the limiting ellipsoid has two sides, which gives

$$H_n = \frac{q^*}{2\pi a \sqrt{a^2 - r^2}} \tag{2.28}$$

The normal magnetic field can also be derived from the fictitious magnetic charge on the aperture (2.9) and (B.16) as follows:

$$H_n = \frac{\rho_s^*}{\mu} = \frac{4\vec{r} \cdot \vec{H}_{t_a}}{\pi(a^2 - r^2)^{1/2}} \tag{2.29}$$

Comparison of (2.28) and (2.29) gives

$$q^* = 8a\vec{r} \cdot \vec{H}_{t_a} \tag{2.30}$$

We thus determine the tangential magnetic field near the edge as

$$H_{tan} \sim \frac{2\vec{r} \cdot \vec{H}_{t_a}}{\pi(r^2 - a^2)^{1/2}} \tag{2.31}$$

Analogously, the tangential electric field on the aperture is

$$\vec{E}_{tan} = \vec{u}_n \times \vec{J}_s^* = \frac{2\,\vec{r}\,E_{n_a}}{\pi(a^2 - r^2)^{1/2}} \tag{2.32}$$

Near the edge, the normal electric field can be approximated by

$$E_n \sim \frac{|\vec{r}|E_{n_a}}{\pi(r^2 - a^2)^{1/2}} \tag{2.33}$$

To summarize, the fields on the aperture are

$$\begin{cases} E_n = E_{n_a} = \dfrac{\rho_s}{2e} \\ \vec{E}_{tan} = \dfrac{\vec{r}\rho_s}{\varepsilon\pi(a^2 - r^2)^{1/2}} \end{cases} \tag{2.34}$$

$$\begin{cases} \vec{H}_{tan} = \vec{H}_{t_a} = \frac{1}{2}\,\vec{u}_n \times \vec{J}_s \\ H_n = \dfrac{2\vec{r} \cdot \vec{u}_n \times \vec{J}_s}{\pi(a^2 - r^2)^{1/2}} \end{cases} \tag{2.35}$$

Here ρ_s, $\vec{J}_s$ are the short circuited surface charge and current.

Near the edge, we have

$$\begin{cases} E_{tan} = 0 \\ E_n \sim \dfrac{|\vec{r}|\rho_s}{2\varepsilon\pi(r^2 - a^2)^{1/2}} \end{cases} \tag{2.36}$$

and

$$\begin{cases} H_n = 0 \\ H_{tan} \sim \dfrac{\vec{r} \cdot \vec{u}_n \times \vec{J}_s}{\pi(r^2 - a^2)^{1/2}} \end{cases} \tag{2.37}$$

Let us also relate the short circuited currents and charges to the incident plane wave. Consider a plane wave incidence on a plane perfectly conducting sheet. The current $\vec{J}_s$ induced on the sheet is given by

$$\vec{J}_s = -\vec{u}_n \times (\vec{H}^i + \vec{H}^r) = -2\vec{u}_n \times \vec{H}^i \tag{2.38}$$

Here $\vec{u}_n$ is the normal to the sheet pointed away from the field as shown in Fig. 2.3. Let us perturb the field by a small opening on the plane conducting sheet as shown in Fig. 2.4. Since there is no current on the aperture, we add a compensation current $\vec{J}_s'$ such that

$$\vec{J}_s + \vec{J}_s' = 0$$

or

$$\vec{J}_s' = -\vec{J}_s \tag{2.39}$$

The compensation current $\vec{J}_s'$ induces perturbed magnetic fields $\vec{H}_{ta}$ and $\vec{H}_{ta}'$. They satisfy

$$\vec{u}_n \times (\vec{H}_{t_a} - \vec{H}_{t_a}') = \vec{J}_s' \tag{2.40}$$

and are shown in Fig. 2.5. Because of the geometrical symmetry, $\vec{H}_{ta} = -\vec{H}_{ta}'$, which leads to

$$\vec{H}_{t_a} = \frac{1}{2}\, \vec{u}_n \times \vec{J}_s = \vec{H}_t^i \tag{2.41}$$

Therefore, for a plane wave incidence $1/2(\vec{u}_n \times \vec{J}_s)$ has to be replaced by $\vec{H}_t^i$. Similarly,

$$E_{n_a} = \frac{\rho_s}{2\varepsilon} = E_n^i \tag{2.42}$$

This conclusion can be arrived at by the decomposition of fields into symmetrical and antisymmetrical components as given by Neugenbauer.[10] However, in a typical measurement one closes the aperture and measures the magnetic field near the surface. The measured value is

$$(\vec{H}_{tot})_t = \vec{H}_t^i + \vec{H}_t^r = 2\vec{H}^i \tag{2.43}$$

$$(E_{tot})_n = \vec{E}_n^i + \vec{E}_n^r = 2E_n^i \tag{2.44}$$

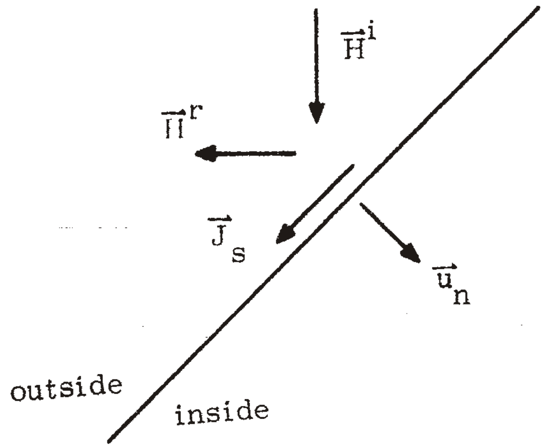


Figure 2.3

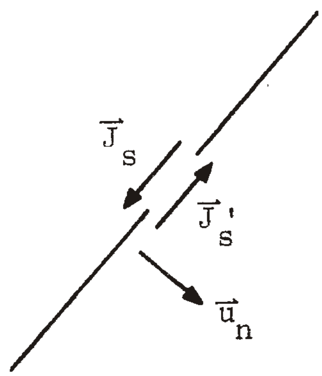


Figure 2.4

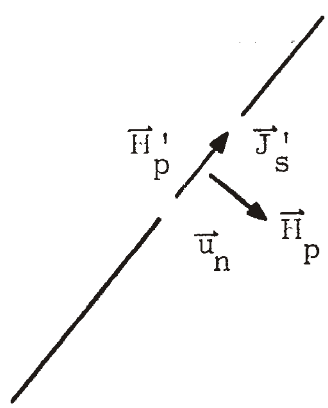


Figure 2.5

It is desirable to express the dipole moments, polarizabilities, and fields on the other side of the aperture in terms of these measured quantities. Thus, the dipole moments are

$$\vec{m} = -\frac{8}{3} a^3 (\vec{H}_{tot})_t \tag{2.45}$$

$$\vec{p} = \frac{4}{3} a^3 (E_{tot})_n \vec{u}_n \tag{2.46}$$

which give modified polarizabilities $\overline{\overline{\alpha}}'_m$ and α'_e for circular apertures as follows:

$$\overline{\overline{\alpha}}'_m = -\frac{8}{3} a^3 \tag{2.47}$$

$$\alpha'_e = \frac{4}{3} a^3 \tag{2.48}$$

The static fields on the other side of the aperture are then given by (2.19) and (2.20).

3. Cavity fields and their matching to an aperture

In this section we concentrate our attention on the solenoidal cavity fields. We give the expansion coefficients of the eigenfunction expansions. In particular, the shifting of the resonant frequencies of the cavity due to the aperture opening is studied in detail. Furthermore, the aperture radiation effect on the cavity is critically studied. It is found that for small apertures the radiation damping is negligible compared to the conducting damping. In brief, we first derive the driving term from the external short circuited surface currents and charges. We then obtain the shifting of the resonant frequencies due to the aperture opening. Finally, we study the aperture radiation effect through the far field radiated field. The formulas obtained for shifting in resonant frequencies and radiation damping are useful for calculating the first order effect of a slightly larger aperture.

In addition to the solenoidal fields, the equivalent electric and magnetic dipoles of the aperture excite electrostatic and magnetostatic fields inside the cavity. They can be calculated by these dipoles and their images in the case of a rectangular cavity; and by solving a boundary value problem in the case of a general cavity.

For a small aperture, the static fields fall off sufficiently rapidly that (2.19) and (2.20) give good approximations. Because of the minor algebraic error which appears in most textbooks in treating the cavity field expansions, we give some discussions[11] in Appendix A on common fallacies. In Appendix B, a derivation of normal modes by the use of scalar and vector potentials is presented.

As shown in Appendix A the solenoidal part of cavity fields can be expanded as follows:

$$\vec{E} = \sum_m a_m \vec{E}_m \tag{3.1}$$

Here $\vec{E}_m$ is a normal mode which satisfies

$$\vec{u}_n \times \vec{E}_m = 0$$

on the boundary. Also,

$$a_m = \int \vec{E} \cdot \vec{E}_m dV = -\frac{c^2}{s_m^2} \int \vec{E} \cdot \vec{\nabla} \times \vec{\nabla} \times \vec{E}_m dV \tag{3.2}$$

since

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}_m = -\frac{s_m^2}{c^2} \vec{E}_m \tag{3.3}$$

$$\int \vec{E}_m \cdot \vec{E}_n dV = \delta_{mn} \tag{3.4}$$

Employing a curl version of integration by parts, which is

$$\int (\vec{\nabla} \times \vec{a}) \cdot \vec{b} dV = \int (\nabla \times \vec{b}) \cdot \vec{a} dV + \int (\vec{a} \times \vec{b}) \cdot u_n da \tag{3.5}$$

one can proceed as

$$-\frac{s_m^2}{c^2} a_m = \int \vec{E} \cdot \vec{\nabla} \times \vec{\nabla} \times \vec{E}_m dV = \int \vec{\nabla} \times \vec{E} \cdot \vec{\nabla} \times \vec{E}_m dV + \int_a \vec{E} \times \vec{\nabla} \times \vec{E}_m \cdot \vec{u}_n da$$

$$= \int \vec{\nabla} \times \vec{\nabla} \times \vec{E} \cdot \vec{E}_m dV + \int_a (\vec{E} \times \vec{\nabla} \times \vec{E}_m + \vec{E}_m \times \vec{\nabla} \times \vec{E}) \cdot \vec{u}_n da \tag{3.6}$$

Furthermore,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{s^2}{c^2} \vec{E} \tag{3.7}$$

$$\int_a (\vec{E}_m \times \vec{\nabla} \times \vec{E}) \cdot \vec{u}_n da = 0 \tag{3.8}$$

The second formula follows from the fact that

$$\vec{E}_m \times \vec{u}_n = 0$$

Therefore, one arrives at a well known formula for the expansion coefficients:

$$a_m = \frac{c^2}{(s^2 - s_m^2)} \int_a (\vec{E} \times \vec{\nabla} \times \vec{E}_m) \cdot \vec{u}_n d\vec{a} \qquad (3.9)$$

This is the cornerstone for the rest of the analysis. First, we apply (3.9) to calculate the coupling of fields outside the cavity to inside. This is carried out with the electric and magnetic dipole approximation of a small aperture discussed in the previous section.

Since $\vec{u}_n \times \vec{E} = \vec{J}_s^*$, one can expand $\vec{\nabla} \times \vec{E}_m$ in terms of Taylor series as follows:

$$\int_{a'} (\vec{E} \times \vec{\nabla} \times \vec{E}_m) \cdot \vec{u}_n da = \int_{a'} \vec{u}_n \times \vec{E} \cdot \vec{\nabla} \times \vec{E}_m da$$

$$= -\int_{a'} \vec{J}_s^* \cdot \vec{\nabla} \times \vec{E}_m da = -\int [\vec{J}_s^* \cdot \vec{\nabla} \times \vec{E}_{m_a}$$

$$+ \frac{1}{2} \vec{r}' \times \vec{J}_s^* \cdot \vec{\nabla} \times \vec{\nabla} \times \vec{E}_{m_a} + \cdots] da$$

$$= -s\mu\vec{m} \cdot \vec{\nabla} \times \vec{E}_{m_a} - \frac{1}{\varepsilon} \vec{p} \cdot \vec{\nabla} \times \vec{\nabla} \times \vec{E}_{m_a}$$

$$+ \cdots \qquad (3.10)$$

Here we have assumed $\vec{\nabla} \times \vec{E}_m$ and $\vec{\nabla} \times \vec{\nabla} \times \vec{E}_m$ as constant on the aperture. We can conveniently use the values at the center of the aperture, which have been denoted as $\vec{\nabla} \times \vec{E}_{ma}$ and $\vec{\nabla} \times \vec{\nabla} \times \vec{E}_{ma}$. We have also made use of

$$\int \vec{J}_s^* da = \int \vec{r} \nabla \cdot \vec{J}_s^* da = -s \int \vec{r} \rho_s^* da$$

$$= -s\mu m = -s\mu \bar{\bar{\alpha}}'_m \cdot (\vec{H}_{tot})_t \qquad (3.11)$$

$$\frac{1}{2} \int \vec{r}' \times \vec{J}_s^* da = -\frac{\vec{p}}{\varepsilon} = \alpha'_e (\vec{E}_{tot})_n \qquad (3.12)$$

$(\vec{H}_{tot})_t$ and $(\vec{E}_{tot})_n$ are the short-circuited tangential magnetic and normal electric field on the aperture. They are previously given in (2.43) and (2.44) in terms of short-circuited surface current and charge as

$$(\vec{H}_{tot})_t = \vec{u}_n \times \vec{J}_s \tag{3.13}$$

$$(\vec{E}_{tot})_n = \frac{\rho_s}{\varepsilon} \tag{3.14}$$

Here we temporarily ignore the reflection of the cavity, which will be taken care of in (3.16). In (3.11) and (3.12) $\bar{\bar{\alpha}}_m'$ and α_e' are the magnetic and electric polarizabilities defined in (2.47) and (2.48). One can further simplify (3.10) by the Maxwell's equations and (3.3) to give

$$\int_{a'} \vec{E} \times \vec{\nabla} \times \vec{E}_m \cdot \vec{u}_n da = s s_m \mu^2 \vec{u}_n \times \vec{J}_s \cdot \bar{\bar{\alpha}}_m' \cdot \vec{H}_{m_a}$$

$$- s_m^2 \mu\varepsilon\alpha_e' \frac{\rho_s}{\varepsilon} \vec{u}_n \cdot \vec{E}_{m_a} + \cdots \tag{3.15}$$

Next, we attempt to use (3.9) to study the damping effect of the aperture radiation. We adapt a procedure similar to that used for studying the effect of a conducting wall.[7] Clearly the tangential electric field of the cavity leaks out of it through the aperture just like the coupling of external field into the cavity. Here the short circuited tangential electric field of the cavity is given by

$$\vec{u}_n \times \vec{E}_{tot} = -\vec{u}_n \times \left(\sum_n a_n \vec{E}_n\right) \tag{3.16}$$

Note that we have a minus sign on the right hand side of (3.16), because the reflected wave has $-\vec{u}_n$ direction. On using (3.16) in (3.10), one arrives at

$$\int_{a'} \vec{u}_n \times \vec{E}_{tot} \cdot \vec{\nabla} \times \vec{E}_m da = s\mu\vec{m}' \cdot \vec{\nabla} \times \vec{E}_{m_a}$$

$$+ \frac{1}{\varepsilon} \vec{p}' \cdot \vec{\nabla} \times \vec{\nabla} \times \vec{E}_{m_a} + \cdots \tag{3.17}$$

Here

$$\vec{m}' = \bar{\bar{\alpha}}_m' \cdot \sum_n a_n (\vec{H}_n)_a \tag{3.18}$$

$$-\frac{1}{\varepsilon_o}\vec{p}' = \alpha_e' \sum_n a_n (\vec{E}_n)_{n_a} \tag{3.19}$$

The simplest way to derive the above two equations is to substitute

$$\sum_n a_n \vec{E}_n$$

and

$$\sum a_n \vec{H}_n$$

for $\vec{E}$ and $\vec{H}$ in (3.11) and (3.12).

As in deriving (3.15) one can simplify (3.17) further to give

$$\int_{a'} \vec{u}_n \times \vec{E} \cdot \vec{\nabla} \times \vec{E}_m da = -ss_m \mu^2 \sum_n a_n \vec{H}_{n_a} \cdot \bar{\bar{\alpha}}_m' \cdot \vec{H}_{m_a}$$

$$+ \mu\varepsilon s_m^2 \alpha_e' \sum_n a_n (E_n)_{n_a} (E_m)_{n_a} + \cdots \tag{3.20}$$

However, we still have not obtained the damping. To obtain the damping, we can study the higher order terms in the quasi-static expansion. Here we rely on the physical nature of the radiation. We calculate the total power flow away from the aperture and infer from the law of energy conservation that the same power also flows out of the aperture.

Let us proceed with the calculation of the radiated field. To begin with, we summarize the fictitious magnetic current sources as follows:

$$\rho_s^* = -\frac{4\mu}{\pi(a^2 - r^2)^{1/2}} \vec{r} \cdot \vec{H}_{t_a} \tag{2.9}$$

$$\vec{J}_s^* = \frac{2}{\pi(a^2 - r^2)^{1/2}} \vec{r} \times \vec{E}_{n_a} \tag{2.10}$$

As before, these fictitious sources can be approximated by the magnetic and electric dipole moments $\vec{m}$ and $\vec{p}$ located at the center of the aperture.

The radiated fields due to these dipoles can be calculated easily with the aid of scalar and vector potentials. They are given by[3]

$$\vec{A} = \frac{s\mu}{4\pi} \vec{p} \frac{e^{-sR/c}}{R} \tag{3.21}$$

$$\phi = \frac{-1}{s\mu\varepsilon} \vec{\nabla} \cdot \vec{A} \tag{3.22}$$

for the electric dipole, and

$$\vec{A} = \frac{s\mu}{4\pi c} (\vec{m} \times \vec{u}_R) \frac{e^{-sR/c}}{R} \tag{3.23}$$

$$\phi = \frac{-1}{s\mu\varepsilon} \vec{\nabla} \cdot \vec{A} \tag{3.24}$$

for the magnetic dipole. As a result, the radiated fields for the electric dipole are

$$\vec{E}_{el} = \frac{s^2}{4\pi\varepsilon c^2} [\vec{u}_R \times (\vec{u}_R \times \vec{p})] \frac{e^{-sR/c}}{R} \tag{3.25}$$

$$\vec{H}_{el} = \frac{s^2}{4\pi c} (\vec{u}_R \times \vec{p}) \frac{e^{-sR/c}}{R} \tag{3.26}$$

Here $\vec{u}_R$ is the unit vector along $\vec{R}$. The radiated field for the magnetic dipole is given by

$$\vec{H}_{ma} = \frac{s^2}{4\pi c^2}[\vec{u}_R \times (\vec{m} \times \vec{u}_R)]\frac{e^{-sR/c}}{R} \tag{3.27}$$

$$\vec{E}_{ma} = \frac{-s^2\mu}{4\pi c}(\vec{u}_R \times \vec{m})\frac{e^{-sR/c}}{R} \tag{3.28}$$

Observe symmetry in the radiated fields of electric and magnetic dipoles. Ignoring ε and μ in (3.25) ~ (3.28), one can replace $\vec{m}$ by $\vec{p}$ to obtain $\vec{E}_{ma}$ and $\vec{H}_{ma}$ from $\vec{H}_{el}$ and $\vec{E}_{el}$. Equivalently, one can derive these radiated fields by the use of the magnetic scalar and vector potentials:

$$\vec{A}^* = -\frac{s}{4\pi c^2}\vec{m}\frac{e^{-sR/c}}{R} \tag{3.29}$$

$$\phi^* = \frac{-c^2}{s}\vec{\nabla}\cdot\vec{A}^* \tag{3.30}$$

and

$$\vec{A}^* = \frac{s\varepsilon}{4\pi c}(\vec{p} \times \vec{u}_R)\frac{e^{-sR/c}}{R} \tag{3.31}$$

$$\phi^* = \frac{-c^2}{s}\vec{\nabla}\cdot\vec{A}^* \tag{3.32}$$

Let us calculate the Poynting vector of the radiation

$$\vec{S} = \vec{E} \times \vec{H}^\dagger = \frac{s^4}{(4\pi)^2\varepsilon c^3}\vec{u}_R[\frac{1}{c}(\vec{u}_R \times \vec{m})$$

$$- \vec{u}_R \times (\vec{u}_R \times \vec{p})]^2\frac{1}{R^2} \tag{3.33}$$

Here $\dagger$ is the complex conjugate. Note that (3.33) is not analytic. The way to circumvent such a difficulty is discussed in Appendix C. Also, we assume the aperture on a plane conducting sheet. However, if the curvature of the exterior conducting surface, on which the aperture is located, is small compared to that of the interior resonating cavity, the approximation is very accurate. Therefore, the total power radiated is

$$S_{tot} = \int \vec{S} \cdot u_n da = \frac{s^4}{(4\pi)^2 \varepsilon c^3} \int_o^{\pi/2} \sin\theta \, d\theta \int_o^{2\pi} d\alpha R^2 |S| \qquad (3.34)$$

where

$$r^2 |S| = (m/c)^2 \cos^2\theta \cos^2\alpha$$

$$+ [\sin\theta \, p - (m/c) \sin\alpha]^2 \qquad (3.35)$$

On carrying out the integration in (3.34), one arrives at

$$S_{tot} = \frac{s^4}{12\pi\varepsilon c^3} [(m/c)^2 + p^2] \qquad (3.36)$$

From the law of energy conservation, (3.36) gives the radiation loss at the aperture. One substitutes the magnetic and electric moments from (3.18) and (3.19) to give

$$\mathrm{Re}\left(\int_{a'} \vec{E} \times \vec{H} \cdot \vec{u}_n da\right) = S_{tot}$$

$$= \frac{s^4}{12\pi\varepsilon c^3} \left[\frac{(\bar{\bar{\alpha}}'_m \cdot \vec{H}_{t_a})^2}{c^2} + (\alpha'_e \vec{E}_{n_a})^2 \right] \qquad (3.37)$$

Specializing to the mth mode of the cavity field, one has

$$\mathrm{Re} \int_{a'} \vec{E}_m \times \vec{H}_m \cdot \vec{u}_n da = \frac{s^4}{12\pi\varepsilon c^3} \left[\left\{ \bar{\bar{\alpha}}'_m \cdot (\vec{H}_m)_{t_a} \right\}^2 / c^2 \right.$$

$$\left. + \left\{ \alpha'_e (E_m)_{n_a} \right\}^2 \right] \qquad (3.38)$$

Finally, in order to use (3.20) and (3.38) to derive a_m including damping, one begins with (3.9), which is equivalent to

$$(s^2 - s_m^2)a_m = c^2 \int_a \vec{E} \times (\vec{\nabla} \times \vec{E}_m) \cdot \vec{u}_n da$$

$$= ss_m \mu^2 \vec{u}_n \times \vec{J}_s \cdot \bar{\bar{\alpha}}'_m \cdot \vec{H}_{m_a}$$

$$- \frac{s_m^2}{c^2} \alpha'_e \frac{\rho_s}{\varepsilon} \vec{u}_n \cdot \vec{E}_{m_a} + ss_m \mu^2 \sum_n a_n (\vec{H}_n)_{t_a} \cdot \bar{\bar{\alpha}}'_m \cdot \vec{H}_{m_a}$$

$$- \frac{s_m^2}{c^2} \alpha'_e \sum_n a_n (E_n)_{n_a} (E_m)_{n_a}$$

$$+ iIm \int_a \sum_n a_n \vec{E}_n \times (\vec{\nabla} \times \vec{E}_m) \cdot \vec{u}_n da \qquad (3.39)$$

Equation (3.39) is a system of equations for a_m. Since the diagonal terms do not vanish simultaneously, we can apply a perturbation scheme to solve for a_m. The result of such a calculation is

$$a_m = \frac{ss_m \mu^2 \vec{u}_n \times \vec{J}_s \cdot \bar{\bar{\alpha}}'_m \cdot \vec{H}_{m_a} - \frac{s_m^2}{c^2} \alpha'_e \frac{\rho_s}{\varepsilon_o} \vec{u}_n \cdot \vec{E}_{m_a}}{\frac{1}{c^2}(s_m^2 - s^2) - ss_m \mu^2 (\vec{H}_m)_a \cdot \bar{\bar{\alpha}}'_m \cdot (\vec{H}_m)_a + \frac{s_m^2}{c^2} \alpha'_e (E_m)^2_{n_a} + D}$$

(3.40)

Here D represents the damping due to aperture radiation, and is given by

$$D = -iIms_m \mu \int_a \vec{E}_m \times \vec{H}_m \cdot \vec{u}_n da$$

$$= - \frac{s_m s^4 \mu}{12\pi c^3 \varepsilon} \left[\frac{\{\bar{\bar{\alpha}}'_m \cdot (\vec{H}_m)_{t_a}\}^2}{c^2} + \{\alpha'_e (E_m)_{n_a}\}^2 \right] \qquad (3.41)$$

In concluding, the expansion coefficients of solenoidal cavity field are obtained in this section. Of particular interest is the analysis of shifting in the resonant frequencies because of the aperture opening. The aperture radiation damping is studied in detail. The damping factor as given in (3.41) is of order $s^3a^6/c^3r_0^3$, and therefore can certainly be ignored in future investigation unless the aperture size is large, or unless the wall damping is neglected.

4. Hemispherical cavity with an opening on the plane wall

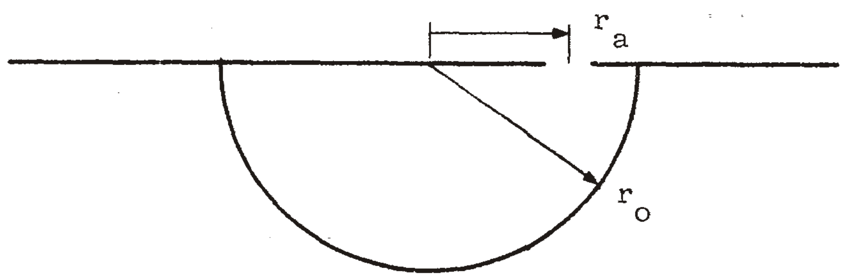


Figure 4.1

In this section we apply the general theory developed previously to a geometry of interest to an EMP system. Such a geometry of a hemispherical cavity with a circular opening on the plane wall is shown in Figure 4.1.

Assuming the opening as closed, we, first of all, study the closed hemispherical cavity. To obtain their normal modes, we follow the procedure given in Appendix B. However, since care must be exercised in the use of differential operators in spherical coordinates, we give a detailed analysis. We adopt the convention in [5].

The time harmonic Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -s\mu\vec{H} \qquad \vec{\nabla} \cdot \vec{E} = 0 \tag{4.1}$$

$$\vec{\nabla} \times \vec{H} = s\varepsilon\vec{E} \qquad \vec{\nabla} \cdot \vec{H} = 0 \tag{4.2}$$

can be solved by the scalar and vector potentials. Let

$$\vec{H}_p = \frac{1}{\mu} \vec{\nabla} \times \vec{A}_p \tag{4.3}$$

$$\vec{E}_p = -\vec{\nabla}\Phi_p - s\vec{A}_p \tag{4.4}$$

Substitution of (4.3) and (4.4) into the first of (4.2) gives

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A}_p = s\mu\varepsilon(-\vec{\nabla}\Phi_p - s\vec{A}_p) \tag{4.5}$$

In spherical coordinates,

$$\vec{\nabla} \times \vec{\nabla} \times (A_r \vec{u}_r) = \left[- \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial A_r}{\partial\theta}\right) - \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 A_r}{\partial\phi^2} \right] \vec{u}_r$$

$$+ \frac{1}{r} \frac{\partial^2 A_r}{\partial r \partial\theta} \vec{u}_\theta + \frac{1}{r \sin\theta} \frac{\partial^2 A_r}{\partial\phi \partial r} \vec{u}_\phi$$

$$= -L(A_r)\vec{u}_r - s\mu\varepsilon\nabla\Phi \tag{4.6}$$

Here

$$L(A_r) = \frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial A_r}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 A_r}{\partial\phi^2} \tag{4.7}$$

$$\Phi_p = \frac{-1}{s\mu\varepsilon} \frac{\partial A_r}{\partial r} \tag{4.8}$$

Equations (4.5) and (4.6) lead to the usual result.

$$L(A_r) + k^2 A_r = 0 \ , \qquad k^2 = -s^2 \mu\varepsilon \tag{4.9}$$

Equation (4.6) is the new gauge condition.

The solution to (4.9) satisfying the boundary conditions

$$\left.\frac{\partial A_r}{\partial r}\right|_{r=r_o} = 0 \ , \qquad A_r\left(r, \frac{\pi}{2}, \phi\right) = 0 \tag{4.10}$$

is

$$A_r = kr\psi_\ell'(kr)P_\ell^m(\cos\theta)\begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix} \tag{4.11}$$

$$\psi_\ell(kr) = \frac{1}{\sqrt{kr}} I_{\ell+1/2}(kr) \tag{4.12}$$

$$k = k_{\ell n}^e = \frac{x_{\ell n}}{r_o} \ , \qquad k = is/c \tag{4.13}$$

$x_{\ell n}$ in the last equation is the nth root of the transcendental equation

$$I_{\ell+1/2}(x) - \frac{x}{\ell} I_{\ell-1/2}(x) = 0 \tag{4.14}$$

which is derived from the first equation of (4.10).

Some explanation of notations is in order. $I_{\ell+1/2}$ is the Bessel function of order $\ell + 1/2$, and P_ℓ^m are the associate Legendre polynomials. Here ℓ is odd so that the second equation of (4.10) can be satisfied.

In order to derive fields from (4.11), we proceed as follows:

$$\vec{H}_p = -\frac{1}{\mu}\vec{\nabla} \times \vec{A}_p = -\frac{1}{\mu}\vec{\nabla}A_r \times \vec{u}_r$$

$$= -\frac{1}{\mu}\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi}\vec{u}_\theta + \frac{1}{\mu r}\frac{\partial A_r}{\partial \theta}\vec{u}_\phi \tag{4.15}$$

$$H_\theta = -\frac{mk}{\mu}\psi_\ell(kr)\frac{1}{\sin\theta}P_\ell^m(\cos\theta)\left\{\begin{matrix}-\sin m\phi\\ \cos m\phi\end{matrix}\right\} \tag{4.16}$$

$$H_\phi = \frac{k}{\mu}\psi_\ell(kr)\frac{d}{d\theta}P_\ell^m(\cos\theta)\left\{\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\right\} \tag{4.17}$$

The scalar potential ϕ_p is

$$\phi_p = \frac{-1}{s\mu\varepsilon}\frac{\partial A_r}{\partial r} = -ic\frac{d}{dr}[r\psi_\ell(kr)]P_\ell^m(\cos\theta)\left\{\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\right\} \tag{4.18}$$

$$\vec{E}_p = -\vec{\nabla}\phi_p - s\vec{A}_p \tag{4.19}$$

$$E_r = -ic\left\{\frac{d^2}{dr^2}[r\psi_\ell(kr)] + k^2 r\psi_\ell(kr)\right\}P_\ell^m(\cos\theta)\left\{\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\right\}$$

$$= -ic\frac{\ell(\ell+1)}{r}\psi_\ell(kr)P_\ell^m(\cos\theta)\left\{\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\right\} \tag{4.20}$$

$$E_\theta = ic\frac{d}{dr}[r\psi_\ell(kr)]\frac{d}{d\theta}P_\ell^m(\cos\theta)\left\{\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\right\} \tag{4.21}$$

$$E_\phi = imc \frac{d}{dr}[r\psi_\ell(kr)]P_\ell^m(\cos\theta)\{{-\sin m\phi \atop \cos m\phi}\} \qquad (4.22)$$

where

$$k = k^e_{\ell m} = \frac{x_{\ell m}}{r_o}$$

Similarly, one can generate $\vec{E}_q$ and $\vec{H}_q$ from A_r^*.

$$A_r^* = kr\psi_\ell(kr)P_\ell^m(\cos\theta)\{{\cos m\phi \atop \sin m\phi}\} \qquad (4.23)$$

Here A_r^* is subject to the boundary condition

$$A_r\Big|_{r=r_o} = 0 \ , \qquad \frac{\partial A_r}{\partial\theta}\Big|_{\theta=\frac{\pi}{2}} = 0 \qquad (4.24)$$

The second equation of (4.24) requires that ℓ be even, while the first equation defines the eigenvalues $k^m_{\ell n}$. It is related to the nth zero of the Bessel functions as follows:

$$\psi_\ell(k^m_{\ell n}a) = \psi_\ell(y_{\ell n}) = \frac{1}{\sqrt{y_{\ell n}}} I_{\ell+1/2}(y_{\ell n}) = 0 \qquad (4.25)$$

The fields can be obtained as follows:

$$\vec{E}_q = -\frac{1}{\varepsilon}\vec{\nabla}\times\vec{A}_q^* = -\frac{1}{\varepsilon}\vec{\nabla}A_r^* \times \vec{u}_r$$

$$= -\frac{1}{\varepsilon}\frac{1}{r\sin\theta}\frac{\partial A_r^*}{\partial\phi}\vec{u}_\theta + \frac{1}{\varepsilon}\frac{1}{r}\frac{\partial A_r^*}{\partial\theta}\vec{u}_\phi \qquad (4.26)$$

$$E_\theta = -\frac{mk}{\varepsilon}\psi_\ell(kr)\frac{1}{\sin\theta}P_\ell^m(\cos\theta)\{{-\sin m\phi \atop \cos m\phi}\} \qquad (4.27)$$

$$E_\phi = \frac{k}{\varepsilon}\psi_\ell(kr)\frac{d}{d\theta}P_\ell^m(\cos\theta)\{{\cos m\phi \atop \sin m\phi}\} \qquad (4.28)$$

$$\phi_q^* = \frac{-1}{s\mu\varepsilon}\frac{\partial A_r^*}{\partial r} = -ic\frac{d}{dr}[r\psi_\ell(kr)]P_\ell^m(\cos\theta)\{{\cos m\phi \atop \sin m\phi}\} \qquad (4.29)$$

$$\vec{H}^*_q = -\nabla\phi^*_q - s\vec{A}^*_q$$

$$H_r = -ic\,\frac{\ell(\ell + 1)}{r}\,\psi_\ell(kr)\,P^m_\ell(\cos\theta)\left\{{\cos m\phi \atop \sin m\phi}\right\} \tag{4.30}$$

$$H_\theta = ic\,\frac{d}{dr}\,[r\psi_\ell(kr)\;\frac{d}{d\theta}\,P^m_\ell(\cos\theta)\left\{{\cos m\phi \atop \sin m\phi}\right\} \tag{4.31}$$

$$H_\phi = imc\,\frac{d}{dr}\,[r\psi_\ell(kr)]\,P^m_\ell(\cos\theta)\left\{{-\sin m\phi \atop \cos m\phi}\right\} \tag{4.32}$$

In contrast to the rectangular cavity, these are the only normal modes supported by hemispherical cavity. This difference lies in the fact that its cavity wall consists of two smooth surfaces, while the rectangular one has six smooth surfaces.

Having obtained the normal modes of the hemispherical cavity, let us normalize the fields.

To do this, we integrate, for p modes,

$$\int \vec{E}_m \cdot \vec{E}_m dV = -\mu^2 c^2 \int \left(H^2_\theta + H^2_\phi\right) r^2 dr\ \sin\theta d\theta d\phi$$

$$= -\mu^2 c^2 \int_0^{r_o} [\psi_\ell(kr)]^2 r^2 dr \int_0^{\pi} \left[\frac{m^2}{\sin^2\theta}\left(P^m_\ell\right)^2 + \left(\frac{dP^m_\ell}{d\theta}\right)^2\right]$$

$$\sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= c^2 \pi k^2 r^3_o [\psi_{\ell-1}(kr_o)\psi_{\ell+1}(kr_o) - \psi_\ell(kr_o)^2]$$

$$\frac{2}{(2\ell + 1)}\,\frac{(\ell + m)!}{(\ell - m)!}\,\ell(\ell + 1) \tag{4.33}$$

Letting (4.33) as $1/A^2$ and multiplying (4.16) to (4.22) by A, we have thus normalized the p mode. For q modes, one arrives at a similar normalizing factor, B,

$$B = \left\{ \pi \frac{k^2}{\varepsilon^2} r_o^3 [\psi_{\ell-1}(kr_o)\psi_{\ell+1}(kr_o) - \psi_\ell(kr_o)^2] \right.$$

$$\left. \frac{2}{(2\ell + 1)} \frac{(\ell + m)!}{(\ell - m)!} \ell(\ell + 1) \right\}^{-1/2} \quad (4.34)$$

For later use, we also write

$$A = \left\{ c^2 \pi k^2 r_o^3 [\psi_{\ell-1}(kr_o)\psi_{\ell+1}(kr_o) - \psi_\ell(kr_o)^2] \right.$$

$$\left. \frac{2}{(2\ell + 1)} \frac{(\ell + m)!}{(\ell - m)!} \ell(\ell + 1) \right\}^{-1/2} \quad (4.35)$$

Having obtained the orthonormal modes for the closed cavity, we are ready to treat the open cavity with a circular opening at (r_a, ϕ_a) on the plane wall. We thus apply (3.40) to calculate the expansion coefficients.

$$a_m = \frac{sJ_{eq} - \rho_{eq}}{\frac{1}{c^2}(s_m^2 - s^2) - ss_m\mu^2\beta_M + \frac{s_m^2}{c^2}\beta_e + iD} \quad (4.36)$$

For p modes,

$$\beta_M = -\frac{8}{3}a^3A^2\left(\frac{k}{\mu}\right)^2 \left[\psi_\ell(kr_a)\frac{d}{d\theta}P_\ell^m(\cos\theta)\Big|_{\theta=\frac{\pi}{2}}\right]^2 \begin{Bmatrix}\cos^2 m\phi_a \\ \sin^2 m\phi_a\end{Bmatrix}$$

$$= -\frac{8}{3}a^3A^2\left(\frac{k}{\mu}\right)^2 [\psi_\ell(kr_a)]^2(\ell + 1)^2 \begin{Bmatrix}\cos^2 m\phi_a \\ \sin^2 m\phi_a\end{Bmatrix} \quad (4.37)$$

$$\beta_e = -\frac{4}{3}a^3c^2A^2\left[\frac{d}{dr}[r\psi_\ell(kr)]\Big|_{r=r_a}\frac{d}{d\theta}P_\ell^m(\cos\theta)\Big|_{\theta=\frac{\pi}{2}}\right]^2\begin{Bmatrix}\cos^2 m\phi_a\\ \sin^2 m\phi_a\end{Bmatrix}$$

$$= -\frac{4}{3}a^3c^2A^2\left[\frac{d}{dr}[r\psi_\ell(kr)]\Big|_{r=r_a}\right]^2(\ell+1)^2\begin{Bmatrix}\cos^2 m\phi_a\\ \sin^2 m\phi_a\end{Bmatrix} \quad (4.38)$$

For q modes,

$$\beta_M = -\frac{8}{3}a^3B^2m^2c^2\left[\frac{d}{dr}[r\psi_\ell(kr)]\Big|_{r=r_a}P_\ell^m(\cos\theta)\Big|_{\theta=\frac{\pi}{2}}\right]^2\begin{Bmatrix}\sin^2 m\phi_a\\ \cos^2 m\phi_a\end{Bmatrix}$$

$$= -\frac{8}{3}a^3B^2m^2c^2\,\frac{d}{dr}[r\psi_\ell(kr)]\Big|_{r=r_a}\begin{Bmatrix}\sin^2 m\phi_a\\ \cos^2 m\phi_a\end{Bmatrix} \quad (4.39)$$

$$\beta_e = -\frac{4}{3}a^3B^2\,\frac{m^2k^2}{\varepsilon^2}\psi_\ell(kr_a)\begin{Bmatrix}\sin^2 m\phi_a\\ \cos^2 m\phi_a\end{Bmatrix} \quad (4.40)$$

To recapitulate the result here, we have studied a small hemispherical cavity backed circular aperture shown in Fig. 4.1. The shift in resonant frequencies is of the order a^3 with a as the radius of aperture.

The radiation damping, D, is given by (3.41) of the previous section and is of the order a^6.

5. Remote sensing of properties of small cavity backed apertures

One additional application of the preceding theory is the determination of the properties of a small cavity backed aperture from the far zone. As mentioned in the introduction the transient field can be separated in two stages in time because of the difference in decay constant of the natural modes of the scatterer and that of the cavity fields. Since the decay of the cavity field is much smaller, the late time signal exhibits the properties of the open cavity through its resonant frequencies and decay constants. However, in order to completely describe the far-zone response, it is necessary to study the transfer functions between the far zone and the aperture on the scatterer. We shall study the problem in six steps.

(1) When a plane wave is incident on the scatterer, the short-circuited current and charge induced on it with the aperture closed are given by $\vec{J}_s^y$, $\vec{J}_s^z$ and ρ_s^y, ρ_s^z. Superscripts y and z refer to the direction of polarization of the incident plane wave. We can relate the induced current and charge on the same scatterer due to a dipole in the far zone to $\vec{J}_s^y$, $\vec{J}_s^z$, and ρ_s^y, ρ_s^z. To explain further this relationship, we study Figs. 5.1 and 5.2. The far field due to an electric dipole is given by

$$\vec{E}_{el} = \frac{s^2}{4\pi\varepsilon c^2} \vec{u}_R \times (\vec{u}_R \times \vec{p}) \frac{e^{-sR/c}}{R} \qquad (3.25)$$

$$\vec{H}_{el} = \frac{s^2}{4\pi c} \vec{u}_R \times \vec{p} \frac{e^{-sR/c}}{R} \qquad (3.26)$$

Let

$$\vec{p} = p_x\vec{u}_x + p_y\vec{u}_y + p_z\vec{u}_z \qquad (5.1)$$

and

$\vec{u}_R$ in (3.25) and (3.26) be $\vec{u}_x$

then p_y radiates a far field near the scatterer with electric field polarized along the y-direction as shown in Fig. 5.1. Similarly, in Fig. 5.2 p_z radiates a far field with polarization along the z-direction. Therefore, the induced short-circuited current and charge on the scatterer due to a far-zone dipole are given by

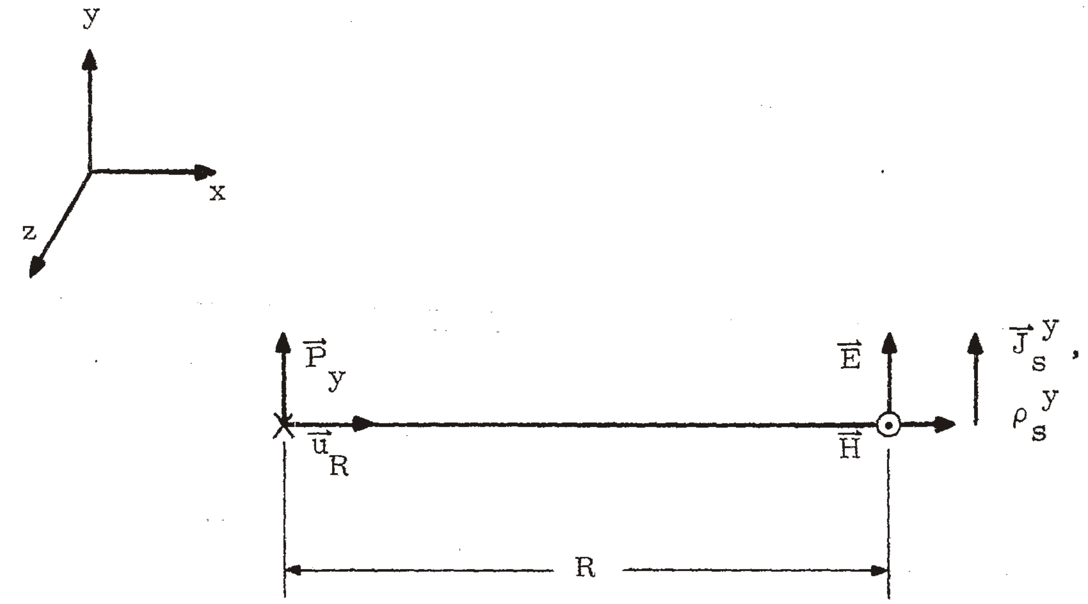


Figure 5.1

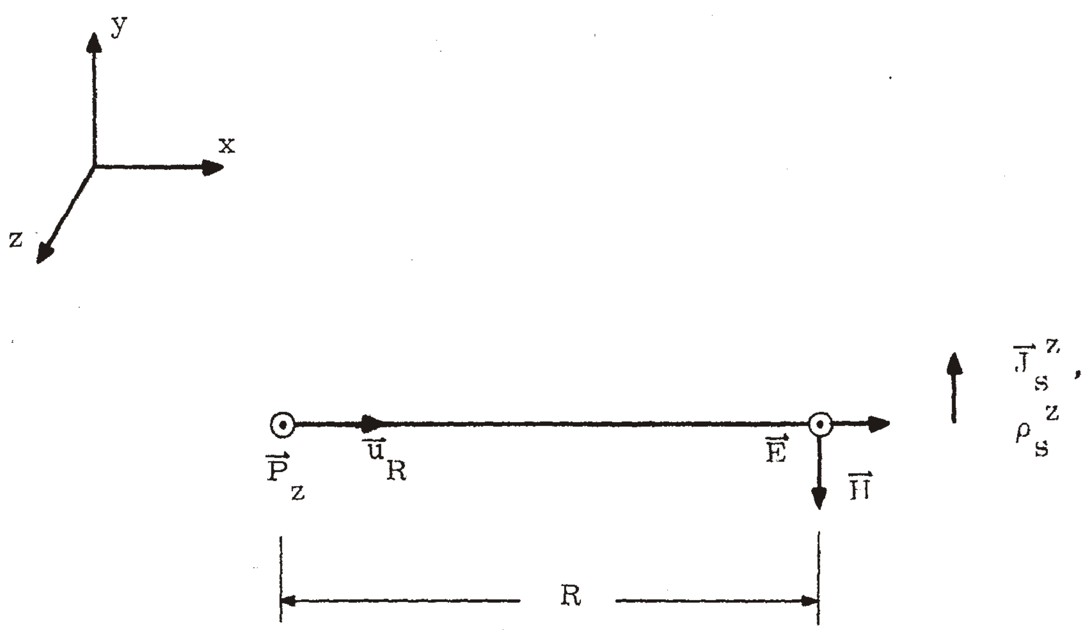


Figure 5.2

$$\vec{J}_s = \frac{s^2}{4\pi\varepsilon c^2} \frac{e^{-sR}}{R}\left(p_y \vec{J}_s^y + p_z \vec{J}_s^z\right) \tag{5.2}$$

$$\rho_s = \frac{s^2}{4\pi\varepsilon c^2} \frac{e^{-sR}}{R}\left(p_y \rho_s^y + p_z \rho_s^z\right) \tag{5.3}$$

Equations (5.2) and (5.3) are the transfer functions of the short-circuited current and charge on the scatterer from a far zone dipole source. Small-wavelength limit has been assumed in deriving (5.2) and (5.3). Here $\vec{J}_s^y$, ρ_s^y, and $\vec{J}_s^z$, ρ_s^z are assumed to have been calculated. They are the first of the three decomposed problems mentioned in the introduction. These induced currents and charges can be numerically computed. They can also be obtained through the measurement of tangential magnetic and normal electric fields; since

$$(\vec{H}_{tot})_t = \vec{u}_n \times \vec{J}_s \tag{5.4}$$

$$(E_{tot})_n = \rho_s/\varepsilon \tag{5.5}$$

$\vec{u}_n$ is as shown on Fig. 2.3.

(2) Having obtained the short-circuited currents and charges, let us proceed to discuss the equivalent polarizabilities of the aperture. In general, these electric and magnetic polarizabilities α_e and $\overline{\overline{\alpha}}_m$ are determined by the geometrical shape of the aperture. The effect of surface curvature on these polarizabilities has been discussed by Latham.[12] Usually, these polarizabilities are computed numerically. They can also be measured by an electrolytic tank.[13] However, the more convenient polarizabilities are

$$\alpha_e' = \frac{\alpha_e}{2}, \qquad \overline{\overline{\alpha}}_m' = \frac{\overline{\overline{\alpha}}_m}{2} \tag{5.6}$$

They are discussed in Section 2, and have the advantage of giving the fields behind the aperture directly. Thus the equivalent dipole moments are given by

$$\vec{m} = \overline{\overline{\alpha}}_m' \cdot (\vec{H}_{tot})_t \tag{5.7}$$

$$\vec{p} = \alpha_e'(E_{tot})_n \vec{u}_n \qquad (5.8)$$

(3) With these dipole moments at hand, the static fields behind the aperture (or inside the cavity) can be obtained by using the simple dipole formulas (2.19) and (2.20). The cavity also supports solenoidal fields:

$$\vec{E} = \sum_m a_m \vec{E}_m \qquad (5.9)$$

$$\vec{H} = -\frac{1}{s\mu} \sum_m a_m \nabla \times \vec{E}_m \qquad (5.10)$$

Here a_m is given by (3.40) and $\vec{E}_m$ the normal modes of the solenoidal cavity field. Because of the complicated derivation which led to (3.40), we devote the next two steps to a brief discussion of (3.40).

(4) The basic identity used for deriving a_m is

$$a_m\left(s^2 - s_m^2\right)/c^2 = \int_a (\vec{E} \times \vec{\nabla} \times \vec{E}_m) \cdot \vec{u}_n da \qquad (3.9)$$

The contribution to the RHS of (3.9) consists of three parts. Firstly, the contribution due to the fields entering the cavity is

$$\int_a \vec{E} \times (\vec{\nabla} \times \vec{E}_m) \cdot \vec{u}_n da = ss_m\mu^2 \vec{u}_n \times \vec{J}_s \cdot \overline{\overline{\alpha}}_m' \cdot \vec{H}_{m_a}$$

$$- s_m^2 \mu\rho_s \alpha_e' \vec{u}_n \cdot \vec{E}_{m_a} + \cdots \qquad (3.15)$$

Secondly, the contribution due to the leaking of cavity field through the aperture is

$$\int_a \left(-\sum_n a_n \vec{E}_n\right) \times \vec{\nabla} \times \vec{E}_m \cdot \vec{u}_n da$$

$$= -ss_m \mu^2 \left[\sum_n a_n (\vec{H}_n)_a\right] \cdot \bar{\bar{\alpha}}'_m \cdot \vec{H}_{m_a}$$

$$+ s_m^2 \mu\varepsilon\alpha'_e \left[\sum_n a_n (E_n)_{n_a}\right] (E_m)_{n_a} + \cdots \qquad (3.20)$$

However, these two contributions are reactive. Using (3.15) and (3.20) in (3.9) results in a solution for cavity fields. Such a solution oscillates forever if excited. Therefore, it is necessary to consider the radiation damping.

(5) Radiation damping cannot be obtained easily by using (3.9) only. Since the order of magnitude of radiation damping is quite high, we calculate it via the radiation of equivalent dipoles of the aperture to infinity. For mth normal modes of the cavity field, it is given by

$$\mathrm{Re}\int_{a'} \vec{E}_m \times \vec{H}_m \cdot \vec{u}_n da = \frac{s^4}{12\pi\varepsilon c^3}\left[\frac{\{\alpha'_m \cdot (H_m)_{t_a}\}^2}{c^2}\right.$$

$$\left. + \{\alpha'_e (E_m)_{t_a}\}^2\right] = -D \qquad (3.38)$$

With the aids of Poynting theorem described in Appendix C, we can show that the contribution of radiation damping to the RHS of (3.9) is

$$\int_a \vec{E} \times (\vec{\nabla} \times \vec{E}_m) \cdot u_n da = i\mathrm{Im}\int_a \sum_a a_n \vec{E}_n \times (\nabla \times \vec{E}_m) \cdot \vec{u}_n da$$

$$= D \qquad (3.39)$$

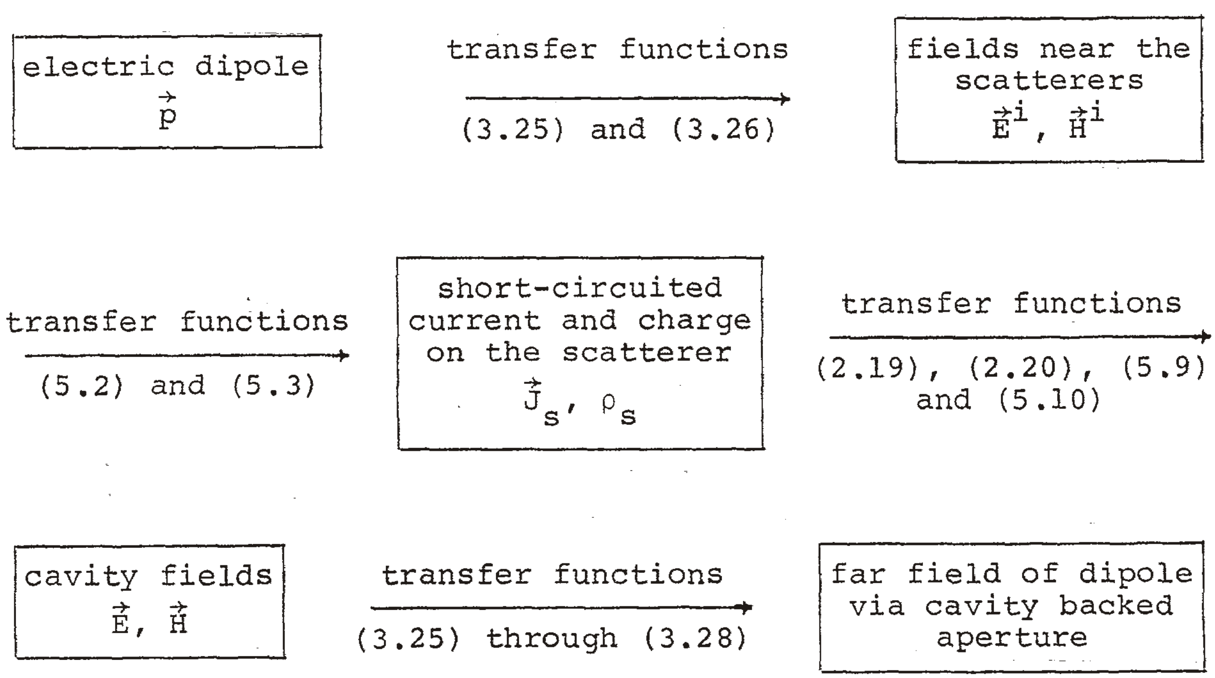
Therefore, (3.15), (3.20) and (3.39) lead to (3.40).

(6) Finally, the formula for the radiated field from the aperture can be used as the transfer function of the far zone from the field inside the cavity via a small aperture. Denoting the equivalent electric and magnetic dipoles as $\vec{p}'$ and $\vec{m}'$, we have the radiated field from these dipoles given in (3.25) through (3.28). For the radiation of the cavity field these equivalent dipoles are

$$\vec{p}' = \alpha_e' \sum_m a_m \vec{E}_m \qquad (5.6)$$

$$\vec{m}' = -\frac{1}{s\mu} \bar{\bar{\alpha}}_m' \cdot \sum_m a_m \nabla \times \vec{E}_m \qquad (5.7)$$

In conclusion, the transfer function from an electric dipole source to the far field via a far zone cavity backed aperture on a scatterer can be obtained in steps. They are best illustrated in the following block diagram:



Note that in the block diagram the transfer functions in the first step are the same as those in the last step for dipole fields. This is because of the reciprocity theorem as is well-known.

Appendix A

A Direct Treatment of Cavity Fields due to Teichmann and Wigner

Teichmann and Wigner[14] showed that a complete expansion of cavity fields consists of not only the solenoidal fields originally given by Weyl, but also zero-frequency fields. They consider a simply connected cavity with perfectly conducting walls, which is excited through holes in these walls. To solve such a problem, they found the set of normal vectors with holes short-circuited. Firstly, the solenoidal part of normal vectors satisfies

$$\begin{aligned} \vec{\nabla} \times \vec{E}_m &= -s_m \mu \vec{H}_m \\ \vec{\nabla} \times \vec{H}_m &= s_m \varepsilon \vec{E}_m \end{aligned} \tag{A.1}$$

in the cavity, and the boundary condition

$$\vec{u}_n \times \vec{E}_m = 0 \tag{A.2}$$

on the wall.

As usual, (A.1) can be reduced to the vector wave equation

$$\nabla^2 \vec{E}_m + k_m^2 \vec{E}_m = 0 \ , \qquad k_m^2 = -s_m^2 \mu\varepsilon \tag{A.3}$$

with boundary condition (A.2). After $\vec{E}_m$ has been found, $\vec{H}_\alpha$ can be obtained from one of (A.1). However, with the presence of holes the set of solenoidal normal modes no longer is complete. To see this, we write the Maxwell's equations

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= -s\mu\vec{H} \qquad & \vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \times \vec{H} &= s\varepsilon\vec{E} \qquad & \vec{\nabla} \cdot \vec{H} &= 0 \end{aligned} \tag{A.4}$$

It is easy to see that the right halves of these equations are satisfied by the gradient of scalar functions. Therefore,

$$\vec{E}_\beta = \vec{\nabla}\phi \ , \qquad \vec{H}_\beta = \vec{\nabla}\psi \tag{A.5}$$

These are the static fields, since the substitution of these fields into the left halves of the Maxwell's equations requires the frequency to be zero.

It is further shown by Teichmann and Wigner that the normal modes, $\vec{E}_\beta = 0$, $\vec{H}_\beta = \nabla\psi$ and $\vec{E}_\beta = \nabla\phi$, $\vec{H}_\beta = 0$, are orthogonal to the solenoidal field. Also, the actual fields are not orthogonal to

$$\vec{E}_\beta = 0\ , \qquad \vec{H}_\beta = \nabla\psi \tag{A.6}$$

but are orthogonal to

$$\vec{E}_\beta = \nabla\phi\ , \qquad \vec{H}_\beta = 0 \tag{A.7}$$

Let us follow their proof first, and then point out the fallacy in the procedure. Here only the second fact is discussed.

$$\begin{aligned}\int \vec{E}_\beta \cdot \vec{E}dV &= \frac{1}{s\varepsilon}\int \vec{E}_\beta \cdot \vec{\nabla} \times \vec{H}dV \\ &= \frac{1}{s\varepsilon}\left\{\int \vec{\nabla} \cdot (\vec{E}_\beta \times \vec{H})dV + \int \vec{H} \cdot \vec{\nabla} \times \vec{E}\, dV\right\} \\ &= \frac{1}{s\varepsilon}\int (\vec{u}_n \times \vec{H}) \cdot \vec{E}_\beta da \\ &= \frac{1}{s\varepsilon}\int (\vec{u}_n \times \vec{E}_\beta \cdot \vec{H}) da \end{aligned} \tag{A.8}$$

They claimed that (A.8) vanishes, since $\vec{E}_\beta \,||\, \vec{u}_n$. Similarly,

$$\int \vec{H}_\beta \cdot \vec{H}dV = \frac{-1}{s\mu}\int (\vec{u}_n \times \vec{E}) \cdot \vec{H}_\beta da \tag{A.9}$$

$\vec{u}_n \times \vec{E}$ is zero on the wall, but not zero on the holes.

Therefore, they gave a complete set of normal modes as follows:

$$\vec{E} = \sum_{\alpha} a_{\alpha}\vec{E}_{\alpha}$$

$$\vec{H} = \sum_{\alpha} a_{\alpha}\vec{H}_{\alpha} + \vec{\nabla}\psi \tag{A.10}$$

Here

$$a_{\alpha} = \int \vec{E} \cdot \vec{E}_{\alpha} dV$$

Eqn. (A.9) appears to be non-zero on the holes. However, on analyzing it more closely, it turns out to be zero. To see this, we study a special case of small circular hole. The field on the hole has been calculated by Bethe. The tangential electric field is

$$\vec{u}_n \times \vec{E} = \frac{\vec{r} \times \vec{E}_o}{2\pi^2 (a^2 - r^2)^{1/2}} \tag{A.11}$$

where $\vec{E}_o$ is in the direction of normal to the hole. The static magnetic field on the hole is constant

$$\vec{H}_{\beta} = \text{constant} \tag{A.12}$$

Therefore,

$$\int \vec{H}_{\beta} \cdot \vec{H} dV = \frac{-1}{s\mu} \int (\vec{u}_n \times \vec{E}) \cdot \vec{H}_{\beta} da$$

$$= \frac{-1}{s\mu} \frac{1}{2\pi^2} \int \frac{\vec{\phi} \cdot \vec{H}_{\beta}}{(a^2 - r'^2)^{1/2}} r'^2 dr' d\phi \tag{A.13}$$

Here $\vec{\phi}$ is the unit vector of polar coordinate of the hole. It is easy to see that the integration in ϕ from 0 to 2π vanishes. Let us also look closely at (A.8). Again from Bethe's quasi-static theory of small holes discussed in Section 2, we note that $\vec{E}_{\beta}$ is not perpendicular to the surface at all. However, if one carries out the integration, (A.8) is found to vanish.

These results are not surprising, since in substituting $\vec{E}$ by $(1/s\varepsilon)\nabla \times \vec{H}$ and $\vec{H}$ by $(-1/s\mu)\vec{\nabla} \times \vec{E}$, one essentially eliminates the static electric and magnetic fields. Therefore, only the solenoidal field remains in the expansion. The correct expansion coefficients should be

$$\int \vec{E}_\beta \cdot \vec{E}dV = \int \vec{\nabla}\phi \cdot \vec{E}dV = \int \vec{\nabla} \cdot (\phi\vec{E})dV$$

$$= \int \phi\vec{E} \cdot \vec{u}_n da \tag{A.14}$$

$$\int \vec{H}_\beta \cdot \vec{H}dV = \int \vec{\nabla}\psi \cdot \vec{H}dV = \int \vec{\nabla} \cdot (\psi\vec{H})dV$$

$$= \int \psi\vec{H} \cdot \vec{u}_n da \tag{A.15}$$

Appendix B

On the Expansion of Cavity Fields through Scalar and Vector Potentials

As is well-known, Maxwell's equations with electric currents and charges

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad \vec{\nabla} \cdot \vec{E} = \rho/\varepsilon$$

$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{J} \qquad \vec{\nabla} \cdot \vec{H} = 0 \tag{B.1}$$

are solved by scalar and vector potentials. Let

$$\vec{H}_e = \frac{1}{\mu} \vec{\nabla} \times \vec{A}$$

$$\vec{E}_e = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{B.2}$$

It can be shown[3] that ϕ and $\vec{A}$ satisfy wave equations

$$\nabla^2 \vec{A} - \frac{\partial^2 A}{\partial t^2} = \mu \vec{J}$$

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = -\rho/\varepsilon \tag{B.3}$$

if $\vec{A}$ and ϕ are related by the Lorentz condition

$$\vec{\nabla} \cdot \vec{A} + \mu\varepsilon \frac{\partial \phi}{\partial t} = 0 \tag{B.4}$$

Similarly, Maxwell's equation with magnetic currents and charges, which is obtained by removing $\vec{J}$ and ρ/ε in (B.1), while inserting $-\vec{J}^*$ and ρ^*/μ into the other halves of Maxwell's equations, are solved by

$$\vec{E}_m = -\frac{1}{\varepsilon}\vec{\nabla} \times \vec{A}^*$$

$$\vec{H}_m = -\vec{\nabla}\phi^* - \frac{\partial \vec{A}^*}{\partial t} \qquad \text{(B.5)}$$

Here $\vec{A}^*$ and ϕ^* again satisfy

$$\nabla^2\vec{A}^* - \frac{\partial^2}{\partial t^2}\vec{A}^* = \varepsilon\vec{J}^*$$

$$\nabla^2\phi^* - \frac{\partial^2}{\partial t^2}\phi^* = -\frac{\rho^*}{\mu} \qquad \text{(B.6)}$$

$$\vec{\nabla} \cdot \vec{A}^* + \mu\varepsilon\frac{\partial\phi^*}{\partial t} = 0$$

A superposition of (B.2) and (B.5) then forms a most general solution of Maxwell's equations in terms of potentials. To employ such a solution, it is important one determines the boundary conditions for the potentials. They are derived from the usual boundary conditions for $\vec{E}$ and $\vec{H}$. For a conducting surface, such conditions are

$$\vec{u}_n \times \vec{H} = \vec{J}_s$$

$$\vec{u}_n \cdot \vec{H} = 0$$

$$\vec{u}_n \times \vec{E} = 0 \qquad \text{(B.7)}$$

$$\vec{u}_n \cdot \vec{E} = \frac{\rho_s}{\varepsilon}$$

where $\vec{J}_s$ and ρ_s are surface electric currents and charges. Thus the source terms $\vec{J}$ and ρ can be considered as

$$\vec{J} = \vec{J}_s\delta(x_n - x_{n_o}) \qquad \text{(B.8)}$$

$$\rho = \rho_s \delta(x_n - x_{n_o}) \tag{B.9}$$

x_n is the coordinate perpendicular to the conducting surface, and x_{n_o} gives the location of the conducting surface. By integrating the second of (B.3) as follows:

$$\int_{x_{n_o}-\delta}^{x_{n_o}+\delta} [\nabla^2\phi + k^2\phi]\,dx_n = -\frac{1}{\varepsilon}\int_{x_{n_o}-\delta}^{x_{n_o}+\delta} \rho_s \delta(x_n - x_{n_o})\,dx_n$$

we obtain

$$\left.\frac{\partial\phi}{\partial n}\right|_{x_{n_{o+}}} = -\frac{\rho_s}{\varepsilon} \tag{B.10}$$

Here the contribution from the lower limit of integration is zero, since it is inside the conductor.

In a similar way $\vec{A} = A_n\vec{u}_n + A_{||}\vec{u}_{||} + A_{\perp}\vec{u}_{\perp}$ can be shown to satisfy the boundary conditions

$$\frac{\partial A_{||}}{\partial x_n} = \mu J_s, \qquad \frac{\partial A_{\perp}}{\partial x_n} = 0, \qquad \frac{\partial A_n}{\partial x_n} = 0 \tag{B.11}$$

$A_{||}$ is the component of $\vec{A}$ parallel to $\vec{J}_s$, $A_{\perp}$ is that perpendicular to $\vec{J}_s$ and $\vec{u}_n$, A_n is that parallel to $\vec{u}_n$.

Furthermore, from the first halves of (B.),

$$\vec{J}_s = \vec{u}_n \times \vec{H} = \frac{1}{\mu}\vec{u}_n \times \vec{\nabla} \times \vec{A}$$

$$= \frac{1}{\mu}\left\{\vec{u}_{\perp}\left(\frac{\partial A_n}{\partial x_{\perp}} - \frac{\partial A_{\perp}}{\partial x_n}\right) + \vec{u}_{||}\left(\frac{\partial A_{||}}{\partial x_n} - \frac{\partial A_n}{\partial x_{||}}\right)\right\} \tag{B.12}$$

$$\vec{u}_n \cdot \vec{H} = \frac{\partial A_{\perp}}{\partial x_{||}} - \frac{\partial A_{||}}{\partial x_{\perp}} = 0$$

Since $\vec{A}$ is quite arbitrary, we can conveniently set

$$\frac{\partial A_n}{\partial x_\perp} = 0 \ , \qquad \frac{\partial A_\perp}{\partial x_{||}} = 0 \ , \qquad \frac{\partial A_{||}}{\partial x_\perp} = 0 \tag{B.13}$$

Finally, we study the last boundary condition:

$$\vec{u}_n \times \vec{E} = -\vec{u}_n \times (\vec{\nabla}\phi + s\vec{A}) = \vec{0}$$

Again we can set

$$\nabla_\perp \phi = 0 \tag{B.14}$$

and

$$\vec{u}_n \times \vec{A} = 0 \tag{B.15}$$

In summary, the boundary conditions for ϕ and $\vec{A}$ are

$$\frac{\partial \phi}{\partial x_n} = \frac{-\rho_s}{\varepsilon} \ , \qquad \nabla_\perp \phi = 0 \tag{B.16}$$

$$\frac{\partial A_{||}}{\partial x_n} = \mu J_s \ , \qquad \frac{\partial A_{||}}{\partial x_{||}} = 0 \ , \qquad \frac{\partial A_{||}}{\partial x_\perp} = 0$$

$$\nabla A_n = 0 \ , \qquad \nabla A_\perp = 0$$

$$A_{||} = 0 \ , \qquad A_\perp = 0 \tag{B.17}$$

Likewise, the boundary conditions for ϕ^* and $\vec{A}^*$ can be derived to give

$$\frac{\partial \phi^*}{\partial x_n} = -\frac{\rho_s^*}{\varepsilon} \ , \qquad \nabla_\perp \phi^* = 0 \tag{B.18}$$

$$\frac{\partial A^*_{||}}{\partial x_n} = \varepsilon J^*_s \,, \qquad \frac{\partial A^*_{||}}{\partial x_{||}} = 0 \,, \qquad \frac{\partial A^*_{|}}{\partial x_{\perp}} = 0$$

$$\nabla A^*_n = 0 \,, \qquad \nabla A^*_{\perp} = 0 \qquad \text{(B.19)}$$

$$A^*_{||} = 0 \,, \qquad A^*_{\perp} = 0$$

Here $\vec{J}^*_s$ and ρ^*_s are the surface magnetic currents and charges.

$A^*_{||}$ is the component parallel to $\vec{J}^*_s$.

$A^*_{\perp}$ is the component perpendicular to both $\vec{J}^*_s$ and $\vec{n}$.

A^*_n is the component parallel to n.

They can be derived from

$$\vec{u}_n \times \vec{H} = 0$$

$$\vec{u}_n \cdot \vec{H} = \rho^*_s/\mu$$

$$\text{(B.20)}$$

$$\vec{u}_n \times \vec{E} = -\vec{J}^*_s$$

$$\vec{u}_n \cdot \vec{E} = 0$$

These are the boundary conditions for the fictitious magnetic conducting surface.

Finally, let us apply the results to study the cavity fields. To begin with, we seek the normal modes of the cavity field. We let ϕ_p and $\vec{A}_p$ be the eigenfunctions with eigenvalues k_p^2 of

$$\nabla^2\phi_p + k_p^2\phi_p = 0 \qquad \text{(B.21)}$$

and

$$\nabla^2\vec{A}_p + k_p^2\vec{A}_p = 0 \qquad \text{(B.22)}$$

satisfying

$$\vec{u}_n \times \vec{A}_p = 0 \qquad (B.23)$$

on the boundary, and

$$\phi_p = \frac{1}{s\mu\varepsilon} \vec{\nabla} \cdot \vec{A}_p \qquad (B.24)$$

everywhere inside the cavity. Then,

$$\vec{E}_p = -(\vec{\nabla}\phi_p + s_p \vec{A}_p)$$

$$\vec{H}_p = \frac{1}{\mu} \nabla \times \vec{A}_p \qquad (B.25)$$

are the normal modes of the cavity. Also,

$$\vec{E}_q = -\frac{1}{\varepsilon} \vec{\nabla} \times \vec{A}_q^*$$

$$\vec{H}_q = -(\vec{\nabla}\phi_q^* + s_q \vec{A}_q^*) \qquad (B.26)$$

are the normal modes where ϕ_q^* and $\vec{A}_q^*$ satisfy equations identical to (B.21) ~ (B.24). Normal modes (B.25) and (B.26) form a complete set for the cavity fields.

Finally, let us discuss the case when $k_p = 0$. Obviously,

$$\nabla^2 \phi_p = 0 \qquad (B.27)$$

and

$$\vec{E}_p = -\nabla\phi_p , \qquad \vec{H}_p = 0 \qquad (B.28)$$

However, (B.18) is not the boundary condition for such a case, since this would lead to $\phi_p = 0$ inside the cavity. These fields exist only in open cavities or multiple-region cavities. Also,

$$\nabla^2 \phi_q^* = 0 \qquad (B.29)$$

$$\vec{H}_q = -\nabla \phi_q^* , \qquad \vec{E}_q = 0 \qquad (B.30)$$

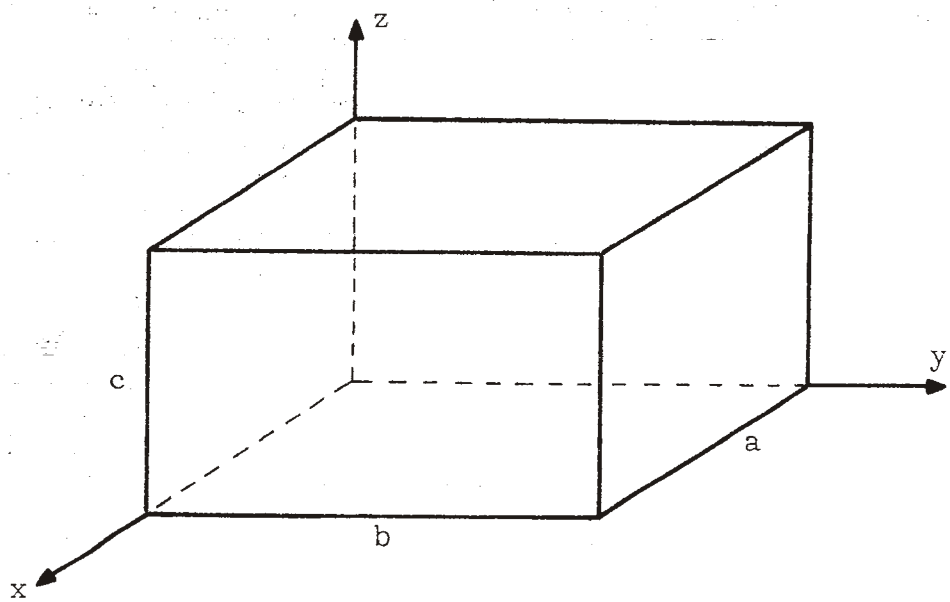


Figure B.1

Let us illustrate the use of these potentials by a closed rectangular cavity shown in Fig. B-1. Since the rectangular cavity has some symmetry, one can conveniently begin by studying a particular solution to (B.22) with (B.23). Such a particular solution is

$$\vec{A}_p = A_z \vec{u}_z = \sin k_\ell x \sin k_m y \cos k_n z \; \vec{u}_z \qquad (B.31)$$

which satisfies

$$\vec{u}_n \times \vec{A} = 0 \qquad (B.32)$$

or

$$A_z(o, y, z) = 0 , \qquad A_z(a, y, z) = 0$$
$$A_z(x, o, z) = 0 , \qquad A_z(x, b, z) = 0 \qquad (B.33)$$

Here and henceforth

$$k_\ell = \frac{\ell\pi}{a} , \qquad k_m = \frac{m\pi}{b} , \qquad k_n = \frac{n\pi}{c} \qquad (B.34)$$

Thus,

$$\vec{H}_p = -\frac{1}{\mu}\nabla \times \vec{A}_p = -\frac{1}{\mu}\nabla A_z \times \vec{u}_z$$
$$= -\frac{1}{\mu}\frac{\partial A_z}{\partial y}\vec{u}_x + \frac{1}{\mu}\frac{\partial A_z}{\partial x}\vec{u}_y \qquad (B.35)$$

$$\phi_p = \frac{1}{s\mu\varepsilon}\nabla \cdot \vec{A}_p = \frac{k_n}{s\mu\varepsilon}\sin k_\ell x \sin k_m y \sin k_n z \qquad (B.36)$$

$$\vec{E}_p = -(\nabla\phi_p + s_p\vec{A}_p)$$
$$= -\frac{\partial\phi_p}{\partial x}\vec{u}_x - \frac{\partial\phi_p}{\partial y}\vec{u}_y + \left(-s_pA_z - \frac{\partial\phi_p}{\partial z}\right)\vec{u}_z \qquad (B.37)$$

Similarly, one can generate $\vec{E}_q$ and $\vec{H}_q$ by a particular solution, A^*_q, which is

$$\vec{A}^*_q = A^*_z\vec{u}_z = \cos k_\ell x \cos k_m y \sin k_n z \ \vec{u}_z \qquad (B.38)$$

Note that A^*_z satisfies $A^*_z(x, y, o) = A^*_z(x, y, c) = 0$. However, a superposition of fields generated by (B.31) and (B.38) is not complete. The symmetry of the geometry allows us to generate more normal modes by assuming

$$\vec{A}_p = A_x\vec{u}_x + A_y\vec{u}_y \qquad (B.39)$$

$$\vec{A}_q = A^*_x\vec{u}_x + A^*_y\vec{u}_y \qquad (B.40)$$

where A_x, A_y, A_x^*, A_y^* can be obtained from (B.31) and (B.38) by rotating as follows:

$$x \to y \to z \qquad \text{(B.41)}$$

Note that a previous study[15] did not include all these symmetric modes.

Appendix C

On Poynting Theorem with Advanced and Retarded Fields

In this appendix we study the removal of a limitation of our derivation of the decay constant for an open cavity in Section 3. In that section we noted that the derivation of the decay constant is made through the use of the complex Poynting theorem. Thus, we had to restrict s to be imaginary (or ω to be real). Here we shall derive a Poynting theorem with advanced and retarded fields. Application of such a theorem does not require any restriction. The exposition of generalization of the conventional Poynting theorem and reciprocity theorem has been given by Baum.[16] We limit our study to the special physical problem treated.

The basic vector identity for deriving the Poynting theorem in its various forms is

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \vec{\nabla} \times \vec{A} - \vec{A} \cdot \vec{\nabla} \times \vec{B} \tag{C.1}$$

Let $\vec{E}^a$, $\vec{H}^a$ and $\vec{E}^r$, $\vec{H}^r$ be the advanced and retarded fields satisfying the source-free Maxwell's equations (A.1). We easily obtain

$$\begin{aligned} \vec{\nabla} \cdot (\vec{E}^a \times \vec{H}^r) &= \vec{H}^r \cdot \vec{\nabla} \times \vec{E}^a - \vec{E}^a \cdot \vec{\nabla} \times \vec{H}^r \\ &= - \mu\vec{H}^r \cdot \frac{d\vec{H}^a}{dt} + \epsilon\vec{E}^a \cdot \frac{dE^r}{dt} \end{aligned} \tag{C.2}$$

A similar equation is obtained by switching advanced and retarded fields to yield

$$\vec{\nabla} \cdot (\vec{E}^r \times \vec{H}^a) = - \mu\vec{H}^a \cdot \frac{d\vec{H}^r}{dt} + \epsilon\vec{E}^r \cdot \frac{d\vec{E}^a}{dt} \tag{C.3}$$

On averaging (C.2) and (C.3), one arrives at

$$\nabla \cdot \frac{1}{2}(\vec{E}^a \times \vec{H}^r + \vec{E}^r \times \vec{H}^a) = - \frac{1}{2} \frac{d}{dt}(\epsilon\vec{E}^r \cdot \vec{E}^a + \mu\vec{H}^r \cdot \vec{H}^a) \tag{C.4}$$

Equation (C.4) gives

$$\int \frac{1}{2}(\vec{E}^r \times \vec{H}^a + \vec{E}^a \times \vec{H}^r) \cdot \vec{u}_n da$$

$$= -\frac{1}{2} \frac{d}{dt} \int (\varepsilon \vec{E}^a \cdot \vec{E}^r + \mu \vec{H}^a \cdot \vec{H}^r) dV \qquad (C.5)$$

Consider an infinite conducting plane with a small aperture in the center as shown in Fig. C.1. Let the electromagnetic wave enter the left half space via the aperture and then radiate to infinity. $\vec{P}$ and $\vec{M}$ are the equivalent electric and magnetic dipole moments of the small aperture. One can show, by using the far field expressions due to $\vec{P}$ and $\vec{M}$, (3.25) through (3.28), that the volume integration on the left hand side of (C.5) diverges. However, if R is kept constant, the left hand side of (C.5) vanishes; because the energy stored in a fixed volume is constant. One can thus separate the right hand side of (C.5) as follows:

$$\int_{R\to\infty} \frac{1}{2}(\vec{E}^a \times \vec{H}^r + \vec{E}^r \times \vec{H}^a) \cdot \vec{u}_n da$$

$$= \int_{hole} \frac{1}{2}(\vec{E}^a \times \vec{H}^r + \vec{E}^r \times \vec{H}^a) \cdot \vec{u}_n da$$

$$= \int_{hole} \vec{E} \times \vec{H} \cdot \vec{u}_n da \qquad (C.5)$$

The last equality follows from the fact that at a small hole the advanced and retarded fields are equal. Equation (C.5) can replace the usual complex Poynting theorem, which involves complex conjugation in the formula.

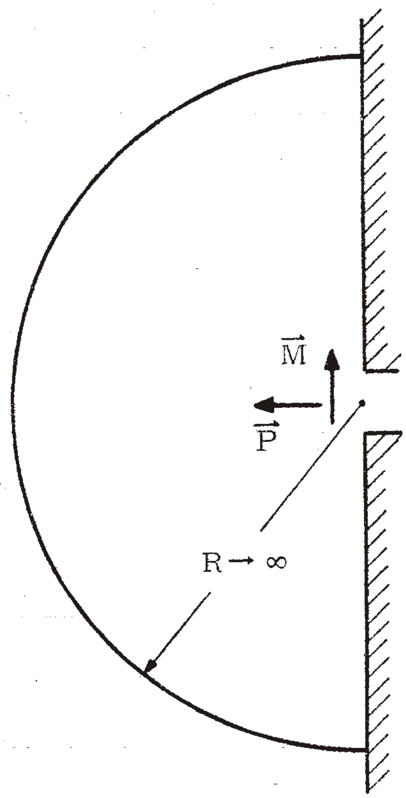


Figure C.1

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