Interaction Notes

Note 212

July 1974

A Numerical Solution Procedure for Small Aperture Integral Equations

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Abstract

A procedure is presented for numerically obtaining solutions of the small aperture integral equations developed in Interaction Note 149. The results of the solution technique provide knowledge of the distribution of electric field (or equivalent magnetic current) in the aperture due to arbitrary plane wave illumination and valid to zeroth and first order in reciprocal wavelength.

This study was performed under subcontract to

The Dikewood Corporation
1009 Bradbury Drive, S.E.
University Research Park
Albuquerque, New Mexico 87106

and has been fully supported by the Defense Nuclear Agency (DNA) under

DNA Subtask EB088
EMP Interaction and Coupling
DNA Work Units 32 and 33
Coupling Characteristics of Apertures
Coaxial Cables

CONTENTS

SECTION I

INTRODUCTION

In this note is presented a procedure for numerically solving a new set of integral equations [1] which characterize the electric field distributions (or equivalent magnetic current) in a small aperture in a planar, perfectly conducting screen of infinite extent. The problem under consideration here is illustrated in Fig. 1 where one sees an incident field ($\overline{E}^i$, $\overline{H}^i$) impinging upon the screen with aperture A. In Note 149 individual integral equations are developed for $\overline{M}_o$ and $\overline{M}_1$ which are, respectively, the zeroth and first order coefficients of a Rayleigh series expansion in k (2π/wavelength) of the equivalent magnetic current in the aperture. In other words, solutions of the above-mentioned integral equations yield $\overline{M}_o$ and $\overline{M}_1$ which one employs in the approximation

$$\overline{M} \doteq \overline{M}_o + k\,\overline{M}_1 \tag{1}$$

where $\overline{M}$ is the equivalent magnetic current in the aperture. For apertures sufficiently small relative to wavelength λ of the time-harmonic fields, the approximation above provides highly accurate results.

In addition to a procedure for solving Equations (47)-(49) of [1], a brief discussion is included of modifications of the equations of [1] together with an outline of how the modified equations may be solved. The two separate procedures yield results which numerically are almost indistinguishable.

The numerical solution scheme outlined in this note has been used by the authors to calculate dipole moments of small apertures and, when specialized to square apertures, the data compare quite favorably with moments of circular apertures (circle inscribed in the square) available in the literature. The technique is essentially the method of moments [2] and

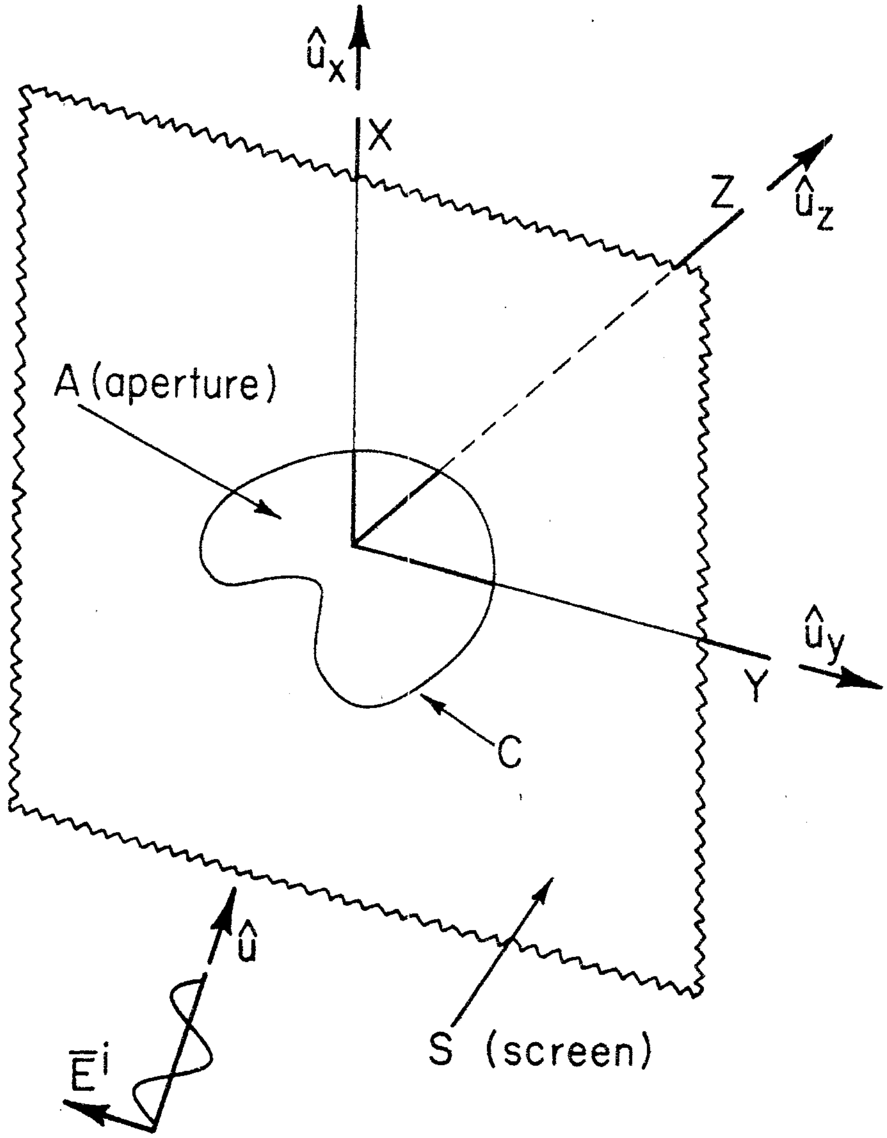

Figure 1. Aperture in Conducting Screen Illuminated by Incident Field

utilizes pulses for representing the unknowns together with point-matching for reducing the integral equations to corresponding matrix equations. The various steps given below are illustrated by a rectangular aperture for ease of presentation but, in general, they apply to any reasonable aperture shape.

SECTION II

ZEROTH ORDER SOLUTION

The integral equation ((47) of [1]) governing in part the zeroth order magnetic current coefficient $\overline{M}_o = M_{ox}\hat{u}_x + M_{oy}\hat{u}_y$ in the two term approximation (1) can be written as below in component form

$$\iint_A M_{ox}(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds' - \oint_C \Psi_{ox}(\overline{r}_c)\, g(\overline{r}_c, \overline{r})d\ell_c$$

$$= -2\pi y\, e_z^i \quad , \qquad \overline{r} \in A \quad , \tag{2a}$$

$$\iint_A M_{oy}(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds' - \oint_C \Psi_{oy}(\overline{r}_c)\, g(\overline{r}_c, \overline{r})d\ell_c$$

$$= 2\pi x\, e_z^i \quad , \qquad \overline{r} \in A \quad , \tag{2b}$$

where e_z^i is the magnitude of the z-component of the incident electric field,

$$\overline{r} = x\hat{u}_x + y\hat{u}_y \quad , \tag{3a}$$

$$\overline{r}' = x'\hat{u}_x + y'\hat{u}_y \quad , \tag{3b}$$

$$R(\overline{r}, \overline{r}') = |\overline{r}-\overline{r}'| = \left[(x-x')^2 + (y-y')^2\right]^{1/2} \tag{3c}$$

and

$$g(\overline{r}_c, \overline{r}) = \frac{1}{2\pi} \ln |\overline{r}_c - \overline{r}| = \frac{1}{2\pi} \ln \left[(x_c - x)^2 + (y_c - y)^2 \right]^{\frac{1}{2}} \tag{3d}$$

with $\overline{r}_c$ on the bounding contour C of the aperture A.

In (2), M_{ox} and M_{oy} are the unknowns which are to be determined but, also, Ψ_{ox} and Ψ_{oy} are unknown auxiliary functions on the contour which must be consistent with the boundary conditions on M_{ox} and M_{oy}:

$$\overline{M}_o \cdot \hat{u}_n = 0 \quad \text{on C.} \tag{4}$$

where $\hat{u}_n$ is the outward normal unit on C.

Expanding M_{ox} and M_{oy} in terms of pulse expansion functions, one has*

$$M_{ox}(\overline{r}') \doteq \sum_{n=1}^{N} M_{oxn} f_n \tag{5a}$$

$$M_{oy}(\overline{r}') \doteq \sum_{n=1}^{N} M_{oyn} f_n \tag{5b}$$

where

$$f_n = \begin{cases} 1 & \text{on } \Delta A'_n \\ 0 & \text{otherwise} \end{cases} \tag{5c}$$

and where $\Delta A'_n$ is the n^{th} "patch" or subarea of the aperture (rectangular) illustrated in Fig. 2. Similar representation of the auxiliary functions Ψ_{ox} and Ψ_{oy} in terms of pulse expansion functions on C lead to

$$\Psi_{ox} \doteq \sum_{q=1}^{Q} X_{oq} P_q \tag{6a}$$

$$\Psi_{oy} \doteq \sum_{q=1}^{Q} Y_{oq} P_q \tag{6b}$$

with

$$P_q = \begin{cases} 1 & \text{on } \Delta C_q \\ 0 & \text{otherwise} \end{cases} \tag{6c}$$

*The use of "n" either to denote normal to C or as an index (5) should be clear from context.

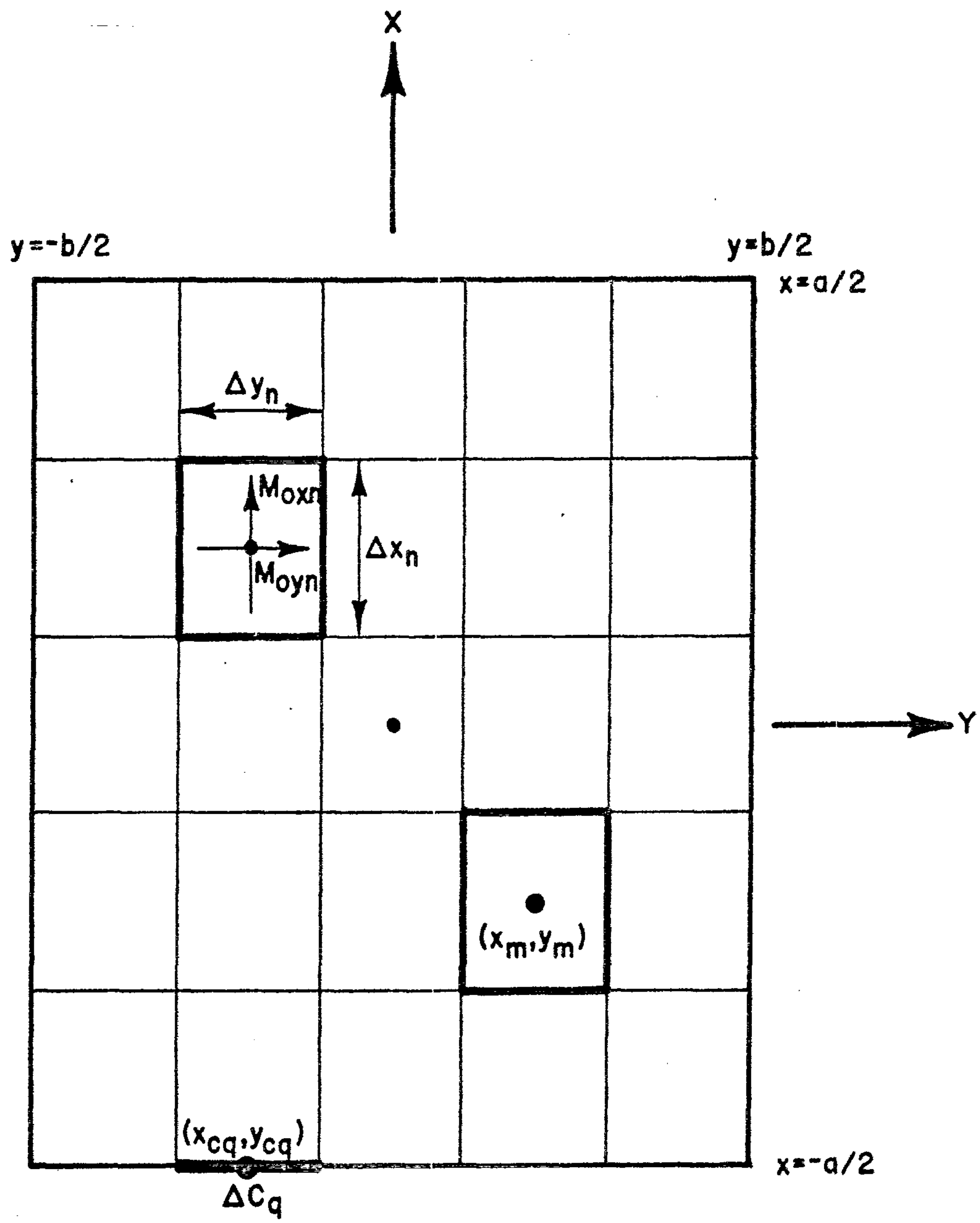

Figure 2. Rectangular Aperture--Patches, Match Points and Contour

where ΔC_q is the $q^{\underline{th}}$ subinterval on C of Fig. 2. Substitution of the expansions (5) and (6) into the integral equations (2) enables one to achieve

$$\sum_{n=1}^{N} M_{oxn} \iint_A f_n R^{-1}(\bar{r}, \bar{r}')ds' - \sum_{q=1}^{Q} X_{oq} \oint_C P_q g(\bar{r}_c \bar{r}) d\ell_c$$

$$= -2\pi y \; e_z^i \tag{7a}$$

and

$$\sum_{n=1}^{N} M_{oyn} \iint_A f_n R^{-1}(\bar{r}, \bar{r}')ds' - \sum_{q=1}^{Q} Y_{oq} \oint_C P_q g(\bar{r}_c, \bar{r}) d\ell_c$$

$$= 2\pi x \; e_z^i \quad . \tag{7b}$$

Enforcing (7) to hold at the center point (x_m, y_m), m=1, 2, . . . , . . . , M(M=N), of each patchillustrated in Fig. 2 generates the following sets of algebraic equations:

$$\sum_{n=1}^{N} M_{oxn} B_{mn} + \sum_{q=1}^{Q} X_{oq} C_{mq}$$

$$= -2\pi y_m e_z^i \quad , \tag{8a}$$

$$m = 1, 2, 3, \ldots\ldots, \quad M = N,$$

and

$$\sum_{n=1}^{N} M_{oyn} B_{mn} + \sum_{q=1}^{Q} Y_{oq} C_{mq}$$

$$= 2\pi x_m e_z^i \quad , \tag{8b}$$

$$m = 1, 2, 3, \ldots\ldots, \quad M = N,$$

where

$$B_{mn} = \iint_{\Delta A'_n} \frac{dx'\ dy'}{[(x_m-x')^2+(y_m-y')^2]^{\frac{1}{2}}} \tag{8c}$$

and

$$C_{mq} = -\frac{1}{4\pi} \int_{\Delta C_q} \ell n \left[(x_c-x_m)^2+(y_c-y_m)^2 \right] d\ell_c \ . \tag{8d}$$

The integral of (8c) can be evaluated analytically. For m = n,

$$B_{nn} = \int_{-\frac{\Delta x_n}{2}}^{\frac{\Delta x_n}{2}} \int_{-\frac{\Delta y_n}{2}}^{\frac{\Delta y_n}{2}} \frac{dxdy}{[x^2+y^2]^{\frac{1}{2}}}$$

$$= \Delta y_n \ \ell n \left[\frac{\frac{\Delta x_n}{2} + R_{xy}}{-\frac{\Delta x_n}{2} + R_{xy}} \right] + \Delta x_n \ \ell n \left[\frac{\frac{\Delta y_n}{2} + R_{xy}}{-\frac{\Delta y_n}{2} + R_{xy}} \right] \tag{9a}$$

where

$$R_{xy} = \left\{ \left[\frac{\Delta x_n}{2}\right]^2 + \left[\frac{\Delta y_n}{2}\right]^2 \right\}^{\frac{1}{2}}$$

with the value

$$B_{nn} = 8w \ \ell n(1 + \sqrt{2}) \tag{9b}$$

for the special case $\Delta x_n = \Delta y_n = 2w$. Similarly, for m ≠ n, (8c) becomes

$$B_{mn} = \int_{x_n - \frac{\Delta x_n}{2}}^{x_n + \frac{\Delta x_n}{2}} \int_{y_n - \frac{\Delta y_n}{2}}^{y_n + \frac{\Delta y_n}{2}} \frac{dx'dy'}{[(x_m-x')^2+(y_m-y')^2]^{\frac{1}{2}}}$$

$$= \Bigg[\ell \cdot \ell n \left[\frac{d + (d^2+\ell^2)^{\frac{1}{2}}}{c + (c^2+\ell^2)^{\frac{1}{2}}} \right]$$

$$+ d \cdot \ell n \cdot [\ell + (d^2+\ell^2)^{\frac{1}{2}}] - c \cdot \ell n\, [\ell + (c^2+\ell^2)^{\frac{1}{2}}] \Bigg]_{x'=x_n - \frac{\Delta x_n}{2}}^{x'=x_n + \frac{\Delta x_n}{2}} \qquad (9c)$$

where

$$\ell = x_m - x' ,$$

$$c = y_m - (y_n - \frac{\Delta y_n}{2}) \quad ,$$

$$d = y_m - (y_n + \frac{\Delta y_n}{2}) \quad .$$

The integral (8d) can be evaluated analytically too. In the case of a rectangular aperture, such as that in Fig. 2, C_{mq} is evaluated individually for ΔC_q on each side; for example when ΔC_q falls on $(-\frac{a}{2}, y) - \frac{b}{2} \leq y \leq \frac{b}{a}$, or on $(+\frac{a}{2}, y)$, $-\frac{b}{2} \leq y \leq \frac{b}{a}$, one obtains

$$C_{mq} = -\frac{1}{4\pi} \int_{\Delta C_q} \ell n[(x_c-x_m)^2 + (y_c-y_m)^2]dy_c$$

$$= -\frac{1}{4\pi} \Bigg[(y_c-y_m)\ \ell n[(x_c-x_m)^2 + (y_c-y_m)^2]$$

$$- 2(y_c-y_m) + 2(x_c-x_m)\ \tan^{-1}\left(\frac{y_c-y_m}{x_c-x_m}\right) \Bigg]_{\Delta C_q(y_c)} \quad .$$

Similarly, C_{mq} can be evaluated readily with ΔC_q on $(x, -\frac{b}{2})$, $-\frac{a}{2} \leq x \leq \frac{a}{2}$, or on $(x, \frac{b}{2})$, $-\frac{a}{2} \leq x \leq \frac{a}{2}$, by interchanging the x and y variables in above.

SECTION III

AUXILIARY EQUATION

Solutions to (8a) and (8b) are not unique unless the auxiliary equation ((49) of [1]) is imposed:

$$\lim_{\overline{r}\uparrow\overline{r}_p} \left\{ \mathrm{div}_t \oint_C \overline{\Psi}_o(\overline{r}_c) \times \hat{u}_z g(\overline{r}_c, \overline{r}) d\ell(\overline{r}_c) \right\} = 0 \quad , \tag{10}$$

for $\overline{r} \in \overline{A}$ and all $\overline{r}_p \in C$, where div_t is the transverse (to z) divergence operator and where $\overline{\Psi}_o$ is the auxiliary vector

$$\overline{\Psi}_o = \Psi_{ox} \hat{u}_x + \Psi_{oy} \hat{u}_y \quad . \tag{11}$$

In view of (10), Ψ_{ox} can in principle be expressed as a function of Ψ_{oy} and thereby a reduction in the number of unknowns from four tb three is achieved in (2).

To circumvent numerical difficulties associated with (10), one converts $\overline{\Psi}_o$ from a vector represented in Cartesian components to one having components normal and tangential to the contour C:

$$\overline{\Psi}_o = \Psi_{ox}\hat{u}_x + \Psi_{oy}\hat{u}_y = \Psi_{on}\hat{u}_n + \Psi_{os}\hat{u}_s \quad , \tag{12}$$

where $\hat{u}_n$ is outward normal to C and $\hat{u}_s$ is positive tangential to C as depicted in Fig. 3. The integrand of (10) can be written as

$$\mathrm{div}_t \left[\overline{\Psi}_o(\overline{r}_c) \times \hat{u}_z g(\overline{r}_c, \overline{r}) \right]_{\overline{r}=\overline{r}_p}$$

$$= \Psi_{on} \frac{\partial}{\partial s} g(\overline{r}_c, \overline{r}_p) - \Psi_{os} \frac{\partial}{\partial n} g(\overline{r}_c, \overline{r}_p) \quad , \tag{13}$$

$$\overline{r}_p \in C$$

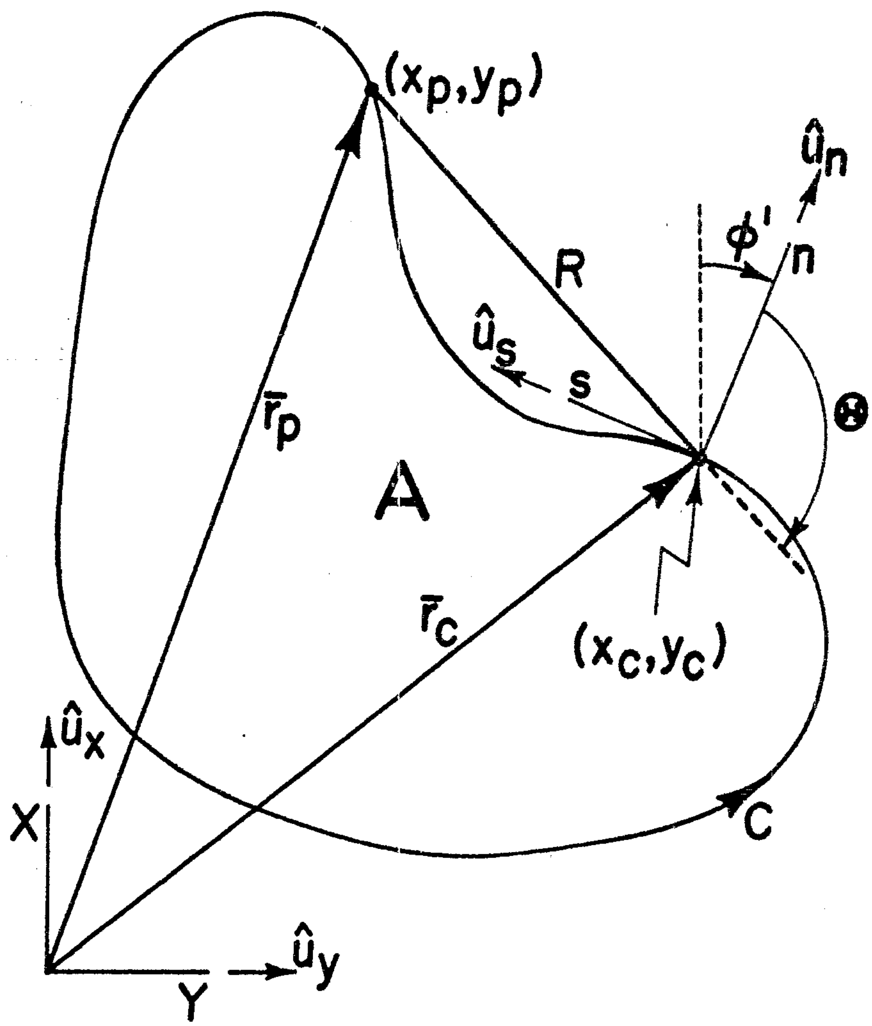

Figure 3. Tangential and Normal Coordinate System

and, furthermore, one can show from

$$\frac{\partial}{\partial n} g = \frac{\partial}{\partial x_c} g \cos \phi' + \frac{\partial}{\partial y_c} g \sin \phi'$$

and

$$\frac{\partial}{\partial s} g = \frac{\partial}{\partial x_c} g \sin \phi' - \frac{\partial}{\partial y_c} g \cos \phi'$$

that

$$\frac{\partial}{\partial n} g(\overline{r}_p, \overline{r}_c) = \frac{\cos \Theta(r_p, r_c)}{2\pi R(\overline{r}_p, \overline{r}_c)} \tag{14a}$$

and

$$\frac{\partial}{\partial s} g(\overline{r}_p, \overline{r}_c) = -\frac{\sin\Theta(\overline{r}_p, \overline{r}_c)}{2\pi R(\overline{r}_p, \overline{r}_c)} \tag{14b}$$

where Θ and ϕ' are defined in Fig. 3. Using (13) and (14) in (10), one can obtain the form below for the auxiliary condition

$$-\frac{1}{2\pi} \oint_C \left\{ \Psi_{on}(\overline{r}_c) \frac{\sin \Theta(\overline{r}_p, \overline{r}_c)}{R(\overline{r}_p, \overline{r}_c)} + \Psi_{os}(\overline{r}_c) \frac{\cos \Theta(\overline{r}_p, \overline{r}_c)}{R(\overline{r}_p, \overline{r}_c)} \right\} d\ell(\overline{r}_c) = 0 \tag{15}$$

which is recognized to be an improper integral and which can be converted to the expression below involving a deleted integral:

$$-\frac{1}{2} \Psi_{os}(\overline{r}_p) - \frac{1}{2\pi} \oint_C \left\{ \Psi_{os}(\overline{r}_c) \frac{\cos \Theta(\overline{r}_p, \overline{r}_c)}{R(\overline{r}_p, \overline{r}_c)} + \Psi_{on}(\overline{r}_c) \frac{\sin \Theta(\overline{r}_p, \overline{r}_c)}{R(\overline{r}_p, \overline{r}_c)} \right\} d\ell(\overline{r}_c) = 0 \quad . \tag{16}$$

Ψ_{0s} and Ψ_{0n} may be expressed in terms of the piecewise constant functions (6c) on the contour as

$$\Psi_{os}(\overline{r}_c) \doteq \sum_{q=1}^{Q} S_{oq} P_q \tag{17a}$$

and

$$\Psi_{on}(\overline{r}_c) \doteq \sum_{q=1}^{Q} N_{oq} P_q \ . \tag{17b}$$

Now one uses (17) in (16) and evaluates the resulting expression at match points (x_{pt}, y_{pt}), $t = 1, 2, \cdots, T(T=Q)$, on the contour C to obtain the matrix equation below relating $\{S_{oq}\}$ to $\{N_{oq}\}$

$$[D_{tq}] \ [S_{oq}] \ + \ [E_{tq}] \ [N_{oq}] \ = \ [0] \tag{18}$$

where

$$D_{tq} = \begin{cases} -\dfrac{1}{2\pi} \displaystyle\int_{\Delta C_q} \dfrac{\cos \Theta(\overline{r}_{pt}, \overline{r}_c)}{R(\overline{r}_{pt}, \overline{r}_c)} \, d\ell(\overline{r}_c) \ , & t \neq q \\ -\dfrac{1}{2} \ , & t = q \end{cases}$$

and

$$E_{tq} = \begin{cases} -\dfrac{1}{2\pi} \displaystyle\int_{\Delta C_q} \dfrac{\sin \Theta(\overline{r}_{pt}, \overline{r}_c)}{R(\overline{r}_{pt}, \overline{r}_c)} \, d\ell(\overline{r}_c) \ , & t \neq q \ , \\ 0 \ , & t = q \end{cases}$$

with

$$\overline{r}_{pt} = x_{pt}\hat{u}_x + y_{pt}\hat{u}_y$$

locating the center of the $t^{\underline{th}}$ interval ΔC_t. Matrix Equation (18) can be used to determine $[S_{oq}]$ in terms of $[N_{oq}]$:

$$[S_{oq}] \quad = \; - \; [D_{tq}]^{-1} \, [E_{tq}] \, [N_{oq}] \; . \tag{19}$$

From (19) and (6) plus the relationships,

$$\Psi_{ox}(\bar{r}_c) = \Psi_{os}(\bar{r}_c) \sin \phi'(\bar{r}_c) + \Psi_{on}(\bar{r}_c) \cos \phi'(\bar{r}_c)$$

and

$$\Psi_{oy}(\bar{r}_c) = - \Psi_{os}(\bar{r}_c) \cos \phi'(\bar{r}_c) + \Psi_{on}(\bar{r}_c) \sin \phi'(\bar{r}_c) \quad ,$$

one may eliminate the unknowns

$\{S_{oq}\}$ and write the matrices $[X_{oq}]$ and $[Y_{oq}]$ in terms of $\{N_{oq}\}$:

$$[X_{oq}] = -[D_{tq}]^{-1} \, [E_{tq}] \, [N_{oq} \sin \phi' \, (\bar{r}_{cq})]$$

$$+ \, [N_{oq} \cos \phi' (\bar{r}_{cq})] \tag{20a}$$

and

$$[Y_{oq}] = [D_{tq}]^{-1} \, [E_{tq}] \, [N_{oq} \cos \phi' \, (\bar{r}_{cq})]$$

$$+ \, [N_{oq} \sin \phi' \, (\bar{r}_{cq})] \quad , \tag{20b}$$

where $\bar{r}_{cq}$ is the location of the midpoint of the $q^{\underline{th}}$ interval ΔC_q on C. The significance of (20) is that the auxiliary condition provides a means of expressing Ψ_{ox} in terms of Ψ_{oy}, thus effectively reducing the number of unknown quantities in (2).

SECTION IV

NUMERICAL SOLUTION

Now attention is turned to the determination of the coefficients $\{M_{oxm}\}$ and $\{M_{oyn}\}$ from knowledge of which one can readily calculate the desired zeroth order magnetic current components M_{ox} and M_{oy}. In matrix form, the set (8a) and (8b) becomes

$$\begin{bmatrix} [B_{mn}] & [C_{mq}] & [0] & [0] \\ [0] & [0] & [C_{mq}] & [B_{mn}] \end{bmatrix} \begin{bmatrix} [M_{oxn}] \\ [X_{oq}] \\ [Y_{oq}] \\ [M_{oyn}] \end{bmatrix} = \begin{bmatrix} [F_{xm}] \\ [F_{ym}] \end{bmatrix} \tag{21}$$

where

$$F_{xm} = -2\pi y_m e_z^i \tag{22a}$$

and

$$F_{ym} = +2\pi x_m e_z^i \ . \tag{22b}$$

Since both $[X_{oq}]$ and $[Y_{oq}]$ depend upon $[N_{oq}]$ via Equations (20a) and (20b), Equation (21) can be simplified to

$$\begin{bmatrix} [B_{mn}] & [C'_{mq}] & [0] \\ [0] & [C''_{mq}] & [B_{mn}] \end{bmatrix} \begin{bmatrix} [M_{oxn}] \\ [N_{oq}] \\ [M_{oyn}] \end{bmatrix} = \begin{bmatrix} [F_{xm}] \\ [F_{ym}] \end{bmatrix} \tag{23}$$

where C'_{mq} and C''_{mq} are defined so that

$$[C_{mq}]\ [X_{oq}] = [C'_{mq}]\ [N_{oq}]$$

and

$$[C_{mq}]\ [Y_{oq}] = [C''_{mq}]\ [N_{oq}]$$

which requires

$$[C'_{mq}] = [C_{mq}] \left\{ -[D_{tq}]^{-1} [E_{tq}] [\sin \phi'(\bar{r}_{cq})] + [\cos \phi'(\bar{r}_{cq})] \right\}$$

and

$$[C''_{mq}] = [C_{mq}] \left\{ [D_{tq}]^{-1} [E_{tq}] [\cos \phi'(\bar{r}_{cq})] + [\sin \phi'(\bar{r}_{cq})] \right\} .$$

In the matrix equation (23), there are three column vectors $[M_{oxn}]$, $[N_{oq}]$, and $[M_{oyn}]$ and the matrix is of the size (N+N)x(N+Q+N) which can be solved subject to the boundary condition (4) which reduces to

$$M_{oxq} \cos \phi'(\bar{r}_{cq}) + M_{oyq} \sin \phi'(\bar{r}_{cq}) = 0 , \qquad q = 1, 2, \cdots, Q \tag{24}$$

where the subscript q on M_{oxq} and M_{oyq} implies that (24) is to be enforced only at the center points of those patches which are adjacent to the contour C. The additional Q equations needed to ensure that (23) is solvable are supplied by (24) and, by standard matrix operations, one may determine $\{M_{oxn}\}$ and $\{M_{oyn}\}$ and, subsequently, M_{ox} and M_{oy}.

SECTION V

FIRST ORDER SOLUTION

For obtaining the first order solution $\overline{M}_1 = \hat{x}M_{1x} + \hat{y}M_{1y}$ one solves the integral equation (40c) and (41) with the corresponding auxiliary condition and boundary condition defined in equations (42) and (43) of [1]. Due to the similarity of the first order integral equations, the method and the solution procedure are identical to that discussed above for the zeroth order solution with the single exception that the forcing functions (right-hand sides) in (2a) and (2b) are replaced, respectively, by

$$j \frac{2\pi}{1-\cos^2\gamma} [\cos\gamma(x\cos\alpha + y\cos\beta)^2 e_y^i + 2xy\, e_z^i \cos\alpha] \tag{25a}$$

and

$$-j \frac{2\pi}{1-\cos^2\gamma} [\cos\gamma(x\cos\alpha + y\cos\beta)^2 e_x^i + 2\,xy\, e_z^i \cos\beta]. \tag{25b}$$

Similarly, for the special case of normal incidence, integral equations (44) and (45) are used instead of (41) and (42) of [1] for the first order solution, and the above procedure is applied. The reader is referred to [1] for definition of symbols in (25).

SECTION VI

ALTERNATE INTEGRAL EQUATIONS

In an effort to improve the convergence rate of solutions for the small aperture problem, modifications have been made in the final integral equations of Note 149. The modifications are minor and are reflected only in the forms of the homogeneous solutions of (26) and (27) of [1] and, also, of the particular solution of (27). Determination of these solutions is discussed in detail in [1] and, therefore, the alternate integral equations are given directly on the following page. Notice that (26b) and (27b), the auxiliary equations, are different from the former expressions. Also, the forcing function of (27a) involves integration of the Green's function g which fortunately can be performed analytically for several shapes of interest. Even though (26) and (27) differ from the integral equations of [1], they yield to the general solution technique presented in this note.

$$\iint_A \bar{M}_o(\bar{r}')R^{-1}(\bar{r}, \bar{r}')ds' - \oint_C \bar{\Psi}_o(\bar{r}_c)g(\bar{r}, \bar{r}_c)d\ell(\bar{r}_c) = \bar{0} \quad , \qquad \bar{r} \in A \quad ; \tag{26a}$$

$$\oint_C [\bar{\Psi}_o(\bar{r}_c) \times \hat{u}_z] \cdot \text{grad}_t\, g(\bar{r}_c, \bar{r})\, d\ell(\bar{r}_c) = 4\pi e_z^i \quad , \qquad \bar{r} \in C \quad ; \tag{26b}$$

and

$$\iint_A \bar{M}_1(\bar{r}')\, R^{-1}(\bar{r}, \bar{r}')ds' - \oint_C \bar{\Psi}_1(\bar{r}_c)g(\bar{r}, \bar{r}_c)\, d\ell(\bar{r}_c) = j4\pi \cos\gamma(\hat{u}_z \times \bar{e}^{\,i}) \iint_A g(\bar{r}', \bar{r})ds' \quad , \quad \bar{r} \in \bar{A} \;; \tag{27a}$$

$$\oint_C [\bar{\Psi}_1(\bar{r}_c) \times \hat{u}_z] \cdot \text{grad}_t\, g(\bar{r}_c, \bar{r})d\ell(\bar{r}_c) = -j4\pi e_z^i(\hat{u} \cdot \bar{r})_o \quad , \qquad \bar{r} \in C \quad . \tag{27b}$$

SECTION VII

SUMMARY

The method outlined here has been implemented on a computer and solutions have been obtained for the original aperture integral equations as well as for the alternate equations. Results are presented in Interaction Note 213.

REFERENCES

1. Butler, C.M., "Formulation of Integral Equations for an Electrically Small Aperture in a Conducting Screen," Interaction Note 149, Air Force Weapons Laboratory, Kirtland, N.M.; Dec., 1973.

2. Harrington, R.F., Field Computations by Moment Methods, MacMillan, New York; 1968.