Interaction Notes

Note 229

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On the Eigenmode Expansion Method for Electromagnetic Scattering and Antenna Problems, Part I: Some Basic Relations for Eigenmode Expansions, and Their Relation to the Singularity Expansion

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Abstract

This note considers the representation of the solution of electromagnetic scattering and antenna problems in terms of eigenvalues and eigenmodes of electromagnetic integral equations and their method of moments numerical approximation. Certain features of the eigenmode expansion are related to pole terms in the singularity expansion method. For impedance operators eigenimpedances are defined and shown to be useful for considering the effects of impedance loading on the object response, thereby giving an approach to synthesizing the object broadband and transient response characteristics.

Foreword

I have been working on this eigenfunction expansion method for some time now and the present note represents a start on the problem. I would like to thank various people with whom I have discussed this topic including R. Garbacz, F. Tesche, K. S. H. Lee, L. Marin, R. Harrington, and R. Mittra.

. . . . . .

I sing of arms and the man who came of old, a fated wanderer, from the coasts of Troy to Italy and the shore of Lavinium; hard driven on land and on the deep by the violence of heaven, by reason of cruel Juno's unforgetful anger, and hard bestead in war also, ere he might found a city and carry his gods into Latium; from whom is the Latin race, the lords of Alba, and high-embattled Rome.

Muse, tell me why, for what attaint of her diety, or in what vexation, did the Queen of heaven urge on a man excellent in goodness to circle through all those afflictions, to face all those toils? Is anger so fierce in celestial spirits?

Vergil
The Aeneid
trans. by J. W. MacKail

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I. Introduction

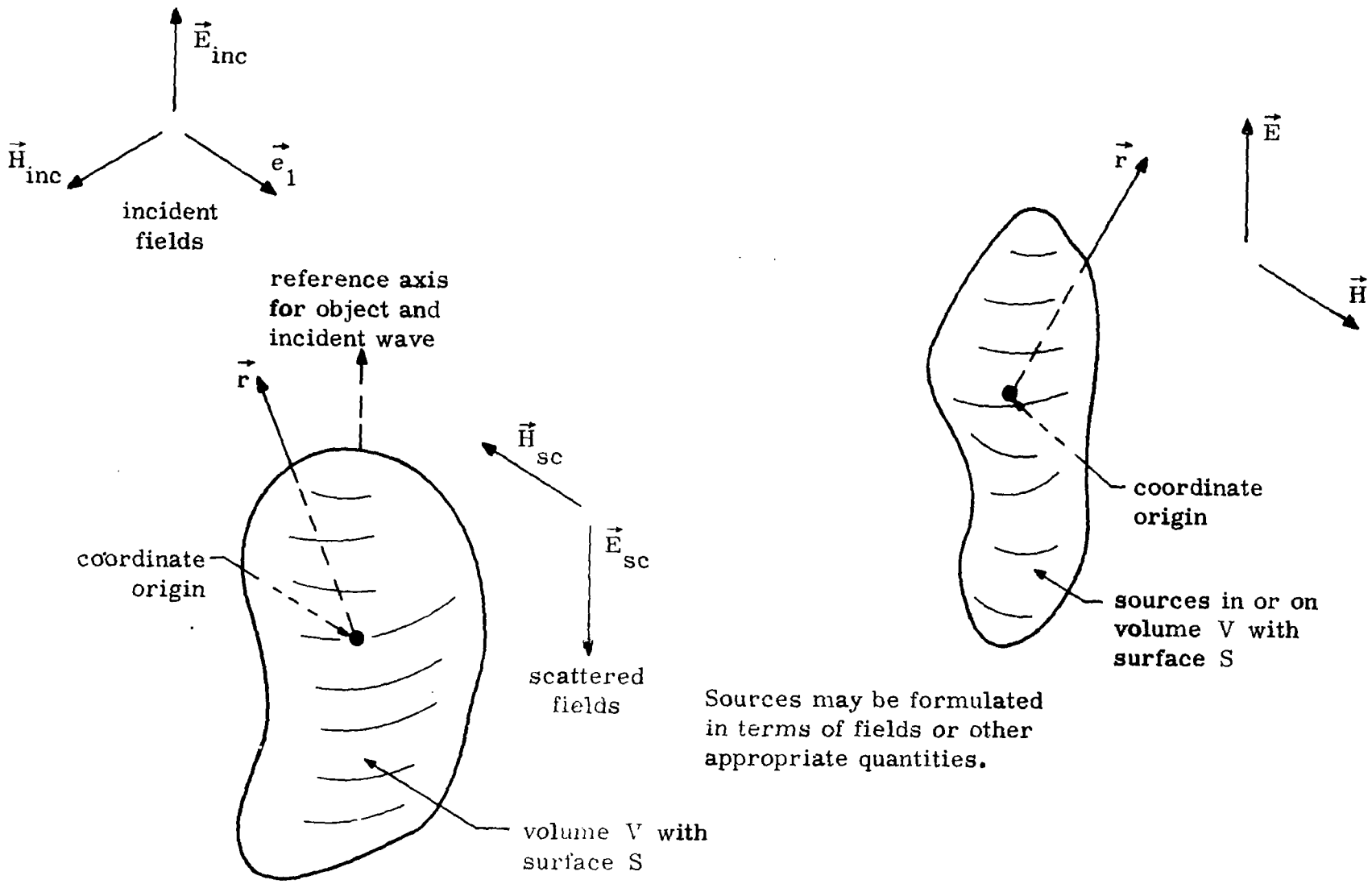
Once upon a time* the investigation of electromagnetic interaction with objects was begun. In the context of the nuclear electromagnetic pulse (EMP) that question has been quite important. Due to the complexity of the problem of EMP interaction with electronic systems techniques are needed to handle such problems with a maximum of insight into how to decompose the problem into some kinds of smaller problems. This decomposition allows parts of the problem to be considered separately and often simplifies the numerical aspects of the computation.

Not too long ago the singularity expansion method (SEM) was introduced to aid in the solution of electromagnetic problems.[1,2,3] In such a representation the solution is expressed as a collection of singularities (poles, branch cuts, entire functions, etc.) in the complex frequency plane. This note deals with what might be called the eigenmode expansion method (EEM). In this representation one finds the eigenmodes and eigenvalues of appropriate electromagnetic integral equations and uses these to represent the solution as an eigenmode series. In considering the eigenmode expansions comparisons of various aspects can be made to the singularity expansion. Thereby more insight is obtained into both approaches.

Much has been learned both theoretically and numerically about SEM but much is still to be learned.[13] Some work has been done on what can be referred to as characteristic modes in which the equation defining the modes is somewhat different from that used later in this note.[7,8,9] The prefix "eigen" is used to distinguish the modes in this note from the "characteristic modes" considered by others.

*This opening was suggested by Maj. Bill Adams, Hq DNA, while we were both flying on Continental flight 94 from San Francisco to Albuquerque on 16 May 1974 as I was writing this note.

In this note we consider the general eigenmode formulation for some antenna or scattering problem as indicated in figure 1.1. While one is often interested in finite size objects in free space as in figure 1.1 the method discussed here applies somewhat more generally to other problems described by the same forms of integral equations. This note begins by defining various terms in the eigenmode expansion and then relates them to pole terms in the singularity expansion. An important application of the eigenmode expansion of impedance operators is made to the synthesis of impedance loading of objects. Of course this note just introduces these topics. Practical examples will need to be considered and the concepts can perhaps be further developed.



A. Scattered Fields

B. Radiated Fields

Figure 1.1. Radiation and Scattering of Electromagnetic Fields by an Object

II. Some Preliminaries from SEM

Let us begin with some basic definitions. Start from an integral equation written in the general form (perhaps normalized and with the space coordinates often suppressed)[6, 11]

$$\left\langle \overset{\sim}{\overrightarrow{\overrightarrow{\Gamma}}}(\vec{r}, \vec{r}'; s) \; ; \; \overset{\sim}{\vec{U}}(\vec{r}', s) \right\rangle = \overset{\sim}{\vec{I}}(\vec{r}, s) \tag{2.1}$$

where $\overset{\sim}{\overrightarrow{\overrightarrow{\Gamma}}}(\vec{r}, \vec{r}'; s)$ is the kernel of the integral equation (related to the dyadic Green's function), $\overset{\sim}{\vec{I}}(\vec{r}, s)$ is the excitation function in some normalized form such as the negative of the incident electric field or an antenna gap source field taken as an appropriate delta function excitation in time domain, and $\overset{\sim}{\vec{U}}(\vec{r}', s)$ is the normalized response related to the current density (or surface current density) on the object of interest. The two-sided Laplace transform variable with respect to time is s, the complex frequency. The domain of integration is the set of volume(s) and/or surface(s) describing the scatterer or antenna of interest. Typically the surrounding medium is taken as free space.

In numerical form from the method of moments[17, 1] the integral equation is written as a matrix equation of the form

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{U}_n(s)) = (\tilde{I}_n(s)) \qquad n, m = 1, 2, \cdots, N \tag{2.2}$$

This is actually only an approximation to the integral equation (equation 2.1), albeit a very useful one. In certain forms of the method of moments (MoM as coined by A. Poggio) the vectors $(\tilde{U}_m(s))$ and $(\tilde{I}_n(s))$ are some discretization of the continuous functions $\overset{\sim}{\vec{U}}(\vec{r}', s)$ and $\overset{\sim}{\vec{I}}(\vec{r}, s)$; such cases are analogous to the trapezoidal rule, Simpson's rule, etc., of numerical integration and for conceptual purposes can be thought to be "physically meaningful." More general functional expansions in MoM are also possible in which the "components" of the numerical vectors $(\tilde{U}_n(s))$ and $(\tilde{I}_n(s))$ are the coefficients in such functional expansions.

The matrix form (equation 2.2) has demonstrated numerical utility. Furthermore it provides a way for one to investigate some of the mathematical properties of antenna, scattering, and other electromagnetic problems. For such theoretical purposes the matrix equation should be in some sense an accurate representation of the integral equation. One criterion of such accuracy is the convergence of the solution $(\tilde{U}_n(s))$ to $\tilde{\tilde{U}}(s)$ as $N \to \infty$. However not all forms of solution are convergent in this sense but are still numerically useful within certain restrictions.

An integral equation which can be approximated by a matrix equation with solution convergent to the true one as $N \to \infty$ can be defined as being matricizable. A matricizable integral equation can be considered exactly solvable by MoM. Of course one should consider for what values of the complex frequency s the integral equation is so matricizable and thereby solvable. There are many forms of integral equations and many types of antenna, scattering, etc., problems to which they can be applied.

An important set of properties of the solutions of electromagnetic problems is their properties as analytic functions (in the usual complex variable sense) of the complex frequency s. These properties form the basis of the singularity expansion method (SEM). For certain types of electromagnetic problems the solution has only poles in the finite s plane.[1,2] In other problems branch cuts are also included in the solution.[4] When considering a matricizable integral equation then one can consider for what portions of the s plane the integral equation is matricizable. Note that analytic continuation is applicable to this question.

SEM, in its basic form to date, has considered the analytic properties of general integral equations from the analytic properties of the determinant

$$\tilde{D}_N(s) \equiv d_N \det\left((\tilde{\Gamma}_{n,m}(s))\right) \tag{2.3}$$

where d_N is some convenient normalization constant which may be 1 or some convenient function of N to put formulas in various convenient forms and/or give $\tilde{D}_N(s)$ a convenient limiting form as $N \to \infty$. For cases that the determinant is an entire function of s the response has only poles s_α in the finite s plane found from the equation

$$\tilde{D}(s_\alpha) = 0 \tag{2.4}$$

where ideally the limit has been taken for $N \to \infty$. For such s_α if the poles are first order we have

$$(\tilde{\Gamma}_{n,m}(s_\alpha)) \cdot (\nu_n)_\alpha = (0_n) = (\mu_n)_\alpha \cdot (\tilde{\Gamma}_{n,m}(s_\alpha))$$

$$\left\langle \vec{\tilde{\Gamma}}(s_\alpha) ; \vec{\nu}_\alpha \right\rangle = \vec{0} = \left\langle \vec{\mu}_\alpha ; \vec{\tilde{\Gamma}}(s_\alpha) \right\rangle \tag{2.5}$$

The solution takes the form

$$(\tilde{U}_n(s)) = \sum_\alpha \tilde{\eta}_\alpha(s)(\nu_n)_\alpha (s - s_\alpha)^{-n_\alpha} + \text{non pole terms}$$

$$\vec{\tilde{U}}(\vec{r}, s) = \sum_\alpha \tilde{\eta}_\alpha(s)\vec{\nu}_\alpha(\vec{r})(s - s_\alpha)^{-n_\alpha} + \text{non pole terms} \tag{2.6}$$

with $n_\alpha = 1$ for first order poles. The coupling coefficient for a first order pole is

$$\tilde{\eta}_\alpha(s_\alpha) = \frac{(\mu_n)_\alpha \cdot (\tilde{I}_n(s_\alpha))}{(\mu_n)_\alpha \cdot (\tilde{\Gamma}_{n,m})_{1_\alpha} \cdot (\nu_n)_\alpha} \tag{2.7}$$

$$\tilde{\eta}_\alpha(s_\alpha) = \frac{\left\langle \vec{\mu} ; \tilde{\vec{I}}(s_\alpha) \right\rangle}{\left\langle \vec{\mu} ; \tilde{\vec{\Gamma}}_{1_\alpha} ; \vec{\nu} \right\rangle}$$

where

$$\begin{aligned} (\tilde{\Gamma}_{n,m})_{\ell_\alpha} &\equiv \frac{1}{\ell!} \frac{d^\ell}{ds^\ell} (\tilde{\Gamma}_{n,m}(s)) \Big|_{s=s_\alpha} \qquad \ell = 1, 2, 3, \cdots \\ \tilde{\vec{\Gamma}}_{\ell_\alpha}(\vec{r}, \vec{r}') &\equiv \frac{1}{\ell!} \frac{d^\ell}{ds^\ell} \tilde{\vec{\Gamma}}(\vec{r}, \vec{r}'; s) \Big|_{s=s_\alpha} \qquad \ell = 1, 2, 3, \cdots \end{aligned} \tag{2.8}$$

Various forms are possible for $s \neq s_\alpha$. Similar formulas can be developed for higher order poles.

If $(\tilde{\Gamma}_{n,m}(s))$ and $\tilde{\vec{\Gamma}}(\vec{r}, \vec{r}'; s)$ are symmetric then one can set

$$\begin{aligned} (\mu_n)_\alpha &= (\nu_n)_\alpha \\ \vec{\mu}_\alpha(r) &= \vec{\nu}_\alpha(\vec{r}) \end{aligned} \tag{2.9}$$

and the formulas and computations simplify somewhat. Note that the E field integral equation operator is symmetric whereas the corresponding one for the H field is not.

III. Eigenmodes: Vector and Dyadic

An interesting approach to the solution of the integral equation 2.1 invokes an eigenmode expansion. Let us define our eigenmode equations as

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta = \tilde{\lambda}_\beta(s)(\tilde{R}_n(s))_\beta$$

$$\left\langle \tilde{\vec{\Gamma}}(s) ; \tilde{\vec{R}}_\beta(s) \right\rangle = \tilde{\lambda}_\beta(s)\tilde{\vec{R}}_\beta(s) \tag{3.1}$$

where $\tilde{\lambda}_\beta$ is the βth eigenvalue and $\tilde{\vec{R}}_\beta$ the corresponding eigenmode. As in the previous section the numerical form from MoM is assumed to converge to the continuous form for $\tilde{\lambda}_\beta$ and the modes as $N \to \infty$. Having defined the right side modes $\tilde{\vec{R}}_\beta(\vec{r}, s)$ the left side modes are

$$(\tilde{L}_n(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) = \tilde{\lambda}_\beta(s)(\tilde{L}_n(s))_\beta$$

$$\left\langle \tilde{\vec{L}}_\beta(s) ; \tilde{\vec{\Gamma}}(s) \right\rangle = \tilde{\lambda}_\beta(s)\tilde{\vec{L}}_\beta(s) \tag{3.2}$$

Note that the left and right side eigenmodes have the same eigenvalues since both cases have the same characteristic equation

$$\det((\tilde{\Gamma}_{n,m}(s)) - \lambda_\beta(s)(\delta_{n,m})) = 0 \tag{3.3}$$

where the Kronecker delta is

$$\delta_{n,m} = \begin{cases} 1 & \text{for} \quad n = m \\ 0 & \text{for} \quad n \neq m \end{cases} \tag{3.4}$$

The analogous operator is $\vec{\vec{I}}\delta(\vec{r} - \vec{r}')$ so that we can symbolically indicate

$$\det(\overleftrightarrow{\Gamma}(\vec{r},\vec{r}';s) - \tilde{\lambda}_\beta(s)\overleftrightarrow{I}\delta(\vec{r}-\vec{r}')) = 0 \tag{3.5}$$

where det operates over the range for $\vec{r},\vec{r}'$ of the integral equation. In a limiting sense as $N \to \infty$ the definition of a matricizable integral equation can be used to define the operation of determinant for the continuous operator case as in equation 3.5.

Consider some s for which

$$d_N \det\left((\tilde{\Gamma}_{n,m}(s))\right) \neq 0 \tag{3.6}$$

or in the limit

$$\tilde{D}(s) \neq 0 \tag{3.7}$$

Then the matrix $(\tilde{\Gamma}_{n,m}(s))$ has a unique inverse. Since the matrix elements are finite then the N eigenvalues $\tilde{\lambda}_\beta(s)$ are all finite. Since

$$\det\left((\tilde{\Gamma}_{n,m}(s))\right) = \prod_{\beta=1}^{N} \tilde{\lambda}_\beta(s) \tag{3.8}$$

then all eigenvalues are non zero for $s \neq s_\alpha$.

The eigenvalues may or may not be all distinct. If the $\tilde{\lambda}_\beta(s)$ are all distinct then the corresponding eigenvectors are unique except for a scaling constant. If some (say M) of the eigenvalues are degenerate (say one set of these degenerate to one value) then the corresponding set of eigenvectors (M_e with $1 \leq M_e \leq M$) can be formed.

For simplicity let the N eigenvalues be distinct for the moment. Then we have for two indices β and β'

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta = \tilde{\lambda}_\beta(s)(\tilde{R}_n(s))_\beta$$

$$(\tilde{L}_n(s))_{\beta'} \cdot (\tilde{\Gamma}_{n,m}(s)) = \tilde{\lambda}_{\beta'}(s)(\tilde{L}_n(s))_{\beta'} \tag{3.9}$$

Dot multiply these two equations by $(\tilde{L}_n(s))_{\beta'}$ and $(\tilde{R}_n(s))_\beta$, respectively, to give

$$(\tilde{L}_n(s))_{\beta'} \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta = \tilde{\lambda}_\beta(s)(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_{\beta'}$$

$$(\tilde{L}_n(s))_{\beta'} \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta = \tilde{\lambda}_{\beta'}(s)(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_{\beta'} \tag{3.10}$$

Subtracting gives

$$0 = [\tilde{\lambda}_\beta(s) - \tilde{\lambda}_{\beta'}(s)](\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_{\beta'} \tag{3.11}$$

and so if $\tilde{\lambda}_\beta(s) \neq \tilde{\lambda}_{\beta'}(s)$ then

$$0 = (\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_{\beta'} \tag{3.12}$$

forming a biorthogonal set of eigenvectors.

In operator form the resulting equations are

$$0 = [\tilde{\lambda}_\beta(s) - \tilde{\lambda}_{\beta'}(s)]\langle \vec{\tilde{R}}_\beta(s) ; \vec{\tilde{L}}_{\beta'}(s) \rangle \tag{3.13}$$

which for $\tilde{\lambda}_\beta$'s $\neq \tilde{\lambda}_{\beta'}(s)$ gives

$$0 = \langle \vec{\tilde{R}}_\beta(s) ; \vec{\tilde{L}}_{\beta'}(s) \rangle \tag{3.14}$$

Note that orthogonality and biorthogonality as in equations 3.12 and 3.14 are often defined with one of the terms in the dot product or symmetric product complex conjugated.[17] The convention in equations 3.12 and 3.14 might be referred to as symmetric biorthogonal or symmetric orthogonal (if the two terms are from the same set of eigenvectors). The common convention of complex conjugating one of the terms gives sets of modes which could be termed conjugate biorthogonal or conjugate orthogonal. For present purposes the symmetric type is established as a convention and referred to as simply biorthogonal or orthogonal as appropriate.

Now define a set of dyadic eigenmodes (eigendyads) as

$$\begin{aligned} (\tilde{D}_{n,m}(s))_{\beta} &\equiv (\tilde{R}_n(s))_{\beta} (\tilde{L}_n(s))_{\beta} \\ \tilde{\vec{\vec{D}}}_{\beta}(\vec{r},\vec{r}';s) &\equiv \tilde{\vec{\vec{R}}}_{\beta}(\vec{r},s)\tilde{\vec{L}}_{\beta}(\vec{r}',s) \end{aligned} \tag{3.15}$$

In normalized form define

$$\begin{aligned} (\tilde{d}_{n,m}(s))_{\beta} &\equiv \frac{(\tilde{R}_n(s))_{\beta} (\tilde{L}_n(s))_{\beta}}{(\tilde{R}_n(s))_{\beta} \cdot (\tilde{L}_n(s))_{\beta}} = \frac{(\tilde{D}_{n,m}(s))_{\beta}}{(\tilde{R}_n(s))_{\beta} \cdot (\tilde{L}_n(s))_{\beta}} \\ \tilde{\vec{\vec{d}}}_{\beta}(\vec{r},\vec{r}';s) &\equiv \frac{\tilde{\vec{\vec{R}}}_{\beta}(\vec{r},s)\tilde{\vec{L}}_{\beta}(\vec{r}',s)}{\langle \tilde{\vec{R}}_{\beta}(s) ; \tilde{\vec{L}}_{\beta}(s) \rangle} = \frac{\tilde{\vec{\vec{D}}}_{\beta}(\vec{r},\vec{r}';s)}{\langle \tilde{\vec{R}}_{\beta}(s) ; \tilde{\vec{L}}_{\beta}(s) \rangle} \end{aligned} \tag{3.16}$$

which require the dot and symmetric products to be non zero (to be discussed later). These dyadic modes satisfy

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{D}_{n,m}(s))_\beta = (\tilde{D}_{n,m}(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) = \tilde{\lambda}_\beta(s)(\tilde{D}_{n,m}(s))_\beta$$

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{d}_{n,m}(s))_\beta = (\tilde{d}_{n,m}(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) = \tilde{\lambda}_\beta(s)(\tilde{d}_{n,m}(s))_\beta$$

(3.17)

$$\langle \tilde{\vec{\Gamma}}(s) ; \tilde{\vec{D}}_\beta(s) \rangle = \langle \tilde{\vec{D}}_\beta(s) ; \tilde{\vec{\Gamma}}(s) \rangle = \tilde{\lambda}_\beta(s)\tilde{\vec{D}}_\beta(s)$$

$$\langle \tilde{\vec{\Gamma}}(s) ; \tilde{\vec{d}}_\beta(s) \rangle = \langle \tilde{\vec{d}}_\beta(s) ; \tilde{\vec{\Gamma}}(s) \rangle = \tilde{\lambda}_\beta(s)\tilde{\vec{d}}_\beta(s)$$

Thus the dyadic modes apply equally to both sides of the integral equation operator or matrix.

There are the orthogonality relations

$$(\tilde{d}_{n,m}(s))_\beta \cdot (\tilde{d}_{n,m}(s))_{\beta'} = \begin{cases} (\tilde{d}_{n,m}(s))_\beta & \text{for} \quad \beta = \beta' \\ 0 & \text{for} \quad \beta \neq \beta' \end{cases}$$

(3.18)

$$\langle \tilde{\vec{d}}_\beta(s) ; \tilde{\vec{d}}_{\beta'}(s) \rangle = \begin{cases} \tilde{\vec{d}}_\beta(s) & \text{for} \quad \beta = \beta' \\ 0 & \text{for} \quad \beta \neq \beta' \end{cases}$$

The normalized dyadic modes have the property

$$(\tilde{d}_{n,m}(s))^n_\beta = (\tilde{d}_{n,m}(s))_\beta$$

(3.19)

$$\tilde{\vec{d}}^{\,n}_\beta(s) = \tilde{\vec{d}}_\beta(s) \qquad n = 1, 2, 3, \cdots$$

where the nth power of a matrix is n - 1 dot products of n such matrices. For operators

$$\overset{\sim}{\vec{\vec{d}}}{}^{n}_{\beta}(s) \equiv \left\langle \overset{\sim}{\vec{\vec{d}}}_{\beta}(s) \,;\, \overset{\sim}{\vec{\vec{d}}}_{\beta}(s) \,;\, \cdots \,;\, \overset{\sim}{\vec{\vec{d}}}_{\beta} \right\rangle \tag{3.20}$$

with the operator included n times with n - 1 operations (indicated by ;).

If $(\tilde{\Gamma}_{n,m}(s))$ and $\overset{\sim}{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}';s)$ are symmetric then certain simplifications can be introduced as

$$(\tilde{L}_{n}(s))_{\beta} = (\tilde{R}_{n}(s))_{\beta}$$

$$\overset{\sim}{\vec{L}}_{\beta}(s) = \overset{\sim}{\vec{R}}_{\beta}(s)$$

$$\left\langle \overset{\sim}{\vec{R}}_{\beta}(s) \,;\, \overset{\sim}{\vec{R}}_{\beta'}(s) \right\rangle = 0 \quad \text{for} \quad \tilde{\lambda}_{\beta}(s) \neq \tilde{\lambda}_{\beta'}(s)$$

$$(\tilde{D}_{n,m}(s))_{\beta} = (\tilde{R}_{n}(s))_{\beta} (\tilde{R}_{n}(s))_{\beta}$$

$$\overset{\sim}{\vec{\vec{D}}}_{\beta}(\vec{r},\vec{r}';s) = \overset{\sim}{\vec{R}}_{\beta}(\vec{r},s)\overset{\sim}{\vec{R}}_{\beta}(\vec{r}',s) \tag{3.21}$$

$$(\tilde{d}_{n,m}(s))_{\beta} = \frac{(\tilde{D}_{n,m}(s))_{\beta}}{(\tilde{R}_{n}(s))_{\beta} \cdot (\tilde{R}_{n}(s))_{\beta}}$$

$$\overset{\sim}{\vec{\vec{d}}}_{\beta}(\vec{r},\vec{r}';s) = \frac{\overset{\sim}{\vec{\vec{D}}}_{\beta}(\vec{r},\vec{r}';s)}{\left\langle \overset{\sim}{\vec{R}}_{\beta}(s) \,;\, \overset{\sim}{\vec{R}}_{\beta}(s) \right\rangle}$$

IV. Operator and Matrix Representation Including Inverse

The inverse matrix or operator can be found by solving the matrix equation in terms of the eigenvectors. Assuming linearly independent eigenvectors $(\tilde{L}_n(s))_\beta$ and $(\tilde{R}_n(s))_\beta$ each span an N dimensional linear vector space which contains both the solution $(\tilde{U}_n(s))$ and the excitation $(\tilde{I}_n(s))$ then we can write

$$(\tilde{U}_n(s)) = \sum_{\beta=1}^{N} \tilde{a}_\beta(s)(\tilde{R}_n(s))_\beta$$

$$(\tilde{I}_n(s)) = \sum_{\beta=1}^{N} \tilde{b}_\beta(s)(\tilde{R}_n(s))_\beta \qquad (4.1)$$

This is merely a property of N linearly independent N dimensional vectors.[16] Using the biorthogonal property (equations 3.12 and 3.14) we have

$$(\tilde{L}_n(s))_\beta \cdot (\tilde{U}_n(s)) = \tilde{a}_\beta(s)(\tilde{L}_n(s))_\beta \cdot (\tilde{R}_n(s))_\beta$$

$$(\tilde{L}_n(s))_\beta \cdot (\tilde{I}_n(s)) = \tilde{b}_\beta(s)(\tilde{L}_n(s))_\beta \cdot (\tilde{R}_n(s))_\beta \qquad (4.2)$$

This gives

$$(\tilde{U}_n(s)) = \sum_{\beta=1}^{N} \frac{(\tilde{L}_n(s))_\beta \cdot (\tilde{U}_n(s))_\beta}{(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta} (\tilde{R}_n(s))_\beta$$

$$(\tilde{I}_n(s)) = \sum_{\beta=1}^{N} \frac{(\tilde{L}_n(s))_\beta \cdot (\tilde{I}_n(s))_\beta}{(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta} (\tilde{R}_n(s))_\beta \qquad (4.3)$$

or

$$(\tilde{U}_n(s)) = \sum_{\beta=1}^{N} (\tilde{d}_{n,m}(s))_\beta \cdot (\tilde{U}_n(s))$$

$$(\tilde{I}_n(s)) = \sum_{\beta=1}^{N} (\tilde{d}_{n,m}(s))_\beta \cdot (\tilde{I}_n(s)) \qquad (4.4)$$

This is a convenient point to consider the product $(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta$. Assume that we can construct N independent eigenvectors $(\tilde{L}_n(s))_\beta$. Consider say the first of equations 4.3. The eigenvectors are not identically zero. Choose $(\tilde{U}_n(s))$ such that for some β we have $(\tilde{L}_n(s))_\beta \cdot (\tilde{U}_n(s)) \neq 0$, say let one component of $(\tilde{U}_n(s))$ be unity and the rest zero and have the non zero component not correspond to a zero component of $(\tilde{L}_n(s))$. Then if for that same β we were to have $(\tilde{R}_n(s)) \cdot (\tilde{L}_n(s)) = 0$ there would be no linear combination of the remaining $(\tilde{R}_n(s))_{\beta'}$ (for $\beta' \neq \beta$) which could equal $(\tilde{R}_n(s))_\beta$ and so necessarily $(\tilde{U}_n(s))$ would not exist ($(\tilde{U}_n(s)) = \infty$) which contradicts our hypothesis for $(\tilde{U}_n(s))$. For linearly independent eigenvectors the coefficients must exist for all expansions of N dimensional vectors. Hence

$$(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta \neq 0 \quad \text{for N linearly independent eigenvectors} \qquad (4.5)$$

Note that if one $\tilde{\lambda}_\beta(s) = 0$ the eigenvalues can still be distinct giving necessarily N independent eigenvectors. This last observation should be useful for SEM application since $s = s_\alpha$ implies at least one $\tilde{\lambda}_\beta(s_\alpha) = 0$ at a pole.

A matrix is said to be of simple structure if it has N linearly independent eigenvectors.[15] Such a matrix is similar to a diagonal matrix and thus can be called a diagonalizable matrix. It is this type of matrix which is assumed for our discussion. The matrix may be singular except

when considering its inverse. The eigenvalues may be degenerate. The important property is that it is diagonalizable, i.e., a matrix of simple structure. If the eigenvalues are all distinct then the matrix is diagonalizable. If the eigenvalues are not all distinct the matrix may still be diagonalizable and so values of s for which the $\tilde{\lambda}_\beta(s)$ are degenerate may require special attention. In this note it is usually assumed that the matrix of interest is diagonalizable, and similarly for the corresponding operator, unless otherwise stated.

Since $(\tilde{U}_n(s))$ and $(\tilde{I}_n(s))$ can separately be considered as arbitrary N dimensional complex vectors it follows from equations 4.4

$$(\delta_{n,m}) = \sum_{\beta=1}^{N} (\tilde{d}_{n,m}(s))_\beta$$

$$= \sum_{\beta=1}^{N} (\tilde{d}_{n,m}(s))^T_\beta \qquad (4.6)$$

where the transpose relation comes from $(\delta_{n,m})$ being symmetric and where superscript T indicates transpose as

$$(\tilde{d}_{n,m}(s))^T_\beta \equiv (\tilde{d}_{m,n}(s))_\beta \qquad (4.7)$$

with elements interchanged about the diagonal. Thus we have convenient representations for the identity matrix $(\delta_{n,m})$ in terms of the eigenvectors. Using the transpose dyadic modes both $(\tilde{U}_n(s))$ and $(\tilde{I}_n(s))$ can be expanded directly in terms of the left eigenvectors $(\tilde{L}_n(s))$. Note also that the identity matrix (or operator) is independent of s; the sum representation is then independent of s and in particular has no poles in the s plane.

Substituting the response vector $(\tilde{U}_n(s))$ from equations 4.1 into the matrix equation 2.2 we have

$$\sum_{\beta=1}^{N} \tilde{a}_\beta(s)(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta = (\tilde{I}_n(s))$$

$$= \sum_{\beta=1}^{N} \tilde{a}_\beta(s)\tilde{\lambda}_\beta(s)(\tilde{R}_n(s))_\beta \qquad (4.8)$$

Taking the dot product with $(\tilde{L}_n(s))_{\beta'}$ and interchanging β and β' gives

$$a_\beta(s)\tilde{\lambda}_\beta(s)(\tilde{L}_n(s))_\beta \cdot (\tilde{R}_n(s))_\beta = (\tilde{L}_n(s))_\beta \cdot (\tilde{I}_n(s)) \qquad (4.9)$$

Hence

$$(\tilde{U}_n(s)) = \sum_{\beta=1}^{N} \frac{1}{\tilde{\lambda}_\beta(s)} \frac{(\tilde{L}_n(s))_\beta \cdot (\tilde{I}_n(s))}{(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta} (\tilde{R}_n(s))_\beta$$

$$= \sum_{\beta=1}^{N} \frac{1}{\tilde{\lambda}_\beta(s)} (\tilde{d}_{n,m}(s))_\beta \cdot (\tilde{I}_n(s))$$

$$= (\tilde{\Gamma}_{n,m}(s))^{-1} \cdot (\tilde{I}_n(s)) \qquad (4.10)$$

Since $(\tilde{I}_n(s))$ can be chosen as an arbitrary N dimensional vector we have a representation of the inverse matrix as

$$(\tilde{\Gamma}_{n,m}(s))^{-1} = \sum_{\beta=1}^{N} \frac{1}{\tilde{\lambda}_\beta(s)} (\tilde{d}_{n,m}(s))_\beta \qquad (4.11)$$

Substituting the response vector $(\tilde{U}_n(s))$ from equations 4.4 into the matrix equation 2.2 we have

$$\sum_{\beta=1}^{N} (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{d}_{n,m}(s))_{\beta} \cdot (\tilde{U}_n(s)) = (\tilde{I}_n(s))$$

$$= \sum_{\beta=1}^{N} \tilde{\lambda}_{\beta}(s)(\tilde{d}_{n,m}(s))_{\beta} \cdot \tilde{U}_n(s)$$

$$= (\tilde{\Gamma}_{n,m}(s)) \cdot \tilde{U}_n(s) \qquad (4.12)$$

Considering $(\tilde{U}_n(s))$ arbitrarily specified as an N dimensional vector (a "source" current) we have

$$(\tilde{\Gamma}_{n,m}(s)) = \sum_{\beta=1}^{N} \tilde{\lambda}_{\beta}(s)(\tilde{d}_{n,m}(s))_{\beta} \qquad (4.13)$$

Using the orthogonality relations for the eigendyads from equations 3.18 one can verify that

$$(\tilde{\Gamma}_{n,m}(s))^{-1} \cdot (\tilde{\Gamma}_{n,m}(s))$$

$$= \left\{ \sum_{\beta=1}^{N} \frac{1}{\tilde{\lambda}_{\beta}(s)} (\tilde{d}_{n,m}(s))_{\beta} \right\} \cdot \left\{ \sum_{\beta=1}^{N} \tilde{\lambda}_{\beta}(s)(\tilde{d}_{n,m}(s))_{\beta} \right\}$$

$$= \sum_{\beta=1}^{N} (\tilde{d}_{n,m}(s))_{\beta}$$

$$= (\delta_{n,m}) \qquad (4.14)$$

By successive matrix multiplication with $(\tilde{\Gamma}_{n,m}(s))$ from equation 4.13 one obtains the nth power of the matrix. Using the expression for the inverse matrix from equation 4.11 with successive matrix multiplication one obtains negative powers as well. The general result is

$$(\tilde{\Gamma}_{n,m}(s))^n = \sum_{\beta=1}^{N} \tilde{\lambda}_\beta^n(s)(\tilde{d}_{n,m}(s))_\beta \tag{4.15}$$

n = any integer (positive or negative including zero)

The orthogonality relations for the eigendyads are used with each multiplication by the matrix $(\tilde{\Gamma}_{n,m}(s))$ or its inverse to simplify the result.. Note that raising $\tilde{\lambda}_\beta(s)$ to the nth power is all that is needed in the eigendyad expansion of the matrix to the nth power.

Let us summarize the corresponding operator equations for this section. Expand the response and excitation functions as

$$\tilde{\vec{U}}(s) = \sum_\beta \frac{\langle \tilde{\vec{L}}_\beta(s) ; \tilde{\vec{U}}(s) \rangle}{\langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{L}}_\beta(s) \rangle} \tilde{\vec{R}}_\beta(s) + \tilde{\vec{U}}_e(s)$$

$$= \sum_\beta \langle \tilde{\vec{d}}_\beta(s) ; \tilde{\vec{U}}(s) \rangle + \tilde{\vec{U}}_e(s) \tag{4.16}$$

$$\tilde{\vec{I}}(s) = \sum_\beta \frac{\langle \tilde{\vec{L}}_\beta(s) ; \tilde{\vec{I}}(s) \rangle}{\langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{L}}_\beta(s) \rangle} \tilde{\vec{R}}_\beta(s) + \tilde{\vec{I}}_e(s)$$

$$= \sum_\beta \langle \tilde{\vec{d}}_\beta(s) ; \tilde{\vec{I}}(s) \rangle + \tilde{\vec{I}}_e(s)$$

where $\tilde{\vec{U}}_e(s)$ and $\tilde{\vec{I}}_e(s)$ account for possible extra terms. The conditions for

$$\langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{L}}_\beta(s) \rangle \neq 0 \tag{4.17}$$

are somewhat more complicated than in the matrix case. However if all $\overleftrightarrow{L}_{\beta'}(s)$ are orthogonal to $\overleftrightarrow{R}_{\beta}$ for $\beta' \neq \beta$ and if any additional terms cannot linearly add to the $\overleftrightarrow{R}_{\beta'}$ to $\overleftrightarrow{R}_{\beta}$ then the non zero characteristic of the symmetric product in equation 4.17 should be maintained. The identity operator is

$$\overleftrightarrow{I}\delta(\vec{r} - \vec{r}') = \sum_{\beta} \overleftrightarrow{d}_{\beta}(\vec{r}, \vec{r}' ; s) + \overleftrightarrow{I}_e$$

$$= \sum_{\beta} \overleftrightarrow{d}_{\beta}^{T}(\vec{r}, \vec{r}' ; s) + \overleftrightarrow{I}_e^{T} \qquad (4.18)$$

The operator of the integral equation to an integer power is

$$\overleftrightarrow{\Gamma}^{n}(\vec{r}, \vec{r}' ; s) = \sum_{\beta} \lambda_{\beta}^{n}(s) \overleftrightarrow{d}(\vec{r}, \vec{r}' ; s) + \overleftrightarrow{\Gamma}_{e,n} \qquad (4.19)$$

n = any integer

In many cases of interest the extra terms should be zero. For example, the sphere response is described only by discrete eigenvalues; there is no continuous spectrum of eigenvalues.[1,5,12] One would suspect that for finite size bodies of well behaved media in free space (or an infinite uniform medium) the extra terms would also not be needed. Note that the finite size of the scattering body is an important consideration since for quantum mechanical problems of scattering from a potential distribution the potential may vary throughout all space. However, some electromagnetic problems (particularly involving infinite regions) may involve other than the presently discussed eigenmode terms. Note that the matrix form necessarily has no continuous spectrum of eigenvalues.

Another interesting question is the convergence of the eigenvalues and eigenvectors to $\tilde{\lambda}_\beta(s)$, $\tilde{\vec{R}}_\beta(s)$, $\tilde{\vec{L}}_\beta(s)$ as $N \to \infty$. On physical grounds one would expect that if say $(\tilde{I}_n(s))$ were a sampling of $\tilde{\vec{I}}(s)$ with appropriately smooth sampling functions in each sample zone then $(\tilde{I}_n(s))$ should converge to $\tilde{\vec{I}}(s)$ and similarly for the other terms. Note that as $N \to \infty$ more and more numerical eigenvectors are involved. One then anticipates an infinite number of eigenmodes of the continuous integral equation operator. To accurately compute the βth eigenvalue/eigenmode one expects to need a significant number of samples over a significant variation of the mode. The number of samples would be proportional to β for a thin wire scatterer and to β^2 for a perfectly conducting fat body such as a sphere unless certain symmetries could be utilized.

The operator can be said to be of simple structure or diagonalizable by analogy with the matrix form if certain conditions are met. In particular the eigenmodes must be all independent, even for cases of eigenvalue degeneracy. Also there must be no extra terms so that only the eigenvalues and eigenmodes are needed to expand the operator. Equations 4.15 for the matrix and 4.19 for the operator are then of corresponding forms. The present discussions apply to such diagonalizable operators.

Note for symmetric $(\tilde{\Gamma}_{n,m}(s))$ and $\tilde{\tilde{\Gamma}}(\vec{r}, \vec{r}\,' ; s)$ we have the interesting result

$$(\tilde{R}_n(s))_\beta \cdot (\tilde{R}_n(s))_\beta \neq 0 \quad \text{for a diagonalizable matrix (matrix of simple structure)}$$

$$\langle (\tilde{\vec{R}}_\beta(s) ; \tilde{\vec{R}}_\beta(s) \rangle \neq 0 \quad \text{for a diagonalizable operator (operator of simple structure)} \qquad (4.20)$$

V. Diagonalization of the Integral Equation Operator or Matrix

Let us construct some convenient combinations of the eigenvectors, eigenmodes, and eigenvalues. Define a matrix (which one might term the right eigenmatrix) as

$$(\tilde{R}_{n,m}(s)) \equiv \left((R_n)_1, (R_n)_2, \cdots, (R_n)_N\right) \tag{5.1}$$

$$n,m = 1, 2, 3, \cdots, N$$

for which each column is a right eigenvector, the vector component subscript (n) is the first index, and the second index (m) corresponds to the eigenvector index β. Similarly define a left eigenmatrix as

$$(\tilde{L}_{n,m}(s)) \equiv \begin{pmatrix} (L_m)_1 \\ (L_m)_2 \\ \vdots \\ (L_m)_N \end{pmatrix} \tag{5.2}$$

for which each row is an eigenvector, the vector component subscript (m) is the second index, and the first index (n) corresponds to the eigenvector index β. Define a diagonal eigenvalue matrix as

$$(\tilde{\Lambda}_{n,m}(s)) \equiv (\tilde{\lambda}_n(s)\delta_{n,m}) = \mathrm{diag}(\tilde{\lambda}_1(s), \tilde{\lambda}_2(s), \cdots, \tilde{\lambda}_N(s))$$

$$= \begin{pmatrix} \tilde{\lambda}_1(s) & & & \mathbf{0} \\ & \tilde{\lambda}_2(s) & & \\ & & \ddots & \\ \mathbf{0} & & & \tilde{\lambda}_N(s) \end{pmatrix} \tag{5.3}$$

With these definitions we have eigenmatrix equations

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_{n,m}(s)) = (\tilde{R}_{n,m}(s)) \cdot (\tilde{\Lambda}_{n,m}(s))$$

$$(\tilde{L}_{n,m}(s)) \cdot (\tilde{\Gamma}_{n,m}(s)) = (\tilde{\Lambda}_{n,m}(s)) \cdot (\tilde{L}_{n,m}(s)) \tag{5.4}$$

The eigenmatrices are related as

$$(\tilde{L}_{n,m}(s)) \cdot (\tilde{R}_{n,m}(s)) = \begin{pmatrix} (\tilde{R}_n(s))_1 \cdot (\tilde{L}_n(s))_1 & & \mathbf{0} \\ & (\tilde{R}_n(s))_2 \cdot (\tilde{L}_n(s))_2 & \\ & & \ddots \\ \mathbf{0} & & (\tilde{R}_n(s))_N \cdot (\tilde{L}_n(s))_N \end{pmatrix}$$

$$= \mathrm{diag}\Big((\tilde{R}_n(s))_1 \cdot (\tilde{L}_n(s))_1, (\tilde{R}_n(s))_2 \cdot (\tilde{L}_n(s))_2,$$

$$\cdots, (\tilde{R}_n(s))_N \cdot (\tilde{L}_n(s))_N\Big) \tag{5.5}$$

Normalize the eigenvectors as

$$(\tilde{r}_n(s))_\beta \equiv \frac{(\tilde{R}_n(s))_\beta}{\left[(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta\right]^{1/2}}$$

$$(\tilde{\ell}_n(s))_\beta \equiv \frac{(\tilde{L}_n(s))_\beta}{\left[(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta\right]^{1/2}} \tag{5.6}$$

where the square root takes the same value in both equations. Then we have normalized eigenmatrices as

$$(\tilde{r}_{n,m}(s)) = \left((\tilde{r}_n)_1, (\tilde{r}_n)_2, \cdots, (\tilde{r}_n)_N\right)$$

$$(\tilde{\ell}_{n,m}(s)) = \begin{pmatrix} (\tilde{\ell}_n(s))_1 \\ (\tilde{\ell}_n(s))_2 \\ \vdots \\ (\tilde{\ell}_n(s))_N \end{pmatrix} \tag{5.7}$$

with the normalized eigenmatrix equations

$$\begin{aligned} (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{r}_{n,m}(s)) &= (\tilde{r}_{n,m}(s)) \cdot (\tilde{\Lambda}_{n,m}(s)) \\ (\tilde{\ell}_{n,m}(s)) \cdot (\tilde{\Gamma}_{n,m}(s)) &= (\tilde{\Lambda}_{n,m}(s)) \cdot (\tilde{\ell}_{n,m}(s)) \end{aligned} \tag{5.8}$$

and the relation

$$(\tilde{\ell}_{n,m}(s)) \cdot (\tilde{r}_{n,m}(s)) = (\delta_{n,m}) \tag{5.9}$$

so that the two eigenmatrices are mutually inverse.

In operator form we have a right eigenvector of vector modes as

$$(\tilde{\vec{R}}_\beta(s)) \equiv (\tilde{\vec{R}}_1(s), \tilde{\vec{R}}_2(s), \tilde{\vec{R}}_3(s), \cdots) \tag{5.10}$$

and a left eigenvector of vector modes as

$$(\tilde{\vec{L}}_\beta(s)) \equiv (\tilde{\vec{L}}_1(s), \tilde{\vec{L}}_2(s), \tilde{\vec{L}}_3(s), \cdots) \tag{5.11}$$

where there are in general an infinite number of components corresponding to an infinite number of βs. Note that each component of the above generalized vectors is itself a vector mode function of position. One can write the above as rows or columns provided one keeps in mind the β index of the components as the important quantity. In normalized form we have

$$\tilde{\vec{r}}_\beta(s) = \frac{\tilde{\vec{R}}_\beta(s)}{\left[\left\langle \tilde{\vec{R}}_\beta(s) \,;\, \tilde{\vec{L}}_\beta(s) \right\rangle\right]^{1/2}}$$

$$\tilde{\vec{\ell}}_\beta(s) = \frac{\tilde{\vec{L}}_\beta(s)}{\left[\left\langle \tilde{\vec{R}}_\beta(s) \,;\, \tilde{\vec{L}}_\beta(s) \right\rangle\right]^{1/2}} \qquad (5.12)$$

$$(\tilde{\vec{r}}_\beta(s)) = (\tilde{\vec{r}}_1(s),\ \tilde{\vec{r}}_2(s),\ \tilde{\vec{r}}_3(s),\ \cdots)$$

$$(\tilde{\vec{\ell}}_\beta(s)) = (\tilde{\vec{\ell}}_1(s),\ \tilde{\vec{\ell}}_2(s),\ \tilde{\vec{\ell}}_3(s),\ \cdots)$$

The eigenvalue matrix takes the form of an infinite matrix as

$$(\tilde{\Lambda}_{n,m}(s)) = (\tilde{\lambda}_n \delta_{n,m}) = \mathrm{diag}(\tilde{\lambda}_1(s),\ \tilde{\lambda}_2(s),\ \tilde{\lambda}_3(s),\ \cdots)$$

$$= \begin{pmatrix} \tilde{\lambda}_1(s) & & & \mathbf{0} \\ & \tilde{\lambda}_2(s) & & \\ & & \tilde{\lambda}_3(s) & \\ \mathbf{0} & & & \ddots \end{pmatrix}$$

$$n, m = 1, 2, 3, \cdots \qquad (5.13)$$

The eigenvectors of modes equations become

$$\left(\left\langle \tilde{\vec{\Gamma}}(s) ; \tilde{\vec{R}}_\beta(s) \right\rangle\right) = (\tilde{\vec{R}}_\beta(s)) \cdot (\tilde{\Lambda}_{n,m}(s)) = (\tilde{\lambda}_\beta(s)\tilde{\vec{R}}_\beta(s))$$

$$= (\tilde{\Lambda}_{n,m}(s)) \cdot (\tilde{\vec{R}}_\beta(s))$$

$$\left(\left\langle \tilde{\vec{L}}_\beta(s) ; \tilde{\vec{\Gamma}}(s) \right\rangle\right) = (\tilde{\Lambda}_{n,m}(s)) \cdot (\tilde{\vec{L}}_\beta(s)) = (\tilde{\lambda}_n(s)\tilde{\vec{L}}_\beta(s)) \tag{5.14}$$

$$= (\tilde{\vec{L}}_\beta(s)) \cdot (\tilde{\Lambda}_{n,m}(s))$$

and in normalized form become

$$\left(\left\langle \tilde{\vec{\Gamma}}(s) ; \tilde{\vec{r}}_\beta(s) \right\rangle\right) = (\tilde{\vec{r}}_\beta(s)) \cdot (\tilde{\Lambda}_{n,m}(s)) = (\tilde{\lambda}_\beta(s)\tilde{\vec{r}}_\beta(s))$$

$$= (\tilde{\Lambda}_{n,m}(s)) \cdot (\tilde{\vec{r}}_\beta(s))$$

$$\left(\left\langle \tilde{\vec{\ell}}_\beta(s) ; \tilde{\vec{\Gamma}}(s) \right\rangle\right) = (\tilde{\Lambda}_{n,m}(s)) \cdot (\tilde{\vec{\ell}}_\beta(s)) = (\tilde{\lambda}_\beta(s)\tilde{\vec{\ell}}_\beta(s)) \tag{5.15}$$

$$= (\tilde{\vec{\ell}}_\beta(s)) \cdot (\tilde{\Lambda}_{n,m}(s))$$

The eigenvectors of modes are related as

$$\left(\left\langle \tilde{\vec{L}}_n(s) \,;\, \tilde{\vec{R}}_m(s) \right\rangle\right)$$

$$= \operatorname{diag}\left(\left\langle \tilde{\vec{R}}_1(s) \,;\, \tilde{\vec{L}}_1(s) \right\rangle, \left\langle \tilde{\vec{R}}_2(s) \,;\, \tilde{\vec{L}}_2(s) \right\rangle, \cdots\right)$$

$$= \begin{pmatrix} \left\langle \tilde{\vec{R}}_1(s) \,;\, \tilde{\vec{L}}_1(s) \right\rangle & & \mathbf{0} \\ & \left\langle \tilde{\vec{R}}_2(s) \,;\, \tilde{\vec{L}}_2(s) \right\rangle & \\ \mathbf{0} & & \ddots \end{pmatrix} \tag{5.16}$$

and in normalized form as

$$\left(\left\langle \tilde{\vec{r}}_n(s) \,;\, \tilde{\vec{\ell}}_m(s) \right\rangle\right) = (\delta_{n,m}) \tag{5.17}$$

The above formalism can be extended to include special product symbols to indicate the types of products for the vector or matrix "elements." These elements are vectors and/or matrices and continuous operation indicated by $\langle\ ,\ \rangle$ is involved. This would seem to be an interesting subject for future development.

With the results of this section the matrix or corresponding integral equation operator can be readily diagonalized. In matrix form we have

$$(\tilde{\ell}_{n,m}(s)) \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{r}_{n,m}(s)) = (\tilde{\ell}_{n,m}(s)) \cdot (\tilde{r}_{n,m}(s)) \cdot (\tilde{\Lambda}_{n,m}(s))$$

$$= (\tilde{\Lambda}_{n,m}(s)) \tag{5.18}$$

and in operator form we have

$$\left(\left\langle \tilde{\vec{\ell}}_n(s) \,;\, \tilde{\vec{\vec{\Gamma}}}(s) \,;\, \tilde{\vec{r}}_m(s) \right\rangle\right) = \left(\left\langle \tilde{\vec{\ell}}_n(s) \,;\, \tilde{\vec{r}}_m(s) \right\rangle\right) \cdot (\tilde{\Lambda}_{n,m}(s))$$

$$= (\tilde{\Lambda}_{n,m}(s)) \tag{5.19}$$

Alternately we can write

$$\begin{aligned} (\tilde{\Gamma}_{n,m}(s)) &= (\tilde{r}_{n,m}(s)) \cdot (\tilde{\Lambda}_{n,m}(s)) \cdot (\tilde{\ell}_{n,m}(s)) \\ \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}';s) &= (\tilde{\vec{r}}_\beta(\vec{r},s)) \cdot (\tilde{\Lambda}_{n,m}(s)) \cdot (\tilde{\vec{\ell}}_\beta(\vec{r}',s)) \end{aligned} \tag{5.20}$$

which leads to

$$\begin{aligned} (\tilde{\Gamma}_{n,m}(s))^n &= (\tilde{r}_{n,m}(s)) \cdot (\tilde{\Lambda}_{n,m}(s))^n \cdot (\tilde{\ell}_{n,m}(s)) \\ \tilde{\vec{\vec{\Gamma}}}^n(\vec{r},\vec{r}';s) &= (\tilde{\vec{r}}_\beta(\vec{r},s)) \cdot (\tilde{\Lambda}_{n,m}(s))^n \cdot (\tilde{\vec{\ell}}_\beta(\vec{r}',s)) \end{aligned} \tag{5.21}$$

These are similarity transformations and we can speak of $(\tilde{\Gamma}_{n,m}(s))$ and $\tilde{\vec{\vec{\Gamma}}}(s)$ as being similar to $(\tilde{\Lambda}_{n,m}(s))$.

Another interesting form of equations 5.18 and 5.19 is equations for the individual eigenvalues. Setting $n = m = \beta$ in $\tilde{\Lambda}_{n,m}(s)$ gives $\tilde{\lambda}_\beta(s)$ as

$$\tilde{\lambda}_\beta(s) = (\tilde{\ell}_n(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{r}_n(s))_\beta$$

$$= \frac{(\tilde{L}_n(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta}{(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta} \tag{5.22}$$

$$\tilde{\lambda}_\beta(s) = \left\langle \tilde{\vec{\ell}}_\beta(s) ; \tilde{\vec{\vec{\Gamma}}}(s) ; \tilde{\vec{r}}_\beta(s) \right\rangle$$

$$= \frac{\left\langle \tilde{\vec{L}}_\beta(s) ; \tilde{\vec{\vec{\Gamma}}}(s) ; \tilde{\vec{R}}_\beta(s) \right\rangle}{\left\langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{L}}_\beta(s) \right\rangle}$$

For symmetric $(\tilde{\Gamma}_{n,m}(s))$ and $\tilde{\vec{\vec{\Gamma}}}(\vec{r}, \vec{r}'; s)$ we have

$$(\tilde{L}_{n,m}(s)) = (\tilde{R}_{n,m}(s))^T$$

$$(\tilde{\vec{L}}_\beta(s)) = (\tilde{\vec{R}}_\beta(s))$$

$$(\tilde{R}_{n,m}(s))^T \cdot (\tilde{R}_{n,m}(s))$$

$$= \mathrm{diag}\left((\tilde{R}_n(s))_1 \cdot (\tilde{R}_n(s))_1 , (\tilde{R}_n(s))_2 \cdot (\tilde{R}_n(s))_2 , \cdots , (\tilde{R}_n(s))_N \cdot (\tilde{R}_n(s))_N \right)$$

$$\left(\left\langle \tilde{\vec{R}}_n(s) ; \tilde{\vec{R}}_m(s) \right\rangle \right)$$

$$= \mathrm{diag}\left(\left\langle \tilde{\vec{R}}_1(s) ; \tilde{\vec{R}}_1(s) \right\rangle , \left\langle \tilde{\vec{R}}_2(s) ; \tilde{\vec{R}}_2(s) \right\rangle , \cdots \right)$$

$$(\tilde{\ell}_n(s))_\beta = (\tilde{r}_n(s))_\beta = \frac{(\tilde{R}_n(s))_\beta}{\left[(\tilde{R}_n(s))_\beta \cdot (\tilde{R}_n(s))_\beta \right]^{1/2}}$$

(5.23)

$$\tilde{\vec{\ell}}_\beta(s) = \tilde{\vec{r}}_\beta(s) = \frac{\tilde{\vec{R}}_\beta(s)}{\left[\left\langle \tilde{\vec{R}}_\beta(s) \,;\, \tilde{\vec{R}}_\beta(s) \right\rangle\right]^{1/2}}$$

$$(\tilde{\ell}_{n,m}(s)) = (\tilde{r}_{n,m}(s))^T$$

$$(\tilde{\vec{\ell}}_\beta(s)) = (\tilde{\vec{r}}_\beta(s))$$

$$(\tilde{r}_{n,m}(s))^T \cdot (\tilde{r}_{n,m}(s)) = (\delta_{n,m})$$

$$\left(\left\langle \tilde{\vec{r}}_n(s) \,;\, \tilde{\vec{r}}_m(s) \right\rangle\right) = (\delta_{n,m})$$

$$(\tilde{r}_{n,m}(s))^T \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{r}_{n,m}(s)) = (\tilde{\Lambda}_{n,m}(s))$$

$$\left(\left\langle \tilde{\vec{r}}_n(s) \,;\, \tilde{\vec{\Gamma}}(s) \,;\, \tilde{\vec{r}}_m(s) \right\rangle\right) = (\tilde{\Lambda}_{n,m}(s))$$

$$(\tilde{\Gamma}_{n,m}(s))^n = (\tilde{r}_{n,m}(s)) \cdot (\tilde{\Lambda}_{n,m}(s))^n \cdot (\tilde{r}_{n,m}(s))^T$$

$$\tilde{\vec{\Gamma}}^n(\vec{r}, \vec{r}'; s) = (\tilde{\vec{r}}_\beta(\vec{r}, s)) \cdot (\tilde{\Lambda}_{n,m}(s))^n \cdot (\tilde{\vec{r}}_\beta(\vec{r}, s))$$

$$\tilde{\lambda}_\beta(s) = (\tilde{r}_n(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{r}_n(s))_\beta$$

$$\tilde{\lambda}_\beta(s) = \left\langle \tilde{\vec{r}}_\beta(s) \,;\, \tilde{\vec{\Gamma}}(s) \,;\, \tilde{\vec{r}}_\beta(s) \right\rangle$$

VI. The Characteristic Equation

The complex natural frequencies and eigenvalues are related through the characteristic determinant which can be expressed in terms of two complex variables, s and λ, as

$$\tilde{D}_N(s,\lambda) \equiv d_N \det\left(\left(\tilde{\Gamma}_{n,m}(s)\right) - \lambda(\delta_{n,m})\right) \tag{6.1}$$

Here d_N is a convenient normalization constant (introduced in equation 2.3) which can be set equal to 1 or to some convenient function of N, say for a convenient limiting form for $\tilde{D}_N$ as $N \to \infty$.

Consider some of the properties of $\tilde{D}_N(s,\lambda)$. This characteristic determinant is a polynomial in λ of degree N and can be written as

$$\tilde{D}_N(s,\lambda) = d_N \det\left(\left(\tilde{\Gamma}_{n,m}(s)\right) - \lambda(\delta_{n,m})\right)$$

$$\det\left(\left(\tilde{\Gamma}_{n,m}(s)\right) - \lambda(\delta_{n,m})\right) = \sum_{p=o}^{N} \tilde{a}_p(s)\lambda^p \tag{6.2}$$

$$= a_N \prod_{\beta=1}^{N} (\lambda - \tilde{\lambda}_\beta(s))$$

$$= \prod_{\beta=1}^{N} (\tilde{\lambda}_\beta(s) - \lambda)$$

Some of the coefficients in this characteristic equation have commonly encountered forms. These include[16]

$$a_N = \tilde{a}_N(s) = (-1)^N$$

$$\tilde{a}_{N-1}(s) = (-1)^{N-1} \mathrm{tr}\left((\tilde{\Gamma}_{n,m}(s))\right)$$

$$= (-1)^{N-1} \sum_{p=1}^{N} \tilde{\Gamma}_{p,p}(s)$$

$$= (-1)^{N-1} \sum_{\beta=1}^{N} \tilde{\lambda}_\beta(s)$$

$$\tilde{a}_o(s) = \det\left((\tilde{\Gamma}_{n,m}(s))\right) = \frac{1}{d_N} \tilde{D}_N(s) \tag{6.3}$$

$$= \prod_{\beta=1}^{N} \tilde{\lambda}_\beta(s)$$

$$= \sum_{(m_1, \cdots, m_N)} \mathrm{sgn}(m_1, \cdots, m_N) \tilde{\Gamma}_{1,m_1}(s) \tilde{\Gamma}_{2,m_2}(s) \cdots \tilde{\Gamma}_{N,m_N}(s)$$

$$= \sum_{(n_1, \cdots, n_N)} \mathrm{sgn}(n_1, \cdots, n_N) \tilde{\Gamma}_{n_1,1}(s) \tilde{\Gamma}_{n_2,2}(s) \cdots \tilde{\Gamma}_{n_N,N}(s)$$

where the summation extends over all possible permutations of $(1, 2, \cdots, N)$ of the n or m indices. The sgn function is defined as the sign of the permutation listed by its N arguments as

$$\mathrm{sgn}(m_1, \cdots, m_N) \equiv \mathrm{sign}\left[\prod_{1 \le p < q \le N} (m_q - m_p)\right]$$

$$= \frac{\prod_{1\le p<q\le N} (m_q - m_p)}{\left|\prod_{1\le p<q\le N} (m_q - m_p)\right|}$$

$$= \prod_{1\le p<q\le N} \frac{(m_q - m_p)}{|m_q - m_p|} \tag{6.4}$$

so that the sgn function is +1 or -1. Permutations with sgn = +1 are called even and those with sgn = -1 are called odd. If r successive interchanges of pairs of numbers in (1, ···, N) give $(m_1, \cdots, m_N)$ the latter permutation has a value of $(-1)^r$ for sgn.

The above considerations can be extended to all the coefficients in the characteristic equation from[15,16]

$$\det\left((\tilde{\Gamma}_{n,m}(s)) - \lambda(\delta_{n,m})\right)$$

$$= \sum_{(m_1,\cdots,m_N)} \operatorname{sgn}(m_1,\cdots,m_N)\left(\tilde{\Gamma}_{1,m_1}(s) - \lambda\delta_{1,m_1}\right)$$

$$\cdots \left(\tilde{\Gamma}_{N,m_N}(s) - \lambda\delta_{N,m_N}\right) \tag{6.5}$$

The coefficient of λ^p in this equation is found by multiplying out the factors and keeping those which have λ exactly p times giving

$$\tilde{a}_p(s) = \sum_{n_1<\cdots<n_{N-p}} \sum_{(m_1,\cdots,m_N)} \operatorname{sgn}(m_1,\cdots,m_N)$$

$$\left[\prod_{n\neq n_1,\cdots,n_{N-p}} \left(-\delta_{n,m_n}\right)\right]\left[\prod_{n=n_1,\cdots,n_{N-p}} \tilde{\Gamma}_{n,m_n}(s)\right]$$

$$= (-1)^p \sum_{n_1 < \cdots < n_{N-p}} \left(\sum_{m_{n_1}, \cdots, m_{n_{N-p}}} \left(\operatorname{sgn} m_{n_1}, \cdots, m_{n_{N-p}} \right) \right.$$

$$\left. \prod_{n = n_1, \cdots, n_{N-p}} \tilde{\Gamma}_{n, m_n}(s) \right) \qquad 0 \leq p < N \tag{6.6}$$

This result takes the form

$$\tilde{a}_p(s) = (-1)^p \sum_{n_1 < \cdots < n_{N-p}} \det(\tilde{\Gamma}_{n,m}(s) \; ; \; n, m = n_1, n_2, \cdots, n_{N-p}) \tag{6.7}$$

$$0 \leq p < N$$

and the summation is over all possible sets of $n_1 < \cdots < n_{N-p}$. The determinants in equations 6.7 are then of N - p by N - p matrices. In effect the new matrices are formed by using only rows $n_1, \cdots, n_{N-p}$ and columns $n_1, \cdots, n_{N-p}$ from the original matrix $(\tilde{\Gamma}_{n,m}(s))$.

Another representation of the coefficients in the characteristic equation is in terms of the eigenvalues. One can expand equation 6.4 and equate coefficients of λ^p. Alternatively one can note that the matrix from equation 5.3

$$(\tilde{\Lambda}_{n,m}(s)) \equiv (\tilde{\lambda}_n(s)\delta_{n,m}) = \operatorname{diag}(\tilde{\lambda}_1(s), \cdots, \tilde{\lambda}_N(s))$$

$$= \begin{pmatrix} \tilde{\lambda}_1(s) & & & \mathbf{0} \\ & \tilde{\lambda}_2(s) & & \\ & & \ddots & \\ \mathbf{0} & & & \tilde{\lambda}_N(s) \end{pmatrix} \tag{6.8}$$

has the same eigenvalues as $(\tilde{\Gamma}_{n,m}(s))$ and hence the same characteristic equation. Since the determinant is the product of the eigenvalues equation 6.7 becomes

$$\tilde{a}_p(s) = (-1)^p \sum_{n_1 < \cdots < n_{N-p}} \det(\Lambda_{n,m}(s) \; ; \; n,m = n_1, n_2, \cdots, n_{N-p}) \qquad (6.9)$$

$$0 \leq p < N$$

or

$$\tilde{a}_p(s) = (-1)^p \sum_{n_1 < \cdots < n_{N-p}} \prod_{\beta = n_1, \cdots, n_{N-p}} \tilde{\lambda}_\beta(s) \qquad (6.10)$$

$$0 \leq p < N$$

For $p = N$ the sums in equations 6.9 and 6.10 can be defined to be 1. Note then that the special results of equations 6.3 follow very simply.

If we have an N by N matrix $(c_{n,m}(w))$ then we have the interesting result

$$\frac{d}{dw} \det\left((c_{n,m}(w))\right) = \sum_{p=1}^{N} \det\left((c_{n,m}(w))_p\right)$$

$$(c_{n,m}(w))_p = \begin{cases} (c_{n,m}(w)) & \text{for } n \neq p \\ \left(\frac{\partial}{\partial w} c_{n,m}(w)\right) & \text{for } n = p \end{cases} \qquad (6.11)$$

which is a row expansion; there is a very similar column expansion as well. Consider the N by N matrix $(b_{n,m})$. A convenient derivative formula with respect to λ is

$$\frac{\partial}{\partial\lambda}\det((b_{n,m}) - \lambda(\delta_{n,m})) = -\sum_{p=1}^{N}\det\left((b_{n,m})_p - \lambda(\delta_{n,m})_p\right)$$

$(b_{n,m})_p$ is an N - 1 by N - 1 matrix with $n,m \neq p$ (6.12)

$(\delta_{n,m})_p$ is an N - 1 by N - 1 matrix with $n,m \neq p$

Another formula for the coefficients in the characteristic equation is obtained from the formula for the coefficients in a Taylor series as

$$\tilde{a}_p(s) = \frac{1}{p!}\frac{\partial^p}{\partial\lambda^p}\det\left((\tilde{\Gamma}_{n,m}(s)) - \lambda(\delta_{n,m})\right)\Big|_{\lambda=o} \tag{6.13}$$

According to the Cayley-Hamilton theorem a matrix satisfies its own characteristic equation. Hence we have

$$\sum_{p=o}^{N}\tilde{a}_p(s)(\tilde{\Gamma}_{n,m}(s))^p = (0_{n,m}) \tag{6.14}$$

For the continuous operator (assumed matricizable) we can write

$$\lim_{N\to\infty}\sum_{p=o}^{N} d_N\tilde{a}_p(s)\vec{\vec{\Gamma}}^p(\vec{r};\vec{r}';s) = \vec{\vec{0}} \tag{6.15}$$

where d_N is chosen such that $d_N\tilde{a}_o(s)$ tend to a finite but not identically zero limit. One way to confirm the above formulas is to write the operator or matrix in diagonalized form from equations 5.21 and thereby convert equations 6.14 and 6.15 into equations involving a diagonal eigenvalue matrix or operator with the βth diagonal element now being the characteristic equation for $\lambda = \tilde{\lambda}_\beta(s)$.

VII. Eigenvalues Related to Natural Frequencies

The characteristic equation can be used to relate natural frequencies and eigenvalues. The natural frequencies are determined from

$$\tilde{D}_N(s_\alpha, 0) = \tilde{D}_N(s_\alpha) = 0 \tag{7.1}$$

The N eigenvalues are determined from

$$\tilde{D}_N(s, \tilde{\lambda}_\beta(s)) = 0 \quad \text{for } \beta = 1, 2, 3, \cdots, N \tag{7.2}$$

These two equations show that the SEM and EEM representations are linked through the determinant function $\tilde{D}_N(s,\lambda)$. The eigenvalues $\tilde{\lambda}_\beta(s)$ (for $\beta = 1, 2, \cdots, N$) and the natural frequencies s_α (for all α where α is some index set) can be interrelated.

Equation 7.2 can be used to establish the analytic properties of the eigenvalues $\tilde{\lambda}_\beta(s)$ as functions of the complex frequency s. These properties can be used to construct a representation of the eigenvalues involving their singularities and their zeros. As indicated in equations 4.15 and 4.19 the inverse of the integral equation operator uses terms involving $\tilde{\lambda}_\beta^{-1}(s)$. Hence the singularity expansions of the reciprocals of the eigenvalues are desired. This section considers some features of the singularity expansion of the reciprocal eigenvalues. An important feature of the relation between eigenvalues and natural frequencies is the typical situation of natural frequencies belonging to particular eigenvalues in the sense of being the zeros of particular eigenvalues. The eigenvalues tend to separate the natural frequencies into separate sets, one set for each eigenvalue.

A. Natural frequencies

Equation 7.1 is the defining equation for the natural frequencies of our object of interest. Since $\tilde{D}_N(s)$ is usually analytic near an s_α of interest it has a power series expansion as

$$\tilde{D}_N(s) = \sum_{\ell=1}^{\infty} D_{N_{\ell_\alpha}} (s - s_\alpha)^\ell$$

(7.3)

$$D_{N_{\ell_\alpha}} \equiv \frac{1}{\ell!} \frac{d^\ell}{ds^\ell} \tilde{D}_N(s) \Big|_{s=s_\alpha} \qquad \text{for } \ell = 0, 1, 2, \cdots$$

Assuming that the numerator (the cofactor matrix)[1] has no zero at s_α, then there is a pole of order n_α at s_α associated with the object response provided

$$D_{N_{\ell_\alpha}} \begin{cases} = 0 & \text{for} \quad \ell = 0, 1, 2, \cdots, n_\alpha - 1 \\ \\ \neq 0 & \text{for} \quad \ell = n_\alpha \end{cases} \tag{7.4}$$

For certain types of bodies $\tilde{D}_N(s)$ is meromorphic (has only pole singularities in the finite s plane) and is hence analytic at its finite zeros.[1,2,4] Such objects include finite size objects in free space, both perfectly conducting and with well behaved properties of the media. For some other types of objects there may be zeros of the "denominator" such as $\tilde{D}_N(s)$ which are not analytic. Such zeros are excluded from our consideration.

B. Eigenvalues near the natural frequencies

From the eigenvalue equation (equation 7.2) the N eigenvalues at the s_α satisfy

$$\tilde{D}_N(s_\alpha, \tilde{\lambda}_\beta(s_\alpha)) = 0 \quad \text{for} \quad \beta = 1, 2, 3, \cdots, N \tag{7.5}$$

Since

$$\tilde{D}_N(s_\alpha) = d_N \prod_{\beta=1}^{N} \tilde{\lambda}_\beta(s_\alpha) = d_N \tilde{a}_o(s_\alpha) = 0 \tag{7.6}$$

then at least one of the eigenvalues must be zero at s_α. Label one of these by $\beta = \beta_1$.

Suppose there is an M-fold eigenvalue degeneracy at s_α

$$\tilde{\lambda}_{\beta_1}(s_\alpha) = \tilde{\lambda}_{\beta_2}(s_\alpha) = \cdots = \tilde{\lambda}_{\beta_M}(s_\alpha) = 0$$

$$\beta_1 < \beta_2 < \cdots < \beta_M , \quad 1 \leq M \leq N \tag{7.7}$$

Writing

$$\tilde{D}_N(s_\alpha, \lambda) = d_N \sum_{p=o}^{N} \tilde{a}_p(s_\alpha)\lambda^p$$

$$= d_N(-1)^N \prod_{\beta=1}^{N} (\lambda - \lambda_\beta(s_\alpha)) \tag{7.8}$$

then $\tilde{D}_N(s_\alpha, \lambda)$ has an M-fold zero at $\lambda = 0$ which implies

$$\tilde{a}_o(s_\alpha) = \tilde{a}_1(s_\alpha) = \cdots = \tilde{a}_M(s_\alpha) = 0 \tag{7.9}$$

In the complex s plane the s_α are discrete and form a set of measure zero (zero area) so one might expect degeneracy at $s = s_\alpha$ to be an unusual occurrence. This would be expected for bodies with rotational symmetry (accidental degeneracy) such as in the case of a sphere.[5] However such bodies tend to have natural frequency degeneracy as well, again as with a sphere.[1] The conditions for such degeneracies need further study as already pointed out in sections IV and V in connection with the matrix (operator) diagonalizability.

It is well known that the eigenvalues are continuous functions of the matrix elements.[15, 16] Except for singularities the matrix elements are continuous functions of s. Hence the $\tilde{a}_p(s)$ are continuous functions of s as are the eigenvalues $\tilde{\lambda}_\beta(s)$. Then except for singularities in the matrix elements small changes in s near s_α produce small changes in the $\tilde{\lambda}_\beta(s)$. If only one eigenvalue is zero at s_α then only that eigenvalue is small in some neighborhood of s_α. In the inverse matrix (equation 4.15) or inverse operator (equation 4.19) this small eigenvalue in the denominator gives the dominant term which is the pole at s_α.

A non degenerate eigenvalue is an analytic function of the matrix elements since it is a non degenerate root of the characteristic equation (a polynomial in λ).[15] Hence non degenerate eigenvalues are analytic functions of s wherever the matrix elements are all analytic functions of s. Points of degeneracy in the s plane are potential points of non analyticity for the corresponding eigenvalues. One might expect branch points and associated branch cuts in the $\tilde{\lambda}_\beta$ as functions in the s plane. However, the degenerate $\tilde{\lambda}_\beta$ may possibly still be analytic at degeneracies.

Let us assume that at some s_α of interest all the $\tilde{\lambda}_\beta(s)$ are analytic. They can then be expanded in a Taylor series convergent in some circle of non zero radius centered on s_α as

$$\tilde{\lambda}_\beta(s) = \sum_{\ell=0}^{\infty} \lambda_{\beta_{\ell_\alpha}} (s - s_\alpha)^\ell$$

$$\lambda_{\beta_{\ell_\alpha}} \equiv \frac{1}{\ell!} \frac{d^\ell}{ds^\ell} \tilde{\lambda}_\beta(s) \Big|_{s=s_\alpha} \qquad (7.10)$$

Then near s_α the matrix determinant is written as

$$\tilde{D}_N(s) = \sum_{\ell=1}^{\infty} D_{N_{\ell_\alpha}} (s - s_\alpha)^\ell$$

$$= d_N a_o(s) = d_N \prod_{\beta=1}^{N} \left\{ \sum_{\ell'=o}^{\infty} \lambda_{\beta_{\ell'_\alpha}} (s - s_\alpha)^{\ell'} \right\} \qquad (7.11)$$

Suppose there is a first order pole ($D_{N1_\alpha} \neq 0$). Then one and only one of the $\tilde{\lambda}_\beta(s)$ has a zero at s_α and s_α can be said to belong to that $\tilde{\lambda}_\beta$.

Suppose that there is a second order pole at s_α. Then equation 7.11 requires two eigenvalues with first order zeros or one eigenvalue with a second order zero. In order to have a second order pole in the inverse matrix (equation 4.15) requires, if the matrix is diagonalizable in a neighborhood of s_α, there must be one and only one eigenvalue with a zero at s_α and the zero is second order.

Similarly (for eigenvalues all analytic at s_α) a pole of order n_α requires one and only one eigenvalue to be zero at s_α and the zero is of order n_α; the natural frequency belongs to only that eigenvalue. Clearly analyticity of the eigenvalues has important implications. Conversely, if more than one eigenvalue is zero at s_α (and if the matrix is diagonalizable) then the eigenvalues (at least two of them) are non analytic at s_α.

C. Eigenvalues away from natural frequencies

With $s \neq s_\alpha$ (for any α) then we have

$$\tilde{D}_N(s, \tilde{\lambda}_\beta(s)) = 0$$

$$\tilde{D}_N(s) = d_N \prod_{\beta=1}^{N} \tilde{\lambda}_\beta(s) = d_N \tilde{a}_o(s) \neq 0 \qquad (7.12)$$

Assume s is chosen such that the matrix elements are all bounded in magnitude. Then the eigenvalues are all finite. This implies from equation 7.12 that away from the s_α (and from infinities) all N eigenvalues must be non zero.

Since the eigenvalues are continuous functions of s (for analytic matrix elements) one can track the variation of the $\tilde{\lambda}_\beta$ as functions of s unambiguously except possibly for points of degeneracy of the $\tilde{\lambda}_\beta$ and discontinuities in the matrix elements. The eigenvalues are in addition analytic functions of s except possibly at points of degeneracy and points of non analyticity of the matrix elements. This allows the use of certain complex variable techniques such as contour integrals and Taylor series in treating the $\tilde{\lambda}_\beta$.

One in general expects points of degeneracy of the $\tilde{\lambda}_\beta$ in the s plane. Consider two eigenvalues, say $\tilde{\lambda}_{\beta_1}(s)$ and $\tilde{\lambda}_{\beta_2}(s)$. We ask whether they are equal for any s. Certainly one can find two analytic functions (entire functions) which are never equal in the finite s plane simply by defining the second as equal to the first plus a non zero constant. However, such cases would seem specially contrived.

In the case of the perfectly conducting sphere eigenvalues are known for both the E field integral equation (EFIE) and the H field integral equation (HFIE).[5] The $\tilde{\lambda}_\beta(s)$ are given by combinations of modified spherical Bessel functions which can be represented by combinations of exponential, polynomial, and reciprocal power functions of s. One can equate two of the $\tilde{\lambda}_\beta(s)$ and solve the transcendental equation for the points of degeneracy in the finite s plane. These eigenvalues are analytic in the entire s plane except for s = 0 in some cases. Certainly then eigenvalues can be analytic at points of degeneracy, at least in some cases.

D. Eigenvalue ordering of the natural frequencies

Since one can think of certain natural frequencies belonging to each eigenvalue as being zeros of that eigenvalue, then one can use this property to order the natural frequencies. For the natural frequencies s_α consider the index set α. Let α be rewritten as

$$\alpha \equiv (\beta, \beta')$$
$$s_\alpha = s_{\beta,\beta'} \tag{7.13}$$

where β indicates the eigenvalue to which s_α belongs. If a particular natural frequency belongs to more than one eigenvalue then it can have more than one set of labels β, β'.

The index β may be a single number (1, 2, 3, $\cdots$) or may be a pair (or larger set) of numbers if the eigenvalues and eigenmodes are divided into different types. Such division might be chargeless modes vs. modes with non identically zero charge or H modes vs. E modes depending on the electromagnetic problem being considered and the conventions used.

The index β' labels the natural frequencies belonging to a particular eigenvalue (specified by β). There are various conventions one may choose for assigning values to β'. A convenient convention is

$$\beta' \begin{cases} > 0 & \text{for } \mathrm{Im}[s_\alpha] > 0 \\ = 0 & \text{for } \mathrm{Im}[s_\alpha] = 0 \\ < 0 & \text{for } \mathrm{Im}[s_\alpha] < 0 \end{cases} \tag{7.14}$$

Of course β' may also be a pair (or larger set) of numbers if convenient. For example, if more than one s_α belonging to $\tilde{\lambda}_\beta$ has $\mathrm{Im}[s_\alpha] = 0$ then β' could conveniently be written as two numbers of the form $(0, \beta'')$ which would be a generalization of the convention of equation 7.14.

Previously[1] the convention for the α index set has been chosen so that the set split into three parts as

$$\alpha \equiv \begin{cases} \alpha_+ & \text{for } \mathrm{Im}[s_\alpha] > 0 \\ \alpha_o & \text{for } \mathrm{Im}[s_\alpha] = 0 \\ \alpha_- & \text{for } \mathrm{Im}[s_\alpha] < 0 \end{cases} \tag{7.15}$$

This can be generalized to the eigenvalue related notation by similarly decomposing β' as

$$\beta' \equiv \begin{cases} \beta'_+ & \text{for } \mathrm{Im}[s_\alpha] > 0 \\ \beta'_o & \text{for } \mathrm{Im}[s_\alpha] = 0 \\ \beta'_- & \text{for } \mathrm{Im}[s_\alpha] < 0 \end{cases} \tag{7.16}$$

This gives

$$\begin{aligned} \alpha_+ &= (\beta, \beta'_+) \\ \alpha_o &= (\beta, \beta'_o) \\ \alpha_- &= (\beta, \beta'_-) \end{aligned} \tag{7.17}$$

Objects of interest can have natural frequencies on both the $i\omega$ axis and in the left half plane. Integral equations for objects comprised of perfectly conducting surfaces apply to both exterior and interior problems. The kernel of the integral equation contains no information regarding the sources; it does contain the information regarding the object (geometry and composition). The kernel then contains the exterior and interior natural frequencies. Both types of natural frequencies are zeros of the eigenvalues.

Upon inspection of the eigenvalues for the sphere derived from the EFIE and HFIE,[5] one observes that they are not the same. In each case they are represented by a product of an interior part and an exterior part where the terms interior and exterior are defined according to whether the respective parts have interior or exterior natural frequencies as their zeros. In the case of the EFIE the two parts have their zeros corresponding in both cases to E modes or in both cases to H modes. In the case of the HFIE the two parts have one part with the zeros of an E mode and the other with the zeros of an H mode. Hence as one might expect the eigenvalues from different integral equations can be different with different sets of natural frequencies. Thus one must distinguish the eigenvalues by integral equation. One might choose

$$\begin{aligned} &\tilde{\lambda}_{e_\beta}(s) \quad \text{for EFIE} \\ &\tilde{\lambda}_{h_\beta}(s) \quad \text{for HFIE} \\ &\tilde{\lambda}_{c_{q_\beta}}(s) \quad \text{for CFIE} \end{aligned} \tag{7.18}$$

and similarly for other types of integral equations. Here CFIE means the combined field integral equation recently introduced.[10] Perhaps by comparing the eigenvalues from different integral equations for various scattering and antenna problems one can develop general procedures for further decomposing the eigenvalues into parts with smaller sets of natural frequencies (such as internal vs. external). Such decomposition can help better define β and β' indices.

One would like a natural ordering principle for assigning numbers (or sets of numbers) to β. Consider the number of s_α belonging to $\tilde{\lambda}_\beta(s)$ as a possible choice for β or for one of the numbers in a set of numbers making up β. However the eigenvalues of a sphere from the EFIE and

HFIE have an infinite number of zeros as can be seen by inspection.[5] This infinite number of zeros is associated with modified Bessel functions of the first kind corresponding to the internal (cavity) resonances. The external part of the eigenvalues for the perfectly conducting sphere has only a finite number of zeros in the left half plane. The number of exterior eigenvalue zeros may then give a better way to choose β.

Another possible way to choose β is based on the magnitude of the smallest internal natural frequency associated with $\tilde{\lambda}_\beta$. Since the interior natural frequencies all lie on the $i\omega$ axis they are uniquely ordered. Note that poles in the response of a passive object on the $i\omega$ axis can only be first order. It is not clear whether the ordering of the $\tilde{\lambda}_\beta$ by this technique gives the same results as using the number of external zeros in $\tilde{\lambda}_\beta$.

The example of the perfectly conducting sphere has been helpful in formulating and suggesting some of the general properties of eigenvalues and natural frequencies. Figure 7.1 shows the exterior natural frequencies and their connection to the eigenvalues for a perfectly conducting sphere. Here $\beta = (q,n)$ and $\beta' = n'$ where $\alpha = q,n,n'$ is discussed in a previous note.[1] Ordering eigenvalues by the number of exterior natural frequencies gives a set of consecutive arcs of natural frequencies of increasing size corresponding to increasing β. The first arc is $n = 1$, the next is $n = 2$, etc. Note that each arc indicated in figure 7.1 might be considered as two arcs, one for E modes and one for H modes; this is accounted for by $q = 1,2$ in the index set (q,n). Note that this approximate geometrical concept of arcs is only for the external natural frequencies. On the $i\omega$ axis the natural frequencies for the different eigenvalues (β values) form trajectories which lie on top of each other. In the left half plane there is some separation of the trajectories (arcs) of eigenvalue zeros.

In some previous considerations of the natural frequencies of thin wire objects the concept of layers of natural frequencies was introduced.[3]

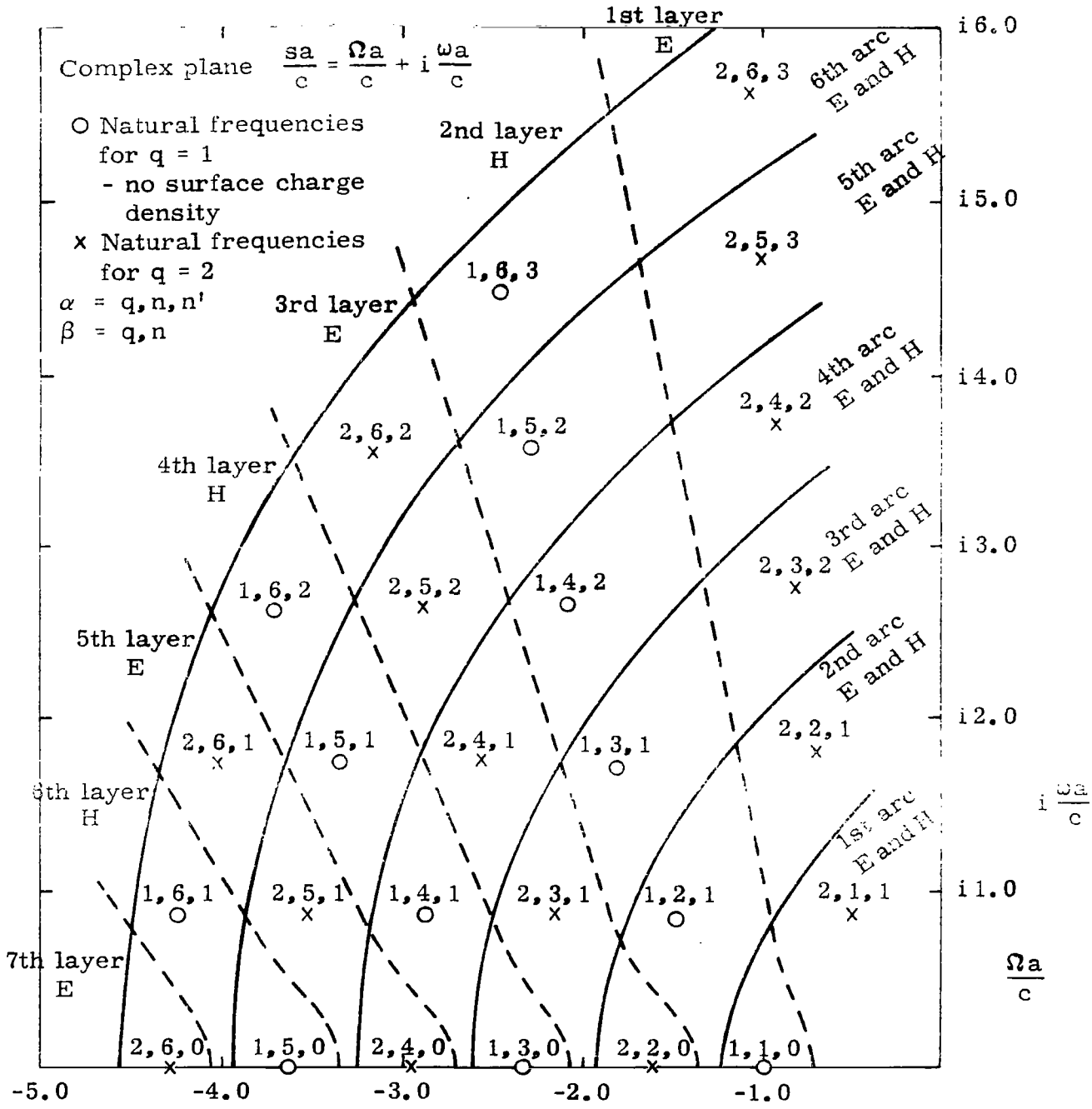


Figure 7.1. Exterior Natural Frequencies $s_{q,n,n'}(a/c)$ of the Perfectly Conducting Sphere for Use With Exterior Incident Wave $1 \leq n \leq 6$

This concept is useful in considering the transient response of thin objects for which the first layer (the external natural frequencies nearest the $i\omega$ axis) seemed to dominate the transient and CW response of the object. One can identify the first layer of natural frequencies for the sphere in figure 7.1 and note that they are all of E-mode type. Successive layers (numbers 2, 3, etc.) are readily identified as one moves to the left in the left half s plane for the perfectly conducting sphere. Note that for the sphere odd layers have E-mode natural frequencies, while even layers have H-mode natural frequencies. Within each layer each natural frequency or conjugate pair of natural frequencies corresponds to different eigenmodes (i.e., different values of β).

Arcs and layers of natural frequencies (poles) would then seem to be complementary. Together they provide a way to label the natural frequencies in the left half plane. Since the left half plane is two dimensional in a geometrical sense a set of two numbers or number sets (arcs and layers) would seem to be a natural way for ordering the natural frequencies.

E. Conjugate symmetry of the eigenvalues

One of the properties of Laplace transformed time functions (operators, etc.) is that of conjugate symmetry if the time domain function is real valued (for real t). If f(t) is a real valued time function then

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt \tag{7.19}$$

$$\bar{\tilde{f}}(s) = \tilde{f}(\bar{s})$$

where f(t) can be a scalar, vector, dyadic, etc., function for present purposes. This is generalized to combined conjugate symmetry as

$$f_q(t) \equiv f(t) + qif'(t)$$

$$q = \pm 1 \qquad (7.20)$$

$$\bar{\tilde{f}}_q(s) = \tilde{f}_{-q}(\bar{s})$$

where both f(t) and f'(t) are real valued time functions. Combined conjugate symmetry then is a generalization of conjugate symmetry to the Laplace transform of complex functions of time.

In the case of the perfectly conducting sphere the eigenvalues of both the EFIE and HFIE have been calculated.[5] In both cases the eigenvalues are conjugate symmetric functions of s. The conjugate symmetric properties of the eigenvalues of general scattering and antenna problems are of concern in understanding and computing these eigenvalues.

Consider an integral equation for which the forcing function (source or incident field) is a real valued time function (Laplace transformed) and likewise for the response (say a current, current density, or surface current density). Examples of these include the EFIE and HFIE. The kernels for such cases are real valued in time domain and conjugate symmetric in complex frequency domain. They consist of the free space dyadic Green's function together with its modifications.

Let a conjugate symmetric integral equation operator be approximated by a conjugate symmetric matrix using the MoM. Then the determinant $\tilde{D}_N(s)$ is a conjugate symmetric function establishing the conjugate symmetry of the placement of the natural frequencies and other singularities. Likewise all the coefficients $\tilde{a}_p(s)$ in the characteristic equation $\tilde{D}_N(s,\lambda)$ are conjugate symmetric functions of s. For real s the $\tilde{a}_p(s)$ are real; $\tilde{D}_N(s,\lambda)$ is then a polynomial in λ with real coefficients and hence a conjugate symmetric function of λ. Similarly, for real λ, $\tilde{D}_N(s,\lambda)$ is a conjugate symmetric function of s. Let us then refer to such a function as a conjugate bisymmetric function of s and λ.

The zeros of $\tilde{D}_N(s,\lambda)$ determine the eigenvalues $\tilde{\lambda}_\beta(s)$. For real s since $\tilde{D}_N(s,\lambda)$ is a conjugate symmetric function of λ the roots $\tilde{\lambda}_\beta(s)$ must be conjugate symmetric in the sense that for every eigenvalue with a non zero imaginary part there must be another eigenvalue which is its conjugate. Furthermore the multiplicity of the eigenvalues must also be preserved in a conjugate symmetric fashion for real s.

Consider some point on the real s axis and the associated N eigenvalues. Since the eigenvalues are continuous functions of the matrix elements and in turn of the complex frequency s (except at singularities and branch cuts of the matrix elements) let us vary s away from the real axis and track the eigenvalues. Eigenvalues which are real on the real s axis can be continued away from the real axis as conjugate symmetric functions since

$$\tilde{D}_N(s,\tilde{\lambda}_\beta(s)) = 0 = \bar{\tilde{D}}_N(s,\tilde{\lambda}_\beta(s))$$

$$= \tilde{D}_N(\bar{s},\bar{\tilde{\lambda}}_\beta(s)) \qquad (7.21)$$

So for every pair $s,\tilde{\lambda}_\beta(s)$ there corresponds the pair $\bar{s},\bar{\tilde{\lambda}}_\beta(s)$. As s varies away from the real axis $\tilde{\lambda}_\beta$ varies continuously and conjugate symmetrically in both the upper and lower half s planes. This allows the same value of β to be applied to an eigenvalue which varies continuously through the real axis as s crosses the real axis. This allows for a type of eigenvalues (conjugate symmetric) as

$$\bar{\tilde{\lambda}}_\beta(s) = \tilde{\lambda}_\beta(\bar{s}) \qquad (7.22)$$

Note that for this development our starting point on the real s axis is assumed to be a point of continuity of the matrix elements as functions of s. If the $\tilde{\lambda}_\beta$ of concern passes through points of degeneracy as s is

varied (including possibly the starting point on the real axis) the choice of which eigenvalue to tag with β going out of the point of degeneracy must be done in a conjugate symmetric manner if equation 7.22 is to apply.

Consider now the case that for s real there may be some non purely real eigenvalues. In this case one follows the more general approach from

$$\tilde{D}_N(s, \tilde{\lambda}_\beta(s)) = 0 = \tilde{D}_N(\bar{s}, \bar{\tilde{\lambda}}_\beta(s))$$

$$= D_N(\bar{s}, \tilde{\lambda}_{\beta'}(\bar{s})) \tag{7.23}$$

where $\tilde{\lambda}_\beta(s)$ is some root for a given s and $\tilde{\lambda}_{\beta'}(\bar{s})$ is some root for the corresponding $\bar{s}$. As indicated by equation 7.23 one allowed choice of $\tilde{\lambda}_{\beta'}(s)$ is

$$\tilde{\lambda}_{\beta'}(\bar{s}) = \bar{\tilde{\lambda}}_\beta(s) \tag{7.24}$$

which can be used to define the index β' for a given β.

For real s then $\tilde{\lambda}_\beta(s)$ and $\tilde{\lambda}_{\beta'}(s)$ are complex conjugates. If for real s we have $\tilde{\lambda}_\beta(s) = \tilde{\lambda}_{\beta'}(s)$ then we can set $\beta' = \beta$ and consider this pair as one eigenvalue which is a continuous function of s as s passes through the real axis. This eigenvalue is conjugate symmetric as in equation 7.22.

If for some real s we have $\tilde{\lambda}_\beta(s) \neq \tilde{\lambda}_{\beta'}(s)$ because $\mathrm{Im}[\tilde{\lambda}_\beta(s)] \neq 0$ then one can use the more general relation of equation 7.24 to group pairs of eigenvalues. However one can still define $\beta' = \beta$ provided a branch cut is placed along the real s axis for those values of real s for which $\tilde{\lambda}_\beta(s) \neq \tilde{\lambda}_{\beta'}(s)$. These "two" eigenvalues can then be written on the real axis as $\tilde{\lambda}_\beta(\Omega + i0)$ and $\tilde{\lambda}_\beta(\Omega - i0)$. We then have

$$\tilde{\lambda}_\beta(\bar{s}) = \bar{\tilde{\lambda}}_\beta(s) \tag{7.25}$$

where say we use $\tilde{\lambda}_\beta$ for Im[s] > 0 and $\tilde{\lambda}_{\beta'}$ for Im[s] < 0 with a discontinuity in crossing the real axis for some real s. Note that for some real s this eigenvalue may be continuous crossing the real s axis. Regions of continuity and discontinuity on the real s axis would be separated by branch points. Adopting such a convention then the eigenvalues of conjugate symmetric integral operators can then each be chosen to be conjugate symmetric functions of s.

Whether or not such eigenvalue discontinuities along the real s axis are necessary to preserve eigenvalue conjugate symmetry in the s plane is an interesting question. The EFIE and HFIE eigenvalues of the perfectly conducting sphere have no such discontinuity while maintaining conjugate symmetry. On the other hand one expects such eigenvalues of the perfectly conducting circular cylinder (infinitely long) to have branch cuts along the negative real s axis due to the branch cut there in the integral equation operator.

The property of conjugate symmetry is a convenience in calculating and labelling eigenvalues. It applies to common integral equations such as the EFIE and the HFIE. However the CFIE deals with combined quantities (equation 7.20) and its eigenvalues will not be conjugate symmetric but rather combined conjugate symmetric by extension of the previous arguments.

VIII. Eigenmodes Related to Natural Modes

A. Normalization of eigenmodes and natural modes

Having considered the eigenvalues as related to the natural frequencies a further step in relating eigenmode expansions to singularity expansions is to relate the eigenmodes to natural modes. The natural modes for the object response (say a surface current density) are determined from

$$(\tilde{\Gamma}_{n,m}(s_\alpha)) \cdot (\nu_n)_\alpha = (0_n)$$

$$\left\langle \vec{\tilde{\Gamma}}(s_\alpha) \,;\, \vec{\nu}_\alpha \right\rangle = \vec{0} \tag{8.1}$$

and the associated coupling vectors are determined from

$$(\mu_n)_\alpha \cdot (\tilde{\Gamma}_{n,m}(s_\alpha)) = (0_n)$$

$$\left\langle \vec{\mu}_\alpha \,;\, \vec{\tilde{\Gamma}}(s_\alpha) \right\rangle = \vec{0} \tag{8.2}$$

For symmetric operators and matrices we have

$$(\mu_n)_\alpha = (\nu_n)_\alpha \,, \qquad \vec{\mu}_\alpha(\vec{r}) = \vec{\nu}_\alpha(\vec{r}) \tag{8.3}$$

if the same normalization is used. Typically (but not necessarily) the normalization is chosen as

$$\max|\nu_{n_\alpha}| = 1 \,, \qquad |\vec{\nu}_\alpha|_{\max} \equiv \max_{\vec{r}} |\vec{\nu}_\alpha(\vec{r})| = 1$$

$$\max|\mu_{n_\alpha}| = 1 \,, \qquad |\vec{\mu}_\alpha|_{\max} \equiv \max_{\vec{r}} |\vec{\mu}_\alpha(\vec{r})| = 1 \tag{8.4}$$

Another kind of normalization is given by

$$(\nu_n)_\alpha \cdot (\mu_n)_\alpha = A_\alpha , \quad \langle \vec{\nu}_\alpha ; \vec{\mu}_\alpha \rangle = A_\alpha \tag{8.5}$$

which for symmetric matrices and operators gives

$$(\nu_n)_\alpha \cdot (\nu_n)_\alpha = A_\alpha , \quad \langle \vec{\nu}_\alpha ; \vec{\nu}_\alpha \rangle = A_\alpha \tag{8.6}$$

where A_α is a convenient normalization constant. One might choose $A_\alpha = 1$ but if the natural mode and coupling vector are to be made dimensionless then A_α has the dimensions of length for one dimensional integrals, length squared for surface integrals, and length cubed for volume integrals. One might set $A_\alpha = Ba$ or $B_\alpha a^2$ or $B_\alpha a^3$ where a is some characteristic dimension of the electromagnetic problem. Note that for $\vec{\mu}_\alpha \neq \vec{\nu}_\alpha$ an additional condition (besides equations 8.5) must be imposed to uniquely specify $\vec{\mu}_\alpha$ and $\vec{\nu}_\alpha$.

The corresponding eigenmodes for the object response are determined from

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta = \tilde{\lambda}_\beta(s)(\tilde{R}_n(s))_\beta$$

$$\langle \tilde{\vec{\Gamma}}(s) ; \tilde{\vec{R}}_\beta(s) \rangle = \tilde{\lambda}_\beta(s)\tilde{\vec{R}}_\beta(s) \tag{8.7}$$

with the associated left side (transpose) modes as

$$(\tilde{\Gamma}_{n,m}(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) = \tilde{\lambda}_\beta(s)(\tilde{L}_n(s))_\beta$$

$$\langle \tilde{\vec{L}}_\beta(s) ; \tilde{\vec{\Gamma}}(s) \rangle = \tilde{\lambda}_\beta(s)\tilde{\vec{L}}_\beta(s) \tag{8.8}$$

For symmetric operators and matrices we have

$$(\tilde{L}_n(s))_\beta = (\tilde{R}_n(s))_\beta , \qquad \tilde{\vec{L}}_\beta(\vec{r},s) = \tilde{\vec{R}}_\beta(\vec{r},s) \tag{8.9}$$

if both modes are normalized in the same way. One choice of normalization is

$$\begin{aligned} \max|\tilde{R}_n(s)|_\beta &= 1 , \qquad |\tilde{\vec{R}}_\beta(s)|_{max} \equiv \max_{\vec{r}}|\tilde{\vec{R}}_\beta(\vec{r},s)| = 1 \\ \max|\tilde{L}_n(s)|_\beta &= 1 , \qquad |\tilde{\vec{L}}_\beta(s)|_{max} \equiv \max_{\vec{r}}|\tilde{\vec{L}}_\beta(\vec{r},s)| = 1 \end{aligned} \tag{8.10}$$

An alternate normalization is

$$(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta = \tilde{A}(s) , \qquad \langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{L}}_\beta(s) \rangle = \tilde{A}(s) \tag{8.11}$$

which for symmetric operators and matrices is

$$(\tilde{R}_n(s))_\beta \cdot (\tilde{R}_n(s))_\beta = \tilde{A}(s) , \qquad \langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{R}}_\beta(s) \rangle = \tilde{A}(s) \tag{8.12}$$

where $\tilde{A}(s)$ is now the normalization constant. Again one could choose $\tilde{A}(s) = 1$. However for dimensionless eigenmodes one might set $\tilde{A}(s) = \tilde{B}(s)a$ or $\tilde{B}(s)a^2$ or $\tilde{B}(s)a^3$ depending on whether surface or volume integrals are involved and where a is some characteristic dimension of the electromagnetic problem. Note that for $\tilde{\vec{L}}_\beta = \tilde{\vec{R}}_\beta$ an additional condition (besides equations 8.11) is needed to uniquely specify $\tilde{\vec{L}}_\beta$ and $\tilde{\vec{R}}_\beta$.

B. Eigenmodes at natural frequencies

In section VII it was noted that at least one of the eigenvalues had a zero at $s = s_\alpha$ and this could form the basis for grouping the natural frequencies with their corresponding eigenvalues. Writing α as β,β' one can note an interesting result for $s = s_\alpha$. Since at $s = s_{\beta,\beta'}$ the eigenvalue $\tilde{\lambda}_\beta(s_{\beta,\beta'}) = 0$ giving

$$
\begin{aligned}
&(\tilde{\Gamma}_{n,m}(s_{\beta,\beta'})) \cdot (\tilde{R}_n(s_{\beta,\beta'}))_\beta = (0_n) \\
&\left\langle \tilde{\vec{\Gamma}}(s_{\beta,\beta'}) ; \tilde{\vec{R}}_\beta(s_{\beta,\beta'}) \right\rangle = \vec{0} \\
&(\tilde{L}_n(s_{\beta,\beta'})) \cdot (\tilde{\Gamma}_{n,m}(s_{\beta,\beta'})) = (0_n) \\
&\left\langle \tilde{\vec{L}}_\beta(s_{\beta,\beta'}) ; \tilde{\vec{\Gamma}}(s_{\beta,\beta'}) \right\rangle = \vec{0}
\end{aligned}
\tag{8.13}
$$

which allows us to set

$$
\begin{aligned}
&(\tilde{R}_n(s_{\beta,\beta'}))_\beta = (\nu_n)_{\beta,\beta'} \\
&\tilde{\vec{R}}_\beta(\vec{r}, s_{\beta,\beta'}) = \vec{\nu}_{\beta,\beta'}(\vec{r}) \\
&(\tilde{L}_n(s_{\beta,\beta'}))_\beta = (\mu_n)_{\beta,\beta'} \\
&\tilde{\vec{L}}_\beta(\vec{r}, s_{\beta,\beta'}) = \vec{\mu}_{\beta,\beta'}(\vec{r})
\end{aligned}
\tag{8.14}
$$

This choice requires that the same normalization be used for both the eigenmodes and the natural modes. If different normalizations are used then the right sides of equations 8.14 are multiplied by non zero complex constants.

For the above results one assumes that neither the natural modes nor the eigenmodes are degenerate at the s_α of interest. If there is degeneracy then one must define a biorthogonal set of natural modes and coupling vectors and have these correspond to an identical set of right and left eigenmodes.

The above equations illustrate that the natural modes and eigenmodes are related in a significant way. At the natural frequencies the associated eigenmodes are the natural modes. Away from the natural frequencies the eigenmodes are then continuations of the natural modes to general complex frequencies. This continuation of the spatial distribution of the eigenmodes leads one to consider the eigenmodes as another way of tracking which natural frequencies "belong" to a particular eigenvalue/eigenmode.

C. Conjugate symmetry of the eigenmodes

In section VII it was shown that for kernels and matrices which were Laplace transforms of real valued time functions (operators) the eigenvalues could be indexed such that each eigenvalue is a conjugate symmetric function of s. Assuming a conjugate symmetric kernel $\overleftrightarrow{\Gamma}(\vec{r},\vec{r}';s)$ and matrix $(\tilde{\Gamma}_{n,m}(s))$ with eigenvalues $\tilde{\lambda}_\beta(s)$ chosen in a conjugate symmetric manner the eigenmodes (both right and left) are also conjugate symmetric as the following development will show.

Starting from the right eigenmode equation

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta = \tilde{\lambda}_\beta(s)(\tilde{R}_n(s))_\beta$$

$$\left\langle \tilde{\vec{\Gamma}}(s) ; \tilde{\vec{R}}_\beta(s) \right\rangle = \tilde{\lambda}_\beta(s)\tilde{\vec{R}}_\beta(s) \tag{8.15}$$

let us form the complex conjugate as

$$(\bar{\tilde{\Gamma}}_{n,m}(s)) \cdot (\bar{\tilde{R}}_n(s))_\beta = \bar{\tilde{\lambda}}_\beta(s)(\bar{\tilde{R}}_n(s))_\beta$$

$$\left\langle \bar{\tilde{\vec{\Gamma}}}(s) ; \bar{\tilde{\vec{R}}}_\beta(s) \right\rangle = \bar{\tilde{\lambda}}_\beta(s)\bar{\tilde{\vec{R}}}_\beta(s) \tag{8.16}$$

Using the conjugate symmetric property of the operator (matrix) and eigenvalues gives

$$(\tilde{\Gamma}_{n,m}(\bar{s})) \cdot (\bar{\tilde{R}}_n(s))_\beta = \tilde{\lambda}_\beta(\bar{s})(\bar{\tilde{R}}_n(s))_\beta$$

$$\left\langle \tilde{\vec{\Gamma}}(\bar{s}) ; \bar{\tilde{\vec{R}}}_\beta(s) \right\rangle = \tilde{\lambda}_\beta(\bar{s})\bar{\tilde{\vec{R}}}_\beta(s) \tag{8.17}$$

However the eigenmode equation for complex frequency $\bar{s}$ is

$$(\tilde{\Gamma}_{n,m}(\bar{s})) \cdot (\tilde{R}_n(\bar{s}))_\beta = \tilde{\lambda}_\beta(\bar{s})(\tilde{R}_n(\bar{s}))_\beta$$

$$\left\langle \tilde{\vec{\Gamma}}(\bar{s}) ; \tilde{\vec{R}}_\beta(\bar{s}) \right\rangle = \tilde{\lambda}_\beta(\bar{s})\tilde{\vec{R}}_\beta(\bar{s}) \tag{8.18}$$

Comparing equations 8.18 to 8.17 allows the choice

$$\left(\bar{\tilde{R}}_n(s)\right)_\beta = \left(\tilde{R}_n(\bar{s})\right)_\beta$$

$$\bar{\tilde{\vec{R}}}_\beta(s) = \tilde{\vec{R}}_\beta(\bar{s}) \tag{8.19}$$

which specifies the choice of normalization for $\tilde{\vec{R}}_\beta(\bar{s})$ in terms of the normalization of $\tilde{\vec{R}}_\beta(s)$. The left eigenvectors have the same type of conjugate symmetry as

$$\left(\bar{\tilde{L}}_n(s)\right)_\beta = \left(\tilde{L}_n(\bar{s})\right)_\beta$$

$$\bar{\tilde{\vec{L}}}_\beta(s) = \tilde{\vec{L}}_\beta(\bar{s}) \tag{8.20}$$

which is obtained by steps identical to equations 8.15 through 8.20 with left eigenvectors instead of right.

One can also note that in normalized form we have (using equations 5.6 and 5.12)

$$\left(\bar{\tilde{r}}_n(s)\right)_\beta = \left(\tilde{r}_n(\bar{s})\right)_\beta$$

$$\bar{\tilde{\vec{r}}}_\beta(s) = \tilde{\vec{r}}_\beta(\bar{s})$$

$$\left(\bar{\tilde{\ell}}_n(s)\right)_\beta = \left(\tilde{\ell}_n(\bar{s})\right)_\beta \tag{8.21}$$

$$\bar{\tilde{\vec{\ell}}}_\beta(s) = \tilde{\vec{\ell}}_\beta(\bar{s})$$

This type of normalization specifies the complex multiplier of the eigenmodes if the left and right modes are related by another condition. It is

the type of normalization in equations 8.11 with $\tilde{A}(s) = 1$. For symmetric operators (matrices) this additional condition is specified by making the left and right modes equal.

IX. Some Derivatives of Eigenmode Expansion Quantities

In studying the relationship of the eigenmode expansion to the singularity expansion the properties of the eigenmode expansion quantities as functions of the complex frequency s are important. The derivatives of these functions with respect to s yield some useful formulas.

Start with the eigenvalues from equations 5.22 as

$$\tilde{\lambda}_\beta(s) = (\tilde{\ell}_n(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{r}_n(s))_\beta$$

$$\tilde{\lambda}_\beta(s) = \left\langle \tilde{\vec{\ell}}_\beta(s) ; \tilde{\vec{\Gamma}}(s) ; \tilde{\vec{r}}_\beta(s) \right\rangle \tag{9.1}$$

Take the s derivative to give

$$\frac{d}{ds}\tilde{\lambda}_\beta(s) = \left[\frac{d}{ds}(\tilde{\ell}_n(s))_\beta\right] \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{r}_n(s))_\beta$$

$$+ (\tilde{\ell}_n(s))_\beta \cdot \left[\frac{d}{ds}(\tilde{\Gamma}_{n,m}(s))\right] \cdot (\tilde{r}_n(s))_\beta$$

$$+ (\tilde{\ell}_n(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot \left[\frac{d}{ds}(\tilde{r}_n(s))_\beta\right]$$

$$= \tilde{\lambda}_\beta(s)\left[\frac{d}{ds}(\tilde{\ell}_n(s))_\beta\right] \cdot (\tilde{r}_n(s))_\beta$$

$$+ (\tilde{\ell}_n(s))_\beta \cdot \left[\frac{d}{ds}(\tilde{\Gamma}_{n,m}(s)\right] \cdot (\tilde{r}_n(s))_\beta$$

$$+ \tilde{\lambda}_\beta(s)(\tilde{\ell}_n(s))_\beta \cdot \left[\frac{d}{ds}(\tilde{r}_n(s))_\beta\right]$$

$$= \tilde{\lambda}_\beta(s) \frac{d}{ds}\left[(\tilde{\ell}_n(s))_\beta \cdot (\tilde{r}_n(s))_\beta\right]$$

$$+ (\tilde{\ell}_n(s))_\beta \cdot \left[\frac{d}{ds}(\tilde{\Gamma}_{n,m}(s))\right] \cdot (\tilde{r}_n(s))_\beta \qquad (9.2)$$

Since

$$(\tilde{\ell}_n(s))_\beta \cdot (\tilde{r}_n(s))_\beta = 1 \qquad (9.3)$$

the eigenvalue derivative is

$$\frac{d}{ds}\tilde{\lambda}_\beta(s) = (\tilde{\ell}_n(s))_\beta \cdot \left[\frac{d}{ds}(\tilde{\Gamma}_{n,m}(s))\right] \cdot (\tilde{r}_n(s))_\beta \qquad (9.4)$$

$$\frac{d}{ds}\tilde{\lambda}_\beta(s) = \left\langle \tilde{\vec{\ell}}_\beta(s) ; \frac{d}{ds}\tilde{\vec{\Gamma}}(s) ; \tilde{\vec{r}}_\beta(s) \right\rangle$$

Another form of equations 9.1 is

$$1 = (\tilde{\ell}_n(s))_\beta \cdot \left[\frac{1}{\tilde{\lambda}_\beta(s)}(\tilde{\Gamma}_{n,m}(s))\right] \cdot (\tilde{r}_n(s))_\beta \qquad (9.5)$$

$$1 = \left\langle \tilde{\vec{\ell}}_\beta(s) ; \frac{1}{\tilde{\lambda}_\beta(s)}\tilde{\vec{\Gamma}}(s) ; \tilde{\vec{r}}_\beta(s) \right\rangle$$

Following the steps in equation 9.2 in differentiating with respect to s gives

$$0 = (\tilde{\ell}_n(s))_\beta \cdot \left\{\frac{d}{ds}\left[\frac{1}{\tilde{\lambda}_\beta(s)}(\tilde{\Gamma}_{n,m}(s))\right]\right\} \cdot (\tilde{r}_n(s))_\beta \qquad (9.6)$$

$$0 = \left\langle \tilde{\vec{\ell}}_\beta(s) \,;\, \frac{d}{ds}\left[\frac{1}{\tilde{\lambda}_\beta(s)}\,\tilde{\vec{\Gamma}}(s)\right] \,;\, \tilde{\vec{r}}_\beta(s) \right\rangle$$

From equations 9.1 and 9.4 one can also form the logarithmic derivative of the eigenvalues as

$$\frac{d}{ds}\left[\ln(\tilde{\lambda}_\beta(s))\right] = \frac{1}{\tilde{\lambda}_\beta(s)}\,\frac{d}{ds}\,\tilde{\lambda}_\beta(s)$$

$$= \frac{(\tilde{\ell}_n(s))_\beta \cdot \left[\frac{d}{ds}(\tilde{\Gamma}_{n,m}(s))\right] \cdot (\tilde{r}_n(s))_\beta}{(\tilde{\ell}_n(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{r}_n(s))_\beta}$$

$$\frac{(\tilde{L}_n(s))_\beta \cdot \left[\frac{d}{ds}(\tilde{\Gamma}_{n,m}(s)\right] \cdot (\tilde{R}_n(s))_\beta}{(\tilde{L}_n(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta}$$

$$= \frac{d}{ds}\ln\left[(\tilde{L}_n(s))_\beta \cdot (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta\right] \qquad (9.7)$$

$$\frac{d}{ds}\left[\ln(\tilde{\lambda}_\beta(s))\right] = \frac{\left\langle \tilde{\vec{\ell}}_\beta(s) \,;\, \frac{d}{ds}\,\tilde{\vec{\Gamma}}(s) \,;\, \tilde{\vec{r}}_\beta(s) \right\rangle}{\left\langle \tilde{\vec{\ell}}_\beta(s) \,;\, \tilde{\vec{\Gamma}}(s) \,;\, \tilde{\vec{r}}_\beta(s) \right\rangle}$$

$$= \frac{\left\langle \tilde{\vec{L}}_\beta(s) \,;\, \frac{d}{ds}\,\tilde{\vec{\Gamma}}(s) \,;\, \tilde{\vec{R}}_\beta(s) \right\rangle}{\left\langle \tilde{\vec{L}}_\beta(s) \,;\, \tilde{\vec{\Gamma}}(s) \,;\, \tilde{\vec{R}}_\beta(s) \right\rangle}$$

$$= \frac{d}{ds} \ln \left[\left\langle \tilde{\vec{L}}_\beta(s) ; \tilde{\vec{\Gamma}}(s) ; \tilde{\vec{R}}_\beta(s) \right\rangle \right]$$

Consider the eigenmodes. From the relation

$$\begin{aligned} &(\tilde{\ell}_n(s))_\beta \cdot (\tilde{r}_n(s))_\beta = 1 \\ &\left\langle \tilde{\vec{\ell}}_\beta(s) ; \tilde{\vec{r}}_\beta(s) \right\rangle = 1 \end{aligned} \tag{9.8}$$

for the normalized eigenmodes we have the derivative relations

$$\begin{aligned} 0 &= \frac{d}{ds} \left[(\tilde{\ell}_n(s))_\beta \cdot (\tilde{r}_n(s))_\beta \right] \\ &= \left[\frac{d}{ds} (\tilde{\ell}_n(s))_\beta \right] \cdot (\tilde{r}_n(s))_\beta + (\tilde{\ell}_n(s))_\beta \cdot \left[\frac{d}{ds} (\tilde{r}_n(s))_\beta \right] \\ 0 &= \frac{d}{ds} \left\langle \tilde{\vec{\ell}}_\beta(s) ; \tilde{\vec{r}}_\beta(s) \right\rangle \\ &= \left\langle \frac{d}{ds} \tilde{\vec{\ell}}_\beta(s) ; \tilde{\vec{r}}_\beta(s) \right\rangle + \left\langle \tilde{\vec{\ell}}_\beta(s) ; \frac{d}{ds} \tilde{\vec{r}}_\beta(s) \right\rangle \end{aligned} \tag{9.9}$$

These relations also apply to $\tilde{\vec{L}}_\beta(\vec{r}, s)$ and $\tilde{\vec{R}}_\beta(\vec{r}, s)$ provided they are normalized such that their symmetric product is independent of s. This would require that $\tilde{A}(s)$ in equation 8.11 be a constant A (or Aa, Aa^2, Aa^3 where a is a characteristic dimension of the electromagnetic problem).

For symmetric operators and matrices the above eigenmode relationship begins from

$$(\tilde{r}_n(s))_\beta \cdot (\tilde{r}_n(s))_\beta = 1$$

$$\left\langle \tilde{\vec{r}}_\beta(s) ; \tilde{\vec{r}}_\beta(s) \right\rangle = 1 \qquad (9.10)$$

and gives

$$0 = \frac{d}{ds}\left[(\tilde{r}_n(s))_\beta \cdot (\tilde{r}_n(s))_\beta\right]$$

$$0 = (\tilde{r}_n(s))_\beta \cdot \left[\frac{d}{ds}(\tilde{r}_n(s))_\beta\right] \qquad (9.11)$$

$$0 = \frac{d}{ds}\left\langle \tilde{\vec{r}}_\beta(s) ; \tilde{\vec{r}}_\beta(s) \right\rangle$$

$$0 = \left\langle \tilde{\vec{r}}_\beta(s) ; \frac{d}{ds}\tilde{\vec{r}}_\beta(s) \right\rangle$$

One can interpret this relation for $d\tilde{\vec{r}}_\beta(\vec{r}, s)/ds$ by noting that the derivative of the eigenmode (normalized in a frequency independent manner) is orthogonal to the eigenmode. In other words the "direction" of the derivative is constrained in this orthogonal fashion for eigenmodes of symmetrical operators and matrices.

X. SEM Coupling Coefficients Related to the Eigenmode Expansion

In comparing the pole terms in the SEM form of the inverse matrix or operator for first order poles as

$$(\tilde{\Gamma}_{n,m}(s))^{-1} = \sum_{\alpha} \frac{1}{s - s_\alpha} \frac{(\nu_n)_\alpha (\mu_n)_\alpha}{(\mu_n)_\alpha \cdot \frac{\partial}{\partial s}(\tilde{\Gamma}_{n,m})\Big|_{s=s_\alpha} \cdot (\nu_n)_\alpha}$$

$$+ \text{ other singularities}$$

$$\tilde{\vec{\Gamma}}^{-1}(\vec{r},\vec{r}';s) = \sum_{\alpha} \frac{1}{s - s_\alpha} \frac{\vec{\nu}_\alpha(\vec{r})\vec{\mu}_\alpha(\vec{r}')}{\left\langle \vec{\mu}_\alpha ; \frac{\partial}{\partial s}\tilde{\vec{\Gamma}}(s)\Big|_{s=s_\alpha} ; \vec{\nu}_\alpha \right\rangle} \tag{10.1}$$

$$+ \text{ other singularities}$$

(which gives the class 2 form of the SEM coupling coefficients)[6] to the eigenmode expansion of the inverse matrix or operator as

$$(\tilde{\Gamma}_{n,m}(s))^{-1} = \sum_{\beta=1}^{N} \frac{1}{\tilde{\lambda}_\beta(s)} \frac{(\tilde{R}_n(s))_\beta (\tilde{L}_n(s))_\beta}{(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta}$$

$$\tilde{\vec{\Gamma}}^{-1}(\vec{r},\vec{r}';s) = \sum_{\beta} \frac{1}{\tilde{\lambda}_\beta(s)} \frac{\tilde{\vec{R}}_\beta(\vec{r},s)\tilde{\vec{L}}_\beta(\vec{r}',s)}{\left\langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{L}}_\beta(s) \right\rangle} \tag{10.2}$$

where the operator is assumed diagonalizable, one notes the similarity in the forms of the terms of the summations. From the matrix or integral equation for the delta function response as

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{U}_n(s)) = (\tilde{I}_n(s))$$

$$\left\langle \tilde{\vec{\Gamma}}(\vec{r},\vec{r}';s) \,;\, \tilde{\vec{U}}(\vec{r}',s) \right\rangle = \tilde{\vec{I}}(\vec{r},s) \tag{10.3}$$

the formal solutions

$$(\tilde{U}_n(s)) = (\tilde{\Gamma}_{n,m}(s))^{-1} \cdot (\tilde{I}_n(s))$$

$$\tilde{\vec{U}}(\vec{r},s) = \left\langle \tilde{\vec{\Gamma}}^{-1}(\vec{r},\vec{r}';s) \,;\, \tilde{\vec{I}}(\vec{r}',s) \right\rangle \tag{10.4}$$

lead to the SEM solution (pole terms exhibited) as

$$(\tilde{U}_n(s)) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)(\nu_n)_\alpha (s - s_\alpha)^{-1} + \text{other singularities}$$

$$\tilde{\vec{U}}(\vec{r},s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha(\vec{r})(s - s_\alpha)^{-1} + \text{other singularities} \tag{10.5}$$

where $\vec{e}_1$ is a unit vector describing the direction of incidence as required and to the EEM solution as

$$(\tilde{U}_n(s)) = \sum_{\beta=1}^{N} \frac{1}{\tilde{\lambda}_\beta(s)} \frac{(\tilde{R}_n(s))_\beta}{(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta} (\tilde{L}_n(s))_\beta \cdot (\tilde{I}_n(s))$$

$$\tilde{\vec{U}}(\vec{r},s) = \sum_\beta \frac{1}{\tilde{\lambda}_\beta(s)} \frac{\vec{R}_\beta(\vec{r},s)}{\left\langle \tilde{\vec{R}}_\beta(s) \,;\, \tilde{\vec{L}}_\beta(s) \right\rangle} \left\langle \tilde{\vec{L}}_\beta(s) \,;\, \tilde{\vec{I}}(s) \right\rangle \tag{10.6}$$

Assuming non degeneracy of the eigenvalues at a natural frequency of interest described as $s_\alpha = s_{\beta,\beta'}$ we have comparing the SEM and EEM representations for the SEM coupling coefficients

$$\tilde{\eta}_{\beta,\beta'}(\vec{e}_1, s_{\beta,\beta'}) = \frac{(\mu_n)_{\beta,\beta'} \cdot (\tilde{I}_n(s_{\beta,\beta'}))}{(\mu_n)_{\beta,\beta'} \cdot \frac{\partial}{\partial s}(\tilde{\Gamma}_{n,m}(s))\Big|_{s=s_{\beta,\beta'}} \cdot (\nu_n)_{\beta,\beta'}}$$

$$= \left[\frac{\partial}{\partial s}\tilde{\lambda}_\beta(s)\Big|_{s=s_{\beta,\beta'}}\right]^{-1} \frac{(\mu_n)_{\beta,\beta'} \cdot (\tilde{I}_n(s_{\beta,\beta'}))}{(\nu_n)_{\beta,\beta'} \cdot (\mu_n)_{\beta,\beta'}} \tag{10.7}$$

$$\tilde{\eta}_{\beta,\beta'}(\vec{e}_1, s_{\beta,\beta'}) = \frac{\langle \vec{\mu}_{\beta,\beta'} ; \tilde{\vec{I}}(s_{\beta,\beta'}) \rangle}{\langle \vec{\mu}_{\beta,\beta'} ; \frac{\partial}{\partial s}\tilde{\vec{\Gamma}}(s)\Big|_{s=s_{\beta,\beta'}} ; \vec{\nu}_{\beta,\beta'} \rangle}$$

$$= \left[\frac{\partial}{\partial s}\tilde{\lambda}_\beta(s)\Big|_{s=s_{\beta,\beta'}}\right]^{-1} \frac{\langle \vec{\mu}_{\beta,\beta'} ; \tilde{\vec{I}}(s_{\beta,\beta'}) \rangle}{\langle \vec{\nu}_{\beta,\beta'} ; \vec{\mu}_{\beta,\beta'} \rangle}$$

The coupling coefficients (at s_α) are then expressible in terms of derivatives of the corresponding eigenvalues with respect to s at the corresponding natural frequencies. This result is found from expanding the terms in equations 10.6 around $s_{\beta,\beta'}$. Alternatively one can use the expression for the derivative of an eigenvalue with respect to s in equations 9.4 evaluated at $s_{\beta,\beta'}$.

Consider one term in the EEM representation of the inverse operator and the corresponding term in the solution of the integral equation. These can be expanded in an SEM form (for first order poles) as

$$\frac{1}{\tilde{\lambda}_\beta(s)} \frac{(\tilde{R}_n(s))_\beta (\tilde{L}_n(s))_\beta}{(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta}$$

$$= \sum_{\beta'} \frac{1}{s - s_{\beta,\beta'}} \left[\frac{\partial}{\partial s} \tilde{\lambda}_\beta(s) \Big|_{s=s_{\beta,\beta'}} \right]^{-1} \frac{(\nu_n)_{\beta,\beta'} (\mu_n)_{\beta,\beta'}}{(\nu_n)_{\beta,\beta'} \cdot (\mu_n)_{\beta,\beta'}}$$

$$+ \text{ other singularities} \qquad (10.8)$$

$$\frac{1}{\tilde{\lambda}_\beta(s)} \frac{\tilde{\vec{R}}_\beta(\vec{r}, s)\tilde{\vec{L}}_\beta(\vec{r}', s)}{\langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{L}}_\beta(s) \rangle}$$

$$= \sum_{\beta'} \frac{1}{s - s_{\beta,\beta'}} \left[\frac{\partial}{\partial s} \tilde{\lambda}_\beta(s) \Big|_{s=s_{\beta,\beta'}} \right]^{-1} \frac{\vec{\nu}_{\beta,\beta'}(\vec{r})\vec{\mu}_{\beta,\beta'}(\vec{r}')}{\langle \vec{\nu}_{\beta,\beta'} ; \vec{\mu}_{\beta,\beta'} \rangle}$$

$$+ \text{ other singularities}$$

and

$$\frac{1}{\tilde{\lambda}_\beta(s)} \frac{(\tilde{R}_n(s))_\beta}{(\tilde{R}_n(s))_\beta \cdot (\tilde{L}_n(s))_\beta} (\tilde{L}_n(s))_\beta \cdot (\tilde{I}_n(s))$$

$$= \sum_{\beta'} \tilde{\eta}_{\beta,\beta'}(\vec{e}_1, s)(\nu_n)_{\beta,\beta'} (s - s_{\beta,\beta'})^{-1}$$

$$+ \text{ other singularities} \qquad (10.9)$$

$$\frac{1}{\tilde{\lambda}_\beta(s)} \frac{\tilde{\vec{R}}_\beta(\vec{r},s)}{\langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{L}}_\beta(s) \rangle} \langle \tilde{\vec{L}}_\beta(s) ; \tilde{\vec{I}}(s) \rangle$$

$$= \sum_{\beta'} \tilde{\eta}_{\beta,\beta'}(\vec{e}_1,s)\vec{\nu}_{\beta,\beta'}(\vec{r})(s - s_{\beta,\beta'})^{-1}$$

+ other singularities

The SEM expansion of the eigenmode terms as in equations 10.8 and 10.9 leads to a sum over β' (a part of the α index set) for the pole terms plus other terms such as branch contributions, etc. In certain cases such as the response of a perfectly conducting sphere to a delta function incident plane wave[1] this sum over β' is a finite sum with its maximum magnitude proportional to the eigenmode index β (= n + a constant in the usual sphere notation). Like the natural frequencies, natural modes, and coupling vectors, the coupling coefficients also associate with the βth eigenvalues and eigenmodes for $\alpha = \beta,\beta'$.

One might take the scalar multiple of $\tilde{\vec{R}}_\beta(\vec{r},s)$ in equations 10.9 and call it the eigenmode coupling coefficient as

$$\tilde{c}_\beta(\vec{e}_1,s) = \frac{1}{\tilde{\lambda}_\beta(s)} \frac{\langle \tilde{\vec{L}}_\beta(s) ; \tilde{\vec{I}}(s) \rangle}{\langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{L}}_\beta(s) \rangle} \tag{10.10}$$

such that

$$\tilde{c}_\beta(\vec{e}_1,s)\tilde{\vec{R}}_\beta(\vec{r},s) = \sum_{\beta'} \frac{1}{s - s_{\beta,\beta'}} \tilde{\eta}_{\beta,\beta'}(\vec{e}_1,s)\vec{\nu}_{\beta,\beta'}(\vec{r})$$

+ other singularities (10.11)

Then since $\tilde{\vec{R}}_\beta \to \vec{\nu}_{\beta,\beta'}$ at $s_{\beta,\beta'}$ one might view $\tilde{\eta}_{\beta,\beta'}/(s - s_{\beta,\beta'})$ in a collective sense (over β') as the eigenmode coupling coefficient in an SEM sense.

XI. Use of Eigenvalues for Synthesis of Antenna and Scatterer Impedance Loading

An interesting aspect of the EEM concerns the properties of the eigenvalues as impedances. In particular certain forms of the matrix or operator admit such an interpretation.

A. Impedance matrix or operator

As discussed by Harrington[17] one can define an impedance matrix based on an E-field type integral equation. Such an equation is one which relates a set of source electric fields to a response current, surface current density, current density, or some combination of these. The current, etc., is the response part of the matrix or integral equation and the source electric field is the excitation function. The integral or matrix equation has a generalized form of Ohm's law as

$$\left\langle \tilde{\vec{Z}}(\vec{r},\vec{r}';s) \,;\, \tilde{\vec{J}}(\vec{r}',s) \right\rangle = \tilde{\vec{E}}_s(\vec{r},s)$$

$$(\tilde{Z}_{n,m}(s)) \cdot (\tilde{J}_n(s)) = (\tilde{E}_{s_n}(s)) \qquad (11.1)$$

The matrix elements $\tilde{Z}_{n,m}(s)$ are referred to as generalized impedances. One can call $\tilde{\vec{Z}}(\vec{r},\vec{r}';s)$ an impedance kernel and with the integration call it an impedance operator. The impedance matrix is $(\tilde{Z}_{n,m}(s))$. The source electric field $\vec{E}_s$ can be an antenna gap field or the incident electric field (or the parallel component in the case of a surface or thin wire) for a scatterer. The response current density is indicated by $\vec{J}$ but this could also be a surface current density or a line current.

For $\vec{r} \neq \vec{r}'$ the usual electric field integral equation (EFIE) for perfectly conducting bodies in free space has

$$\overset{\sim}{\overset{\rightarrow}{\vec{Z}}}(\vec{r},\vec{r}';s) = s\mu_o \overset{\sim}{\overset{\rightarrow}{\vec{G}}}_o(\vec{r},\vec{r}';s) \tag{11.2}$$

(with only components tangential to the surface considered) where the free space dyadic Green's function is

$$\overset{\sim}{\overset{\rightarrow}{\vec{G}}}_o(\vec{r},\vec{r}';s) = \left[\overset{\rightarrow}{\vec{I}} - \frac{1}{\gamma^2}\nabla\nabla\right]\tilde{G}_o(\vec{r},\vec{r}';s) \tag{11.3}$$

where $\overset{\rightarrow}{\vec{I}}$ is the 3×3 identity matrix and the scalar free space Green's function is

$$\tilde{G}_o(\vec{r},\vec{r}';s) = \frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \tag{11.4}$$

with

$$\gamma \equiv \frac{s}{c}$$

$$Z_o \equiv \sqrt{\frac{\mu_o}{\epsilon_o}} \tag{11.5}$$

The singularity at $\vec{r} = \vec{r}'$ is rather troublesome. There are other ways to formulate the operator to attempt to avoid this problem such as by writing the fields due to the source electric field as integrals over both current and charge in the form of vector and scalar potentials; this leads to an integro-differential equation. It is not the purpose of this section to consider the various ways of formulating the EFIE either analytically or in a method of moments (MoM) numerical format. What is of concern here is what the form of an impedance operator implies.

B. Eigenimpedances

Inasmuch as this note is concerned with eigenmode expansions let us consider the eigenmode quantities concerned with impedance operators. In particular note that the eigenvalues of an impedance operator are appropriately identified as eigenimpedances $\tilde{Z}_\beta(s)$ from

$$(\tilde{Z}_{n,m}(s)) \cdot (\tilde{R}_n(s))_\beta = \tilde{Z}_\beta(s)(\tilde{R}_n(s))_\beta$$

$$\langle \tilde{\vec{Z}}(s) ; \tilde{\vec{R}}_\beta(s) \rangle = \tilde{Z}_\beta(s)\tilde{\vec{R}}_\beta(s) \tag{11.6}$$

where now

$$\tilde{Z}_\beta(s) \equiv \tilde{\lambda}_\beta(s) \tag{11.7}$$

Why these are termed eigenimpedances can be seen by choosing

$$\tilde{\vec{J}}(\vec{r}, s) \equiv \tilde{C}(s)\tilde{\vec{R}}_\beta(\vec{r}, s) \tag{11.8}$$

so that the corresponding source electric field is

$$\tilde{\vec{E}}_s(\vec{r}, s) = \tilde{Z}_\beta(s)\tilde{C}(s)\tilde{\vec{R}}_\beta(\vec{r}, s)$$

$$= \tilde{Z}_\beta(s)\tilde{\vec{J}}(\vec{r}, s) \tag{11.9}$$

This relation might be referred to as Ohm's law for impedance eigenmodes.

The eigenimpedances have dimensions of ohms times appropriate powers of length (meters). If the integral equation uses a surface integral so that the response is a surface current density $\vec{J}_s$ (amperes/m)

then $\tilde{Z}_\beta(s)$ is a simple impedance (ohms or ohms per square as a sheet impedance). If the integral equation uses a volume integral so that the response is a current density $\vec{J}$ (amperes/m^2) then $\tilde{Z}_\beta(s)$ has dimensions ohm meters. If a line integral is used so that the response is a current I then $\tilde{Z}_\beta(s)$ has dimensions ohms/m (impedance per unit length).

The reciprocals of the eigenimpedances are the eigenadmittances as

$$\tilde{Y}_\beta(s) \equiv \frac{1}{\tilde{Z}_\beta(s)} \tag{11.10}$$

The eigenadmittances are the eigenvalues of the inverse matrix or operator

$$\begin{aligned} \tilde{\vec{\vec{Y}}}(\vec{r},\vec{r}';s) &\equiv \tilde{\vec{\vec{Z}}}^{-1}(\vec{r},\vec{r}';s) \\ (\tilde{Y}_{n,m}(s)) &\equiv (\tilde{Z}_{n,m}(s))^{-1} \end{aligned} \tag{11.11}$$

so that we have

$$\begin{aligned} \tilde{\vec{J}}(\vec{r},s) &= \left\langle \tilde{\vec{\vec{Y}}}(\vec{r},\vec{r}';s) \,;\, \tilde{\vec{E}}_s(\vec{r},s) \right\rangle \\ (\tilde{J}_n(s)) &= (\tilde{Y}_{n,m}(s)) \cdot (\tilde{E}_{s_n}(s)) \end{aligned} \tag{11.12}$$

and

$$\tilde{\vec{J}}(\vec{r},s) = \tilde{Y}_\beta(s)\tilde{\vec{E}}_s(\vec{r},s) \tag{11.13}$$

for current or source electric field chosen as an eigenmode. The impedance eigenmodes can then also be referred to as admittance eigenmodes. The inverse of an impedance operator (matrix) is of course designated an admittance operator (matrix).

C. Impedance loading of antennas and scatterers

Suppose now that we have formulated a matrix or integral equation of the impedance type for a perfectly conducting object excited by some incident wave or other source electric field. Let this case be denoted by

$$(\tilde{Z}_{n,m}(s)) \cdot (\tilde{J}_n(s)) = (\tilde{E}_{s_n}(s))$$

$$\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) \,;\, \tilde{\vec{J}}(\vec{r}',s) \right\rangle = \tilde{\vec{E}}_s(\vec{r},s) \tag{11.14}$$

Now let us load the object with an impedance function so that

$$\tilde{\vec{E}}(\vec{r},s) = \tilde{\vec{E}}_{inc}(\vec{r},s) + \tilde{\vec{E}}_{sc}(\vec{r},s) = \tilde{\vec{\vec{Z}}}_\ell(\vec{r},s) \cdot \tilde{\vec{J}}(\vec{r},s) \tag{11.15}$$

where the impedance loading $\tilde{\vec{\vec{Z}}}_\ell(\vec{r},s)$ may in its general form be a dyadic function of position and frequency. The electric field in this relation is the total electric field including the incident plus scattered electric field for cases that both are present. For surface type objects the electric fields and surface current densities have only their components parallel to the surface considered.

Now the total electric field is

$$\tilde{\vec{E}}(\vec{r},s) = \tilde{\vec{E}}_s(\vec{r},s) + \tilde{\vec{E}}_r(\vec{r},s) = \tilde{\vec{\vec{Z}}}_\ell(\vec{r},s) \cdot \tilde{\vec{J}}(\vec{r},s) \tag{11.16}$$

where the four terms are, respectively, the total electric field, the source field (or the incident field in a scattering problem), the radiated (or scattered) field, and the impedance loading field (IR). If

$$\vec{\vec{T}} = \vec{\vec{I}} - \vec{n}\vec{n} \tag{11.17}$$

is the tangential dyadic to the surface of an object where $\vec{n}$ is the outward pointing unit normal vector, then for a perfectly conducting object we have on the surface

$$\vec{\vec{T}}(\vec{r}) \cdot \tilde{\vec{E}}(\vec{r}, s) = \vec{0} \tag{11.18}$$

For zero loading impedance (a perfectly conducting body) then we have (where $\tilde{\vec{E}}_s$ is by definition tangential to the surface)

$$\tilde{\vec{E}}_s(\vec{r}, s) = -\vec{\vec{T}}(\vec{r}) \cdot \tilde{\vec{E}}_r(\vec{r}, s)$$

$$= \left\langle \tilde{\vec{Z}}(\vec{r}, \vec{r}'; s) ; \tilde{\vec{J}}_s(\vec{r}', s) \right\rangle \tag{11.19}$$

For the loaded surface type body ($\tilde{\vec{Z}}_\ell$ being a sheet impedance) equation 11.16 takes the form

$$\tilde{\vec{E}}_s(\vec{r}, s) = -\vec{\vec{T}}(\vec{r}) \cdot \tilde{\vec{E}}_r(\vec{r}, s) + \tilde{\vec{Z}}_\ell(\vec{r}, s) \cdot \tilde{\vec{J}}_s(\vec{r}, s)$$

$$= \left\langle \tilde{\vec{Z}}(\vec{r}, \vec{r}'; s) ; \tilde{\vec{J}}_s(\vec{r}', s) \right\rangle + \tilde{\vec{Z}}_\ell(\vec{r}, s) \cdot \tilde{\vec{J}}_s(\vec{r}, s)$$

$$= \left\langle \tilde{\vec{Z}}(\vec{r}, \vec{r}'; s) + \tilde{\vec{Z}}_\ell(\vec{r}, s)\delta_s(\vec{r} - \vec{r}') ; \tilde{\vec{J}}_s(\vec{r}', s) \right\rangle \tag{11.20}$$

where δ_s represents a surface (two dimensional) delta function or in matrix form

$$\left(\tilde{E}_{s_n}(s)\right) = \left[\left(\tilde{Z}_{n,m}(s)\right) + \left(\tilde{Z}_{\ell_{n,m}}(s)\right)\right] \cdot \left(\tilde{J}_{s_n}(s)\right) \tag{11.21}$$

Thus the impedance loading changes the impedance matrix or operator from the unloaded case to a new loaded impedance matrix or operator.

This type of result applies not only to sheet impedance loading of perfectly conducting surfaces but also to volume loading as well. To see this write the radiated (or scattered) electric field as a volume integral over the current density as

$$\tilde{\vec{E}}_r(\vec{r},s) = -\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) \,;\, \tilde{\vec{J}}(\vec{r}';s) \right\rangle \tag{11.22}$$

where $\tilde{\vec{\vec{Z}}}$ is an impedance operator as before with volume integration used. Again the operator is assumed properly defined to take care of the integration near $\vec{r} = \vec{r}'$. A generalized Ohm's law of the medium is

$$\begin{aligned}\tilde{\vec{J}}(\vec{r},s) &= \tilde{\vec{\vec{Y}}}_\ell(\vec{r},s) \cdot \tilde{\vec{E}}(\vec{r},s) \\ &= \tilde{\vec{\vec{Y}}}_\ell(\vec{r},s) \cdot \left[\tilde{\vec{E}}_s(\vec{r},s) + \tilde{\vec{E}}_r(\vec{r},s)\right]\end{aligned} \tag{11.23}$$

where for a simple conducting dielectric

$$\begin{aligned}\tilde{\vec{\vec{Y}}}_\ell(\vec{r},s) &= [\sigma + s(\epsilon - \epsilon_o)]\vec{\vec{I}} \\ &\equiv \tilde{\vec{\vec{Z}}}_\ell^{-1}(\vec{r},s)\end{aligned} \tag{11.24}$$

where σ and ϵ are the conductivity and permittivity, respectively, of the medium. More general forms of this admittance loading are also possible.

Assuming the permeability of the medium is the same as free space ($\mu = \mu_o$) then we can combine equations 11.22 and 11.23 to give

$$\tilde{\vec{E}}_s(\vec{r},s) = \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) \,;\, \tilde{\vec{J}}(\vec{r},s) \right\rangle + \tilde{\vec{\vec{Y}}}_\ell^{-1}(\vec{r},s) \cdot \tilde{\vec{J}}(\vec{r},s)$$

$$= \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) + \tilde{\vec{\vec{Z}}}_\ell(\vec{r},s)\delta(\vec{r}-\vec{r}') \,;\, \tilde{\vec{J}}(\vec{r}',s) \right\rangle \qquad (11.25)$$

which is of the same form as equation 11.20 except that it is a volume integral equation. It is this type of equation which is interesting for our present development. In matrix form it is

$$\left(\tilde{E}_{s_n}(s)\right) = \left[\left(\tilde{Z}_{n,m}(s)\right) + \left(\tilde{Z}_{\ell_{n,m}}(s)\right)\right] \cdot \left(\tilde{J}_n(s)\right) \qquad (11.26)$$

D. Eigenimpedance modification by uniform scalar impedance loading

The basic idea is to now have the impedance loading appear as a change in the eigenvalues. Taking equations 11.25 and 11.26 as the general form of our equation of interest let us first choose a simple type of impedance loading specified by

$$\tilde{\vec{\vec{Z}}}_\ell(\vec{r},s) \equiv \vec{Z}_\ell(s)\vec{\vec{I}} \quad \text{(over the object)} \qquad (11.27)$$

In this form the loading impedance is scalar and independent of the spatial coordinates over the domain of integration (the object of interest) with appropriate modification to allow for sheet impedances on surface

type objects. Note that the scalar $\tilde{Z}_\ell(s)$ is still allowed to be a function of the complex frequency.

Find eigenimpedances and eigenmodes as before for the impedance operator (or matrix) without loading as

$$\left\langle \tilde{\vec{\vec{Z}}}(s) ; \tilde{\vec{R}}_\beta(s) \right\rangle = \tilde{Z}_\beta(s)\tilde{\vec{R}}_\beta(s)$$

$$\left\langle \tilde{\vec{L}}_\beta(s) ; \tilde{\vec{\vec{Z}}}(s) \right\rangle = \tilde{Z}_\beta(s)\tilde{\vec{R}}_\beta(s) \qquad (11.28)$$

Next apply the loaded impedance operator to the eigenmodes in equation 11.28 to obtain

$$\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) + \tilde{Z}_\ell(s)\vec{\vec{I}}\delta(\vec{r}-\vec{r}') ; \tilde{\vec{R}}_\beta(\vec{r}',s) \right\rangle$$

$$= \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) ; \tilde{\vec{R}}_\beta(\vec{r}',s) \right\rangle + \tilde{Z}_\ell(s)\tilde{\vec{R}}_\beta(\vec{r},s)$$

$$= [\tilde{Z}_\beta(s) + \tilde{Z}_\ell(s)]\tilde{\vec{R}}_\beta(\vec{r},s) \qquad (11.29)$$

$$\left\langle \tilde{\vec{L}}_\beta(\vec{r},s) ; \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) + \tilde{Z}_\ell(s)\vec{\vec{I}}\delta(\vec{r}-\vec{r}') \right\rangle$$

$$= \left\langle \tilde{\vec{L}}_\beta(\vec{r},s) ; \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) \right\rangle + \tilde{Z}_\ell(s)\tilde{\vec{L}}_\beta(\vec{r}',s)$$

$$= [\tilde{Z}_\beta(s) + \tilde{Z}_\ell(s)]\tilde{\vec{L}}_\beta(\vec{r}';s)$$

Hence the loaded impedance operator has the same eigenmodes as the unloaded impedance operator. Furthermore, the eigenimpedances of the loaded impedance operator are equal to those of the unloaded impedance operator plus the loading impedance. This transformation can be indicated symbolically by

$$\tilde{\vec{R}}_\beta \xrightarrow[\text{loading}]{\text{impedance}} \tilde{\vec{R}}_\beta$$

$$\tilde{\vec{L}}_\beta \xrightarrow[\text{loading}]{\text{impedance}} \tilde{\vec{L}}_\beta \qquad (11.30)$$

$$\tilde{Z}_\beta \xrightarrow[\text{loading}]{\text{impedance}} \tilde{Z}_\beta + \tilde{Z}_\ell$$

Write the solution of our impedance integral equation (11.25) as

$$\tilde{\vec{J}}(\vec{r}, s) = \sum_\beta \frac{1}{\tilde{Z}_\beta(s) + \tilde{Z}_\ell(s)} \frac{\tilde{\vec{R}}_\beta(\vec{r}, s)}{\langle \tilde{\vec{R}}_\beta(s); \tilde{\vec{L}}_\beta(s) \rangle} \langle \tilde{\vec{L}}_\beta(s); \tilde{\vec{E}}_s(s) \rangle \qquad (11.31)$$

One can now see that the solution for our impedance loaded object can be obtained for a wide variety of loading impedances $\tilde{Z}_\ell(s)$ using the once-obtained eigenimpedances and eigenmodes of the unloaded object.

E. Eigenimpedance modification by nonuniform dyadic impedance loading

Now choose the loading impedance of the type

$$\tilde{\vec{\vec{Z}}}_\ell(\vec{r}, s) \equiv \tilde{Z}_\ell(s) \vec{\vec{f}}(\vec{r}) \qquad (11.32)$$

as a scalar impedance function of the complex frequency times a dyadic space function with the property

$$\vec{\vec{h}}(\vec{r}) \cdot \vec{\vec{f}}(\vec{r}) \cdot \vec{\vec{h}}^{-1}(\vec{r}) = \begin{pmatrix} f_1(\vec{r}) & 0 & 0 \\ 0 & f_2(\vec{r}) & 0 \\ 0 & 0 & f_3(\vec{r}) \end{pmatrix} \tag{11.33}$$

$$f_n(\vec{r}) > 0 \quad \text{for } n = 1, 2, 3$$

i.e., that $\vec{\vec{f}}(\vec{r})$ is diagonalizable and positive definite (to assure impedance realizability).

Our integral equation now has the form

$$\tilde{\vec{E}}_s(\vec{r}, s) = \left\langle \tilde{\vec{\vec{Z}}}(\vec{r}, \vec{r}'; s) + \tilde{Z}_\ell(s)\vec{\vec{f}}(\vec{r})\delta(\vec{r} - \vec{r}') \,;\, \tilde{\vec{J}}(\vec{r}', s) \right\rangle \tag{11.34}$$

There are many approaches to manipulating this equation into a form in which the eigenvalues are modified by the "addition" of the loading impedance. Consider one such approach in which we multiply through by $\vec{\vec{f}}^{-1}$ to give

$$\tilde{\vec{E}}'_s(\vec{r}, s) = \left\langle \tilde{\vec{\vec{Z}}}'(\vec{r}, \vec{r}'; s) + \tilde{Z}_\ell(s)\vec{\vec{f}}^{-1}\delta(\vec{r} - \vec{r}') \,;\, \tilde{\vec{J}}(\vec{r}, s) \right\rangle \tag{11.35}$$

where

$$\tilde{\vec{E}}'_s(\vec{r}, s) \equiv \vec{\vec{f}}^{-1}(\vec{r}) \cdot \tilde{\vec{E}}_s(\vec{r}, s)$$

$$\tilde{\vec{\vec{Z}}}'(\vec{r}, \vec{r}'; s) \equiv \vec{\vec{f}}^{-1}(\vec{r}) \cdot \tilde{\vec{\vec{Z}}}(\vec{r}, \vec{r}'; s) \tag{11.36}$$

Equation 11.35 is now of the same form as equation 11.25.

We have modified unloaded eigenimpedances and eigenmodes from

$$\left\langle \tilde{\vec{Z}}'(s) ; \tilde{\vec{R}}'_\beta(s) \right\rangle = \tilde{Z}'_\beta(s)\tilde{\vec{R}}'_\beta(s)$$

$$\left\langle \tilde{\vec{L}}'_\beta(s) ; \tilde{\vec{Z}}'(s) \right\rangle = \tilde{Z}'_\beta(s)\tilde{\vec{R}}'_\beta(s) \qquad (11.37)$$

For $\tilde{Z}_\ell(s) \neq 0$ we have

$$\tilde{\vec{R}}'_\beta \xrightarrow[\text{loading}]{\text{impedance}} \tilde{\vec{R}}'_\beta$$

$$\tilde{\vec{L}}'_\beta \xrightarrow[\text{loading}]{\text{impedance}} \tilde{\vec{L}}'_\beta \qquad (11.38)$$

$$\tilde{Z}'_\beta \xrightarrow[\text{loading}]{\text{impedance}} \tilde{Z}'_\beta + \tilde{Z}_\ell$$

just as in equation 11.30. This gives a solution

$$\tilde{\vec{J}}(\vec{r},s) = \sum_\beta \frac{1}{\tilde{Z}'_\beta(s)+\tilde{Z}_\ell(s)} \frac{\tilde{\vec{R}}'_\beta(r,s)}{\left\langle \tilde{\vec{R}}_\beta(s) ; \tilde{\vec{L}}_\beta(s) \right\rangle} \left\langle \tilde{\vec{L}}'_\beta(s) ; \vec{\vec{f}}^{-1} \cdot \tilde{\vec{E}}_s(s) \right\rangle \qquad (11.39)$$

as an expansion in terms of the modified impedance eigenmodes. Again a wide variety of choices is possible for $\tilde{Z}_\ell(s)$, the frequency dependent part of the loading impedance.

There are other ways to define modified impedance operators and associated modified eigenimpedances and eigenmodes. Some other ways

may prove more useful than the present one. The present derivation shows that the basic equation for the change of the eigenimpedances with loading as in equations 11.30 carries through to the case of spatially varying loading impedance as in equations 11.32 and 11.38.

F. Eigenimpedance synthesis

Synthesis is an established discipline in the field of electric circuits. One purpose of this note is to extend some circuit synthesis concepts to antennas and scatterers. There are various types of synthesis one might speak of for antennas and scatterers depending on what is being synthesized. Some attention has been given by various other investigators to pattern synthesis (spatial distribution of the scattering at some frequency or frequency band). However, in this note a different type of synthesis is considered.

What the eigenmode expansion appears particularly suited for is synthesis of the frequency dependence of the radiated or scattered fields. This type of synthesis is directly related to classical circuit synthesis for which one is referred to various texts such as that by Guillemin.[14]

The essence of the synthesis technique discussed here is to synthesize the loading impedance to make the eigenimpedances have desired features. Note that for a chosen form of the spatial dependence $\vec{\vec{f}}(\vec{r})$ of the eigenmode invariance and eigenimpedance transformation of equations 11.38 apply. For our known set of eigenmodes obtained from the object geometry and $\vec{\vec{f}}(\vec{r})$ we can vary the scalar impedance function $\tilde{Z}_\ell(s)$ to modify the eigenvalues to some new frequency dependence while keeping the other terms in the solution (equation 11.39) unchanged.

There are various criteria one might place on $\tilde{Z}'_\beta(s) + \tilde{Z}_\ell(s)$ for obtaining some optimum frequency dependence. If the antenna or scatterer were driven so as to excite only one eigenmode then the problem would be somewhat simplified. However in the general case all the eigenmodes and the associated eigenimpedances are included.

An interesting approach to the eigenimpedance synthesis is obtained from SEM considerations. Assuming that in some region of interest in the s plane there are only pole singularities in the response then one can choose the loading impedance so as to move the poles to desired locations. For broadband or transient antenna and scattering problems the solution may be dominated by a few poles for the frequency band or time regime of interest. For such problems a pole synthesis would appear to have some desirable features.

The poles of the response are the zeros of the eigenimpedances, i.e.,

$$\tilde{Z}_\beta(s_{\beta,\beta'}) + \tilde{Z}_\ell(s_{\beta,\beta'}) = 0 \tag{11.40}$$

In this synthesis procedure one can regard the eigenimpedance zeros (the $s_{\beta,\beta'}$) as variables to be altered depending on the given $\tilde{Z}_\beta(s)$ and choosable $\tilde{Z}_\ell(s)$. To do this consider some set of eigenimpedances to be of interest as designated say by $\beta = 1, 2, \cdots, M$. This choice might be based on the known positions of the zeros of the unloaded eigenimpedances. Then for each β choose where its set of zeros should be, i.e., choose the $s_{\beta,\beta'}$. Then set

$$\tilde{Z}_\ell(s_{\beta,\beta'}) = -\tilde{Z}_\beta(s_{\beta,\beta'}) \tag{11.41}$$

and see whether or not an impedance function $\tilde{Z}_\ell(s)$ can be synthesized with this set of values. This procedure can be thought of as an M fold circuit synthesis since in effect, while one impedance loading is chosen, M eigenimpedances are synthesized.

Of course there are some difficulties with this approach. While the $\tilde{Z}_\ell(s)$ synthesized will satisfy equation 11.40 at the chosen $s_{\beta,\beta'}$ there may be other zeros introduced as well. Furthermore the eigenimpedances for $\beta > M$ will also be affected. Thus one may have to be careful with the degrees of freedom allowed to the loading impedance.

As part of a synthesis of $\tilde{Z}_\ell(s)$ one must find a realizable $\tilde{Z}_\ell(s)$. The loading impedance might be active or passive as desired within the constraint of stable eigenimpedances. Circuit theory needs to be applied to this synthesis with perhaps a few twists not found in conventional applications.

Concerning the practical realization of $\tilde{Z}_\ell(s)$ in terms of lumped elements one must consider that the loading impedance must approximate the continuous loading impedance chosen for the synthesis. Hence $\tilde{Z}_\ell(s)\vec{\vec{f}}(\vec{r})$ is approximated as many small lumped element networks (perhaps including distributed elements such as transmission lines) inserted into the antenna or scatterer at many places. Questions of how many circuits and how small they should be can then also be asked.

It would be helpful in this general synthesis theory to understand the general properties of the unloaded eigenimpedances $\tilde{Z}_\beta(s)$. Their passive nature will have much to say about their zeros, poles, and entire functions, and even branch cuts in objects require their introduction. For surface type objects one can consider the unloaded eigenimpedances with relation to the exterior and interior fields. One may be able to regard the eigenimpedances as the parallel combination of an exterior and an interior part where the interior part satisfies Foster's theorem (poles and zeros alternating on the $i\omega$ axis). There is clearly much to be explored here.

The eigenimpedance synthesis would seem to apply to some important practical problems. In EMP simulator design it can be used to synthesize waveforms on simulators that utilize impedance loading. In radar scattering it gives some control over the radar cross section of the target and allows one to vary certain of the aspect invariant properties of the target (in particular the poles of the response) in a controlled manner. In EMP interaction it may give an optimization procedure in certain cases to minimize the currents and charges on the objects over some frequency band.

XII. Summary

This note has covered much ground. Expansions in terms of eigenmodes have various interesting and useful properties for describing the electromagnetic properties of objects. The modes form biorthogonal sets and can be physically constructed, at least approximately. Eigenmode representations can also be singularity expanded, thereby providing a further degree of splitting up (factorization) of the solution.

There is much to be learned about the EEM in terms of special theorems of an energy or reciprocity nature. The various types of electromagnetic integral equations also define their own types of eigenmodes and eigenvalues with presumably their own special characteristics. Various types of objects need to be considered to see what special characteristics these objects give to the EEM and SEM. These objects can be both those with separable coordinates for which "analytic" results can be obtained, and those of more general shape for which "numerical" formulations are needed. The implications of object geometrical symmetry on the EEM and SEM could be considered in some detail. The eigenmodes can be extended away from the object to represent the fields and far fields.

The eigenvalues order the natural frequencies but since the eigenvalues are different for different integral equations one would like to know how such differences affect the natural frequency ordering. The asymptotic forms of the eigenvalues and eigenmodes for large s are also of interest. Impedance eigenvalues (eigenimpedances) are useful for synthesizing loading impedances; perhaps other types of eigenvalues are also useful in this regard. In the method of moments (MoM) there is some error in this numerical formulation for a given choice of basis functions and testing functions. The EEM should give some approaches to quantifying such errors in terms of eigenvalue errors and eigenmode errors; such errors may then also be interpretable in terms of changes in the SEM quantities such as natural frequencies.

XIII. Epilogue

After thinking about the EEM and SEM for some time now and having learned a few things in the process it is apparent that there is much more yet to learn. This is a humbling experience to realize more and more how little one knows and how much more there is to be done. It would seem that such is the nature of science and that God wanted it that way. Now it is time to take yet more steps.

. . . . . .

The woods are lovely, dark and deep.
But I have promises to keep,
And miles to go before I sleep,
And miles to go before I sleep.

Robert Frost,
Stopping by Woods on a
Snowy Evening

XIV. References

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