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CHARGE DENSITY INDUCED ON THE SURFACE OF A PROLATE SPHEROID ILLUMINATED BY A PLANE WAVE ELECTROMAGNETIC FIELD

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ABSTRACT

The charge density induced on the surface of a highly conducting prolate spheroid in the presence of a plane wave electromagnetic field is obtained. The derived formula is approximate except at the tips of the spheroid, where it yields exact results. In the theoretical development shadowing effect is ignored. One application of the theory is the determination of the total permissible current in a missile whose axis is perpendicular to the ground plane, for a vertically polarized incident field, before dielectric breakdown of the ambient medium occurs at the tip of the missile.

INTRODUCTION

A few years ago an exact formula was obtained for the total axial current in a prolate spheroidal receiving and scattering antenna when the incident field is directed parallel to the major axis of the spheroid.[1] In the theoretical development the circumferential variation in surface current density was not considered, i.e. shadowing effect was ignored.[2] Nevertheless, when the ratio of the major to minor axis of the prolate spheroid is large accurate values for the surface current density are obtained because the phase shift of the incident field along any diameter of the structure is small.

Both analytical and numerical solution techniques are presented in this paper for determining the charge density induced on the surface of a prolate spheroid by an incident plane wave with the electric field directed parallel to the axis of the spheroid. Although the shadowing effect is ignored, the analytical formula becomes exact at the tips of the spheroid where the charge density generally reaches a maximum. The analytical formula is obtained via a boundary value problem solution and in most cases is less convenient to evaluate than the numerical solution. However the analytical solution is the more accurate. But a comparison of the two results indicates that the numerical solution may be sufficiently accurate in many cases.

Penetration of electromagnetic energy through electrically small openings in otherwise shielded enclosures has been studied recently.[3-6] In the sequence of steps to solve the problem it is necessary to determine the charge and current densities established on the structure by the total electric field and magnetic field at the position of the opening when metallically closed.[3] Since a prolate spheroid is geometrically similar to a missile the charge density expression obtained in this paper is directly applicable to the shielding problem.

In concluding this section of the paper, the writers wish to mention that a method for estimating the electric field near the end of an antenna has been developed by King.[7] This result has application in the study of the proximity field hazard of linear antennas and in the study of the power handling limitation due to dielectric breakdown of the ambient medium. A comparison of King's expression with the corresponding result of the presented formulation is made.

ANALYSIS

1. ANALYTIC SOLUTION

The electric field incident on a prolate spheroid may be expressed by the relation

$$\overline{E}^i = -j \frac{1}{k} E_o \sum_{m,\ell} A_{m\ell} \, {}^{y}_{e}\overline{M}^{(1)}_{m\ell} \qquad (1)$$

and the scattered field may be written

$$\overline{E}^s = -j \frac{E_o}{k} \sum_{m,\ell} [\alpha_{m\ell} \, {}^{x}_{o}\overline{M}^{(4)}_{m\ell} + \beta_{m\ell} \, {}^{y}_{e}\overline{M}^{(4)}_{m\ell}] \qquad (2)$$

where the expansion coefficients $[A_{m\ell}, \alpha_{m\ell}, \beta_{m\ell})$ and the solenoidal vector wave functions $[{}^{x}_{o}\overline{M}^{(4)}_{m\ell}, {}^{y}_{e}\overline{M}^{(1)}_{m\ell}, {}^{y}_{e}\overline{M}^{(4)}_{m\ell}]$ are defined in the literature.[1,8] Here $E_o = E^{inc}_{direct}$ for an isolated structure; for a hemispheroid with image $E_o = E^{inc}_{direct} + E^{inc}_{image} = 2E^{inc}_{direct}$. The radian wave number is $k = 2\pi/\lambda$. Since the prolate spheroid is perfectly conducting, the induced surface charge density is

$$\rho_s = \epsilon_o \hat{\xi} \cdot (\overline{E}^i + \overline{E}^s) \quad . \qquad (3)$$

where ϵ_o is the dielectric constant of free space. In the rationalized mks system of units, $\epsilon_o = 8.85 \times 10^{-12}$ F/m. Using (1) and (2) in (3) yields,

$$\rho_s(\eta,\phi) = -j\,\frac{\varepsilon_o E_o}{k}\,\frac{(1-\eta^2)^{\frac{1}{2}}}{F(\xi^2-\eta^2)^{\frac{1}{2}}}\left\{\sum_{m,\ell} A_{m\ell}\left[-S'_{m\ell}R^{(1)}_{m\ell}\cos\phi\cos m\phi + \frac{m\,\eta}{1-\eta^2}S_{m\ell}R^{(1)}_{m\ell}\sin\phi\sin m\phi\right]\right.$$

$$+\sum_{m,\ell}\alpha_{m\ell}\left[S'_{m\ell}R^{(4)}_{m\ell}\sin\phi\sin m\phi - \frac{m\eta}{1-\eta^2}S_{m\ell}R^{(4)}_{m\ell}\cos\phi\cos m\phi\right]$$

$$\left.+\sum_{m,\ell}\beta_{m\ell}\left[-S'_{m\ell}R^{(4)}_{m\ell}\cos\phi\cos m\phi + \frac{m\eta}{1-\eta^2}S_{m\ell}R^{(4)}_{m\ell}\sin\phi\sin m\phi\right]\right\} \tag{4}$$

where (ξ,η,ϕ) are the prolate spheroidal coordinates.

Of considerable interest is the charge density at the ends of the spheroid, i.e., $\rho_s(\pm 1,\phi)$. Before evaluating (4) note that

$$\lim_{\eta\to 1}(1-\eta^2)^{-\frac{1}{2}}\,S_{m\ell}(c,\eta) = \begin{cases}\displaystyle\sum_{n=0,1}{}' d_n^{1\ell}(c)\,\frac{(2+n)!}{2n!} & m = 1\\ 0 & \text{otherwise}\end{cases} \tag{5}$$

$$\lim_{\eta\to 1}(1-\eta^2)^{\frac{1}{2}}\,\frac{d}{d\eta}\,S_{m\ell}(c,\eta) = \begin{cases}\displaystyle -\sum_{n=0,1}{}' d_n^{1\ell}(c)\,\frac{(n+2)!}{2n!} & m = 1\\ 0 & \text{otherwise}\end{cases} \tag{6}$$

where $c = kF$ and $d_n^{1\ell}(c)$ are defined in the literature.[8] (The interfocal distance of a prolate spheroid is 2F.) Here and in the sequel, the prime over the summation sign indicates that the summation is over only even values of n when ℓ is odd, and over only odd values of n when ℓ is even. Using (5) and (6) in (4) yields

$$\rho_s(\eta,\phi)\Big|_{\eta=\pm 1} = -j\,\frac{\varepsilon_o E_o}{c}\,\frac{1}{\sqrt{\xi^2-1}}\sum_{\ell=0}^{\infty}\Big\{[A_{1\ell}R_{1\ell}^{(1)}(c,\xi)$$

$$+ (\beta_{1\ell}-\alpha_{1\ell})R_{1\ell}^{(4)}(c,\ )]\cdot\sum_{n=0,1}{}' d_n^{1\ell}(c)\,\frac{(n+2)!}{2n!}\Big\} \quad . \tag{7}$$

The expansion coefficients of the scattered field are[1]

$$\alpha_{1\ell}-\beta_{1\ell} = \frac{\xi R_{1\ell}^{(1)}(c,\xi) + (\xi^2-1)R_{1\ell}^{(1)'}(c,\xi)}{\xi R_{1\ell}^{(4)}(c,\xi) + (\xi^2-1)R_{1\ell}^{(4)'}(c,\xi)}\,A_{1\ell} \tag{8}$$

where

$$A_{1\ell} = j^{\ell}\,\frac{4S_{1\ell}(c,0)}{\sum\limits_{n=0,2,4}\frac{2(n+2)!}{(2n+3)!}\,[d_n^{1\ell}(c)]^2} \qquad \text{odd } \ell$$

$$= 0 \qquad \text{even } \ell \tag{9}$$

with

$$S_{1\ell}(c,0) = \frac{(-1)^{\frac{\ell-1}{2}}\,(\ell+1)!}{2\ell\left(\frac{\ell-1}{2}\right)!\left(\frac{\ell+1}{2}\right)!} \qquad \text{odd } \ell \tag{10}$$

Substituting (8) into (7) yields

$$\rho_s(\eta,\phi)\Big|_{\eta=\pm 1} = -j\,\frac{\varepsilon_o E_o}{c}\,\frac{1}{\sqrt{\xi^2-1}}\sum_{\ell=1,3,4,\cdots}$$

$$\cdot\Big\{A_{1\ell}\,\frac{(\xi^2-1)[R_{1\ell}^{(4)'}R_{1\ell}^{(1)} - R_{1\ell}^{(1)'}R_{1\ell}^{(4)}]}{[\xi R_{1\ell}^{(4)} + (\xi^2-1)R_{1\ell}^{(4)'}]}$$

$$\cdot\sum_{n=0,2,4,\cdots} d_n^{1\ell}(c)\,\frac{(n+2)!}{2n!}\Big\} \quad .$$

Using the Wronskian relation[8]

$$R_{1\ell}^{(4)'} R_{1\ell}^{(1)} - R_{1\ell}^{(1)'} R_{1\ell}^{(4)} = -j \frac{1}{c(\xi^2 - 1)}$$

it follows that

$$\rho_s(\eta,\phi)\Big|_{\eta=\pm 1} = - \frac{\varepsilon_o E_o}{c^2(\xi^2 - 1)^{\frac{1}{2}}} \sum_{\ell=1,3,5,\cdots} \cdot \left\{ \frac{A_{1\ell}}{[\xi R_{1\ell}^{(4)}(c,\xi) + (\xi^2 - 1) R_{1\ell}^{(4)'}(c,\xi)]} \cdot \sum_{h=0,2,4,\cdots} d_n^{1\ell}(c) \frac{(n+2)!}{2n!} \right\} . \quad (11)$$

Note that (11) is an exact equation for the charge distribution at the tips of the spheroid. However, it is not possible to develop a corresponding relationship for the charge distribution over the complete surface of the spheroid, but it is possible to obtain an approximation to the charge distribution which applies for electrically thin spheroids.

For an electrically thin prolate spheroid the surface current density is approximately

$$\vec{J}_s(\eta) \simeq \frac{I(\eta)}{2\pi r(\eta)} \hat{\eta} \quad (12)$$

where $I(\eta)$ is the total current through the cross section of the prolate spheroid at η with cross section radius $r(\eta)$ given by[4]

$$r(\eta) = F[(\xi^2 - 1)(1 - \eta^2)]^{\frac{1}{2}} .$$

From the equation of continuity the charge distribution is obtained as[9]

$$\rho_s(\eta) \simeq \frac{j}{\omega} \nabla_s \cdot \vec{J}_s$$

$$\simeq \frac{j}{\omega} \frac{1}{h_\xi h_\eta h_\phi} \frac{\partial}{\partial \eta} (h_\xi h_\phi \hat{\eta} \cdot \vec{J}_s) \tag{13}$$

where

$$h_\xi h_\eta h_\phi = F^3(\xi^3 - \eta^2) \tag{14}$$

$$h_\xi h_\phi = F^2[(\xi^2 - \eta^2)(1 - \eta^2)]^{\frac{1}{2}} \quad . \tag{15}$$

The total axial current induced on a prolate spheroid is[1]

$$I(\eta) = -j\pi \frac{E_o r(\eta)}{c\xi_o} \sum_{\ell=1,3,5,\cdots} \frac{A_{1\ell} S_{1\ell}(c,\eta)}{[\xi R_{1\ell}^{(4)}(c,\xi) + (\xi^2 - 1) R_{1\ell}^{(4)'}(c,\xi)]} \quad . \tag{16}$$

Substituting (12), (14), (15), and (16) into (13) yields for the surface charge density

$$\rho_s(\eta) \simeq \frac{\varepsilon_o E_o}{2c^2(\xi^2 - \eta^2)} \sum_{\ell=1,3,5} \frac{A_{1\ell} \frac{d}{d\eta} [(\xi^2 - \eta^2)^{\frac{1}{2}} (1 - \eta^2)^{\frac{1}{2}} S_{1\ell}(c,\eta)]}{[\xi R_{1\ell}^{(4)}(c,\xi) + (\xi^2 - 1) R_{1\ell}^{(4)'}(c,\xi)]} \quad . \tag{17}$$

Note that if (17) is evaluated at the tips of the spheroid

$$\rho_s(\eta)\Big|_{\eta=\pm 1} \simeq \pm \frac{\varepsilon_o \mathcal{E}_o}{c^2 - (\xi^2 - 1)^{\frac{1}{2}}} \sum_{\ell=1,3,5,\cdots}$$

$$\cdot \left\{ \frac{A_{1\ell}}{[\xi R_{1\ell}^{(4)}(c,\xi) + (\xi^2 - 1) R_{1\ell}^{(4)'}(c,\xi)]} \sum_{n=0,2,4,\cdots} d_n^{1\ell}(c) \frac{(n+2)!}{2n!} \right\} \quad . \tag{18}$$

As expected, (18) is in exact agreement with (11). This occurs since the electrically thin condition assumed for (13) is always satisfied in the vicinity of the tip of the spheroid.

Whenever the interfocal distance of the spheroid assumes an odd integer multiple of a half wave length the expressions for the current and charge on the spheroid assume particularly simple forms. For $c = m\pi/2$, where m is an odd integer,[1]

$$I(\eta) = I(0) \cos\left(\frac{m\pi}{2}\eta\right) \tag{19}$$

Using eqn. (19) in eqns. (12) and (13) yields

$$\rho_s(\eta) \simeq -j \frac{60\ \varepsilon_o\ I(0)}{F\sqrt{(\xi^2 - \eta^2)(\xi^2 - 1)}} \left\{ \sin \frac{m\pi}{2}\eta + \frac{2\eta}{m\pi} \frac{\cos \frac{m\pi}{2}\eta}{\xi^2 - \eta^2} \right\} \tag{20}$$

In terms of the axial coordinate, z, the total axial length, 2h, and the midsection radius, a, the foregoing becomes

$$I(z) = I(0) \cos\left(\frac{m\pi}{2}\frac{z}{h}\right) \tag{21}$$

$$\rho_s(z) \simeq -j \frac{60\ \varepsilon_o I(0)}{a\sqrt{\xi^2 - (z/h)^2}} \left\{ \sin \frac{m\pi}{2}\frac{z}{h} + \frac{2z}{m\pi h} \frac{\cos \frac{m\pi}{2}\frac{z}{h}}{\xi^2 - (z/h)^2} \right\} \tag{22}$$

where

$$\xi^2 = \frac{h/a}{\sqrt{(h/a)^2 - 1}} \tag{23}$$

Keep in mind that (22) becomes an exact expression for $z = \pm h$ and (21) is always exact.

If eqn. (22) is applied to a driven antenna where I(0) becomes the antenna terminal current then eqn. (22) can be used to estimate the intensity of the fields near the ends of a driven antenna. Using a highly approximate methodology King[7] has developed an estimate of these fields for a cylindrical antenna that is very thin and is operating near resonance, i.e. $kF = m\pi/2$. King's estimate agrees exactly with (22) although it is determined by a completely different procedure.

2. NUMERICAL SOLUTION

By using the so called method of moments an integral equation can be solved to obtain the axial current induced on a prolate spheroid. A comparison of the numerically obtained results with analytical results yielded agreement to about three significant figures (see ref. 10). However there were no charge density comparisons.

A knowledge of the axial current is sufficient to determine the surface charge density provided the spheroid is electrically thin. The surface charge density on a body of revolution such as a prolate spheroid is[10]

$$\rho_s(z) = \frac{1}{-j\omega}\,\frac{1}{a(z)}\,\frac{d}{dt}\,[a(z)J_s(z)] \tag{24}$$

where a(z) is the radius of the body at the axial coordinate z and t is the surface coordinate measured from one tip of the spheroid. It is readily shown that the foregoing yields

$$\rho_s(z) = j\,\frac{1}{\omega}\,\frac{I'(z)}{2\pi}\,\{[a(z)]^2 + [a(z)a'(z)]^2\}^{-\frac{1}{2}} \tag{25}$$

where the prime indicates a derivative with respect to the axial coordinate z. For the prolate spheroid

$$a(z) = \frac{\sqrt{\xi^2 - 1}}{\xi}\,[h^2 - z^2] \quad , \tag{26}$$

noting that $h = F\xi$ and $a = F\sqrt{\xi^2 - 1}$.

In ref. 10 the current distribution is represented by a piecewise sinusoid, i.e.

$$I(z) = \sum_{n=1}^{N} f_n(z)\ \chi(z;z_n,z_{n+1}) \tag{27}$$

where

$$f_n(z) = \frac{I(z_{n+1}) \sin k(z - z_n) + I(z_n) \sin k(z_{n+1} - z)}{\sin k\,(z_{n+1} - z_n)}$$

$$X(z;\, z_n,\, z_{n+1}) = 1 \qquad z_n \le z \le z_{n+1}$$

$$= 0 \qquad \text{otherwise}$$

The derivative of the foregoing,

$$I'(z) = \sum_{n=1}^{N} f'_n(z)\, X(z; z_n, z_{n+1}) \quad , \tag{28}$$

exhibits discontinuities at the z_n points. If the current derivative at the discontinuity is defined to be the arithmetic mean then one obtains

$$I'(z_n) = \frac{k}{2 \sin k(z_{n+1} - z_n)} [I(z_{n+1}) - I(z_{n-1})] \tag{29}$$

for $n = 2, 3, \cdots, N$. No special procedure is required to obtain the current derivatives at the end points z_1 and z_{N+1}.

Using eqns. (26), (28) and (29) in eqn. (25) allows the numerical determination of the surface charge density induced on a prolate spheroid. It is of interest to compare the numerical results with data obtained from the analytical expression. In particular it is important to determine the accuracy of the numerical formulation inasmuch as the current expansion used yields a discontinuous charge distribution.

NUMERICAL RESULTS

First the charge densities from the analytical and numerical solution techniques are compared. The following dimensions are considered: h = 10.125 in and a = 0.9965 in. Hence $\xi = 1.0048787$ and from the numerical solution technique for f = 290.2 mHz ($kF = \pi/2$),

$$I(0) = (4.5168 + j\ 0.22131)E^{inc}\ \mathrm{m\ A}$$

where E^{inc} is the amplitude of the incident electric field directed parallel to the axis of the spheroid. Using the foregoing in (22) allows the computation of the charge density from the analytical formula. The corresponding numerically obtained charge density is compared with the results from the analytical formula and exhibited in Table 1. Of course the analytical results are more accurate but the agreement between the numerical and analytical results indicates that for most practical applications the numerical solution may be sufficiently accurate.

A second calculation is presented to illustrate the determination of the maximum current (and incident field) that occurs just before corona breakdown appears. Consider now $kF = \pi/2$ and $\xi = 1.00063$. At sea level corona breakdown occurs at about 3×10^6 V/m. The maximum current on the spheroid is that which would produce a charge density

$$\rho_s(1) = \varepsilon_o(3 \times 10^6\ \mathrm{V/m})$$

at the tip of the spheroid. Thus from (22) the maximum current occuring just before corona breakdown is

$$|k\ I(0)| = 98.99\ \mathrm{A/m} \qquad (30)$$

TABLE 1: Surface Charge Density Induced on a Prolate Spheroid with the Incident Electric Flux Density, D^{inc}, parallel to the Axis. $\xi = 1.0048787$, $2h = 0.51435$ m and $f = 2.902$ mHz

ρ_s/D^{inc}

$\eta = z/h$	ANALYTICAL SOLUTION	NUMERICAL SOLUTION
0	0	0
0.1	0.11507 −j 2.3486	0.04954 −j 1.6825
0.2	0.23114 −j 4.7174	0.10318 −j 3.3761
0.3	0.34954 −j 7.1340	0.16548 −j 5.0951
0.4	0.47248 −j 9.6429	0.24181 −j 6.8656
0.5	0.60385 −j 12.324	0.34025 −j 8.7425
0.6	0.75140 −j 15.336	0.47301 −j 10.828
0.7	0.93158 −j 19.013	0.66316 −j 13.348
0.8	1.1877 −j 24.240	0.96699 −j 16.912
0.9	1.6858 −j 34.407	1.5840 −j 23.932
1.0	5.3045 −j 108.26	7.7833 −j 109.62

To determine the incident field required to produce such a current (16) is used. From (16)

$$|kI(0)| = 0.02737\, E^{inc} \tag{31}$$

Combining (30) and (31) yields the maximum incident field

$$E^{inc} = 3.6 \text{ kV/m}$$

above which the induced charge distribution causes corona breakdown to occur. A corresponding computation can be made for determining the maximum input power to a spheroidal antenna occurring just before corona breakdown.

CONCLUSION

Using the boundary-value-problem approach an exact solution is obtained for the surface charge density induced on the tips of a prolate spheroid by an incident electric field directed parallel to the axis of the spheroid. Also obtained for the total surface charge density is an expression approximately valid for electrically thin spheroids. Data from this approximation is used to verify the method of moments solution for the charge density.

Also a sample calculation is presented for determining the maximum current (and accompanying incident field) supported on the prolate spheroid just before corona breakdown occurs.

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